SOLUTIONS FOR PRACTICE EXAM #2 FOR THIRD MIDTERM EXAM

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1. Notes

This practice exam has more problems than the real exam will. To make most effective use of this document, take the exam under conditions simulating the real exam — no book, no calculator.

(1) Short answer questions:

(a) Suppose that $f'(x) = g'(x)$. What can we say about f and g?

Proof. The mean value theorem implies that $f(x) = g(x) + k$, where k is a constant. □

(b) Define the definite integral $\int_a^b f(x)dx$.

Proof. The definite integral is given by the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{(b-a)}{n}(f(a + \frac{(b-a)i}{n})).$$

More generally, the definite integral can be defined as

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{(b-a)}{n}(f(x_i)),$$

where $x_i \in [a + \frac{(b-a)i}{n}, a + \frac{(b-a)(i+1)}{n}]$. □

(c) Explain what the second derivative tells us about maxima and minima.

Proof. The second derivative tells us about concavity; so the sign of the second derivative tells us whether a critical point is a maximum or a minimum (specifically, if $f''(x) > 0$ then f is concave up and so x is a local minimum). □

(d) Riemann sums can be computed using the left endpoint, the midpoint, and the right endpoint. Which is better? Explain.

Proof. It depends on f; one can construct examples for which each of these choices is better than the others, and examples for which they are all the same. □

(2) Sketch the curve $f(x) = \frac{e^x}{1-e^x}$ by finding maxima, minima, concavity, inflection points, and so forth.

Proof. Omitted. □

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(3) Define the function $f(x) = \frac{e^x - 1}{x}$ for $x \neq 0$, and $f(0) = 1$. Compute $f'(0)$ using L'Hopital's rule. (Hint: use the definition of the derivative.)

Proof. Recall that the derivative at 0 is defined to be

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Plugging in, we get

$$\lim_{h \to 0} \frac{\frac{e^h - 1}{h} - 1}{h} = \lim_{h \to 0} \frac{\frac{e^h - 1}{h} - \frac{h}{h}}{h} = \lim_{h \to 0} \frac{e^h - 1 - h}{h^2}.$$

This is of the form $\frac{0}{0}$, so we can apply L'Hopital's rule. Differentiating, we get the limit

$$\lim_{h \to 0} \frac{e^h - 1}{2h}.$$

This is still of the form $\frac{0}{0}$, so we differentiate again and get

$$\lim_{h \to 0} \frac{e^h}{2} = \frac{1}{2}.$$

□

(4) Find the derivative of

$$\int_{-x^2}^{x^2} \ln(\sin x) dx.$$

(Hint: split this up!)

Proof. The trick is to observe that we can rewrite the definite integral as

$$\int_{-x^2}^{0} \ln(\sin x) dx + \int_{0}^{x^2} \ln(\sin x) dx.$$

(Note that $-x^2 \leq 0 \leq x^2$, so 0 is a particularly good choice; other points would allow the split, but provide problems for some values of x.) Then the fundamental theorem of calculus tells us that the derivative of

$$\int_{-x^2}^{0} \ln(\sin x) dx$$

is $-(-2x)(\ln \sin(x^2))$ and of

$$\int_{0}^{x^2} \ln(\sin x) dx$$

is $(2x)(\ln \sin(-x^2))$. □

(5) Solve the following optimization problems:

(a) Find the area of the rectange of maximal area which has two vertices on the x-axis and two vertices with $x > 0$ and lying on the curve $9 - x^2$.

Proof. The rectangle is specified by a point $(x, 9 - x^2)$. The length of the side of the rectangle perpendicular to the x-axis is $2x$ and length of the side of the rectangle perpendicular to the y-axis is $2(9 - x^2)$.

Therefore, the area of the rectangle is $4x(9-x^2)$. To maximize, we find the critical points. Solving, we have

$$4x(9-x^2)=0,$$

which implies that the critical points are at 3, -3, and 0. Clearly 0 is a minimum, and since $x>0$ we are only interested in the point $x=3$. The second derivative of the area is $36-12x^2$, and at 3 this is $36-12(9)<0$, which implies we have a maximum. □

(b) I want to make a cylindrical barrel to hold 10 cubic meters of molasses. Assume that the sides cost 36 per square meter and the (circular) ends cost 18 per square meter. What should the dimensions of the cheapest barrel with the required volume be?

Proof. The volume of the cylinder is $V=\pi r^2 h$. The cost of the barrel is described by the function

$$C(r,h)=2\pi r^2+2\pi rh.$$

Rewriting the volume expression, we have $10=\pi r^2 h$, which we can rewrite as $\pi rh=\frac{10}{r}$. Substituting, we have a cost function in terms of r given as

$$C(r)=2\pi r^2+\frac{20}{r}.$$

Differentiating, we have

$$C'(r)=4\pi r-\frac{20}{r^2}.$$

Solving for the critical points, we're looking at

$$4\pi r=\frac{20}{r^2},$$

or

$$r^3=\frac{5}{\pi}.$$

This implies that $r=(\frac{5}{\pi})^{\frac{1}{3}}$. Checking the second derivative, we see that this is a local maximum, and look at $C'(r)$ tells us that it is a global max. □

(6) Let $f(x)=\frac{1}{x}$.

(a) Approximate the definite integral $\int_1^5 f(x)dx$ using a Riemann sum with 2 intervals and using the lefthand side of the rectangle.

Proof. The intervals are $[1,3]$ and $[3,5]$, of width 2. Therefore, the Riemann sum is

$$(2)(\frac{1}{1})+(2)(\frac{1}{3})=2+\frac{2}{3}=\frac{8}{3}.$$

□

(b) Approximate the definite integral $\int_1^5 f(x)dx$ using a Riemann sum with 4 intervals and using the lefthand side of the interval.

Proof. The intervals are $[1,2]$, $[2,3]$, $[3,4]$, and $[4,5]$, of width 1. Therefore, the Riemann sum is

$$(1)(\frac{1}{1}) + (1)(\frac{1}{2}) + (1)(\frac{1}{3}) + (1)(\frac{1}{4}).$$

$\square$

(c) Compute the definite integral using the fundamental theorem of calculus.

Proof. Computing, we have

$$\int_1^5 \frac{1}{x} dx = \ln(5) - \ln(1) = \ln(5).$$

$\square$

(7) Compute the following definite integrals.

(a) $\int_0^{100} 8x^2 e^{x^3+5} dx$.

Proof. We use substitution; set $u = x^3 + 5$. Then $du = 3x^2 dx$, and we get

$$\int_5^{10005} \frac{8}{3} e^u du = \frac{8}{3}(e^{10005} - e^5).$$

$\square$

(b) $\int_{-1}^1 e^x \sin(e^x) dx$.

Proof. We use substitution; set $u = e^x$. Then $du = e^x dx$, and we get

$$\int_{e^{-1}}^{e^1} \sin(u) du = \sin(e^1) - \sin(e^{-1}).$$

$\square$

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