SOLUTIONS TO PRACTICE EXAM #1 FOR THIRD MIDTERM EXAM

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1. Notes

This practice exam has more problems than the real exam will. To make most effective use of this document, take the exam under conditions simulating the real exam — no book, no calculator.

(1) Short answer questions:

(a) Why doesn't L'Hopital's rule imply that

$$\lim_{x\to 0} \frac{x+3}{x^2+3x+1} = \lim_{x\to 0} \frac{1}{2x+3} = \frac{1}{3}?$$

Proof. This limit is not in the admissible form for L'Hopital's rule; on the contrary, evaluating at $x = 0$ we get $\frac{3}{1} = 3$. □

(b) Please state the mean value theorem.

Proof. The mean value theorem says that if f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in [a, b]$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. □

(c) We developed the theory of integration using rectangles. Why didn't we use circles instead?

Proof. Circles are much harder to work with; it isn't at all clear how to concisely describe an efficient tiling of a region with circles. □

(2) Sketch the curve $\ln(x^2 - 3x + 2)$ using derivative information.

Proof. Omitted. □

(3) Compute the following limits:

(a) $\lim_{x\to\infty} x \sin(\frac{1}{x})$.

Proof. Rewriting, we get

$$\lim_{x\to\infty} \frac{\sin(\frac{1}{x})}{\frac{1}{x}}.$$

This is in the form $\frac{0}{0}$, so L'Hopital's rule applies. Differentiating, we find that the limit can be computed as

$$\lim_{x\to\infty} \frac{\cos(\frac{1}{x}) - x^{-2}}{-x^{-2}} = \lim_{x\to\infty} \cos(\frac{1}{x}).$$

Evaluating, this is 1. □

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(b) $\lim_{x\to 0} \frac{x^2+2x+1}{3x^2-4x+5}$.

Proof. Both the numerator and the denominator are continuous, so evaluating at $x = 0$ gives us $\frac{1}{5}$. □

(c) $\lim_{x\to\infty} \frac{\ln(\sqrt{x})}{x^2}$.

Proof. As $x \to \infty$, this is of the form $\frac{\infty}{\infty}$, so we can apply L'Hopital's rule. Differentiating, we get

$$\lim_{x\to\infty} \frac{x^{-\frac{1}{2}}\frac{1}{2}x^{-\frac{1}{2}}}{2x} = \lim_{x\to\infty} \frac{1}{4x^2} = 0.$$

□

(4) Find the following derivatives.

(a) $f(x) = \int_3^{x^2} \ln y dy$.

Proof. Using the fundamental theorem of calculus and the chain rule, we have

$$f'(x) = (2x)\ln(x^2).$$

□

(b) $f(x) = \int_5^{\ln(x^2+x+1)} e^z dz$.

Proof. Similarly, we have

$$f'(x) = \frac{2x+1}{x^2+x+1} e^{\ln(x^2+x+1)} = (2x+1).$$

□

(5) A rope which is 20 feet long is cut into two pieces; one piece is used to make a circle, and one piece is used to make a square. How should the rope be cut in order to maximize the area enclosed? To minimize the area enclosed?

Proof. We cut the rope into pieces of length c and $20 - c$. The radius of the circle is given by solving the equation $c = 2\pi r$ for r, and the area of the square is $(\frac{20-c}{4})^2$. Assembling, the function for the area is then

$$A(r) = \pi(\frac{c}{2\pi})^2 + \frac{(20-c)^2}{16} = \frac{c^2}{4\pi} + \frac{(20-c)^2}{16}$$

$$= \frac{4c^2}{16\pi} + \frac{\pi(400 - 40c + c^2)}{16\pi} = \frac{(4+\pi)c^2 - 40\pi c + 400\pi}{16}.$$

Differentiating, we find

$$A'(r) = \frac{(8+2\pi)c - 40\pi}{16}.$$

Solving for the critical points, we get

$$(8+2\pi)c = 40\pi,$$

which implies $c = \frac{40\pi}{8+2\pi}$. The second derivative test tells us that this is a minimum. To maximize, we look at the endpoints of $[0, 20]$ — $A(0) = \frac{200}{8} = 25$, and $A(20) = \frac{400-400+200}{8} = 25$. Thus, the maximum is achieved at either endpoint. □

(6) For the function $f(x) = e^{-x^2}$,

(a) Approximate the definite integral $\int_{-1}^{4} f(x)dx$ using a Riemann sum with 5 intervals and using the lefthand side of the rectangle.

Proof. Dividing the interval $[-1, 4]$ into 5 pieces, we have endpoints $-1, 0, 1, 2, 3, 4$, and each piece has width 1. Therefore, the desired Riemann sum is computed as

$$(1)(e^{0^2}) + (1)(e^{1^2}) + (1)(e^{2^2}) + (1)(e^{3^2}) + (1)(e^{4^2}).$$

□

(b) Approximate the definite integral $\int_{-1}^{4} f(x)dx$ using a Riemann sum with 5 intervals and using the midpoint of the rectangle.

Proof. Similarly, we have

$$(1)(e^{(\frac{1}{2})^2}) + (1)(e^{(\frac{3}{2})^2}) + (1)(e^{(\frac{5}{2})^2}) + (1)(e^{(\frac{7}{2})^2}) + (1)(e^{(\frac{9}{2})^2}).$$

□

(c) Write the definite integral as a limit expression.

Proof. The definite integral can be computed as

$$\lim_{n\to\infty} \sum_{i=0}^{n} \frac{5}{n}(e^{-(-1+\frac{5i}{n})}).$$

□

(d) Can you use substitution to find an antiderivative for this function?

Proof. No, this integral can't be evaluated using substitution (you can try all the possiblities!). □

(7) Compute the following definite integrals:

(a) $\int_{-3}^{3} x\sqrt{x-1}dx$.

Proof. Substitute $u = x - 1$. Then we find that we are computing the definite integral

$$\int_{-4}^{2} (u+1)\sqrt{u}du = \int_{-4}^{2} u^{\frac{3}{2}} + u^{\frac{1}{2}}du.$$

(Notice we have transformed the limits.) Evaluating, an antiderivative is

$$\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}$$

and so the fundamental theorem of calculus tells us that we have

$$(\frac{2}{5}2^{\frac{5}{2}} + \frac{2}{3}2^{\frac{3}{2}}) - (\frac{2}{5}(-4)^{\frac{5}{2}} + \frac{2}{3}(-4)^{\frac{3}{2}}).$$

□

(b) $\int_{0}^{10} \frac{e^{\frac{1}{x}}}{x^2}dx$.

Proof. Here, we substitute $u = \frac{1}{x}$. Then $du = -\frac{1}{x^2}dx$, so the integral transforms to

$$\int -e^u du.$$

Notice that I've left off the limits, because there's an issue with u at $x = 0$. An antiderivative is given by $-e^u$, and to study the definite integral we could investigate

$$\lim_{u \to \infty} \int_{\infty}^{\frac{1}{10}} -e^u du.$$

We'll return to these sorts of calculations, but for now if you got this far or noted the issue, you did this correctly. □

(c) $\int_{-5}^{0}(x \sin(3x^2 + 4) + 2^x)dx$.

Proof. Splitting this into pieces, we have

$$\int_{-5}^{0} x \sin(3x^2 + 4)dx + \int_{-5}^{0} 2^x dx.$$

The second we can simply evaluate directly, and for the first we use the substitution $u = 3x^2 + 4$. Then $du = 6xdx$, so we have

$$\int_{29}^{4} \frac{1}{6} \sin(u)du.$$

This then becomes $-\frac{1}{6}(\cos(29) - \cos(4))$. The first part has antiderivative $\frac{2^x}{\ln 2}$, and so we get $\frac{1}{\ln 2} - \frac{2^{-5}}{\ln 2}$. □

(8) Suppose that $-1 \leq f(x) \leq 1$. What can we say about $\int_{-3}^{3} f(x)dx$?

Proof. We know that

$$-1(6) \leq \int_{-3}^{3} f(x)dx \leq (1)(6).$$

□

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