TILING SPACES ARE CANTOR SET FIBER BUNDLES

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ABSTRACT. We prove that fairly general spaces of tilings of $\mathbb{R}^d$ are fiber bundles over the torus T^d, with totally disconnected fiber. In fact, we show that each such space is homeomorphic to the d-fold suspension of a $\mathbb{Z}^d$ subshift (or equivalently, a tiling space whose tiles are marked unit d-cubes). The only restrictions on our tiling spaces are that 1) the tiles are assumed to be polygons (polyhedra if $d > 2$) that meet full-edge to full-edge (or full-face to full-face), 2) only a finite number of tile types are allowed, and 3) each tile type appears in only a finite number of orientations. The proof is constructive, and we illustrate it by constructing a "square" version of the Penrose tiling system.

Barge, Jacklitch, and Vago [BLV] used this bundle structure in classifying tiling spaces of dimension 1, where the bundle structure was already known to exist.

1. Introduction and results

The paper of Anderson and Putnam [A-P] inspired this work. These authors show that *substitution* tiling spaces are a special case of expanding attractors, a concept introduced [W1], to study the dynamics of diffeomorphisms. It is well known in both camps that these spaces are locally the topological product of a Cantor set and a disk of the appropriate dimension. But what are they globally? Perhaps a bundle over a manifold with fiber a Cantor set? Though false for expanding atractors, [F-J] this is true (Theorem 1) for our tiling spaces, which in particular are *flat*, and thus one sees that the appropriate base space will be the torus.

Fiber bundles are used throughout mathematics. For a bundle over the d-dimensional torus, there are d commuting, "characteristic homeomorphisms" $f_i : F \to F$, of the fiber, F, which do characterize the bundle, (as a bundle). And though the Cantor set fiber makes these

Mathematics subject classification(1985 revision). Primary 58F13, 54F15. Key words: tiling, tiling space, branched surface, Cantor fiber bundle.

R.F.W gratefully acknowledges helpful conversations with Marcy Barge and hospitality of the Mathematics Department of Montana State University. This work is partially supported by ...

harder to deal with than, say, plane bundles, this is a start and has been effective in the one dimesnional case (Franks, [F], Barge and Diamond [B-D], and [BLV]). In higher dimensions, these bundles are generically, (e.g., not products) less "flabby" and thus, hopefully, allow stronger invariants than Parry-Sullivan and Bowen-Franks types.

The basic ingredients here are tiling systems P of $\mathbb{R}^d$, and the corresponding tiling spaces $X(P)$. These spaces are assumed to satisfy the following hypotheses:

1. The tiles are (triangulated) polyhedra that meet full-face to full-face.
2. Only a finite number of tile types appear. In this counting, tiles that are translations of one another are considered to be the same type, but tiles that are rotations of one another are considered to be different.
3. The space $X(P)$ is a closed and nonempty subset of the space of all tilings that can be formed from the tiles in P.
4. The space $X(P)$ is invariant under translation.

We will henceforth refer to both the tiling system and the associated topological space by the same letter P.

Our first result is:

Theorem 1.1. *A tiling space that satisfies the above hypotheses is a fiber bundle over the torus, with totally disconnected fiber.*

Note that we do not assume that the tilings are quasiperiodic, or generated by a substitution, or even that they are nonperiodic. The only difference between these cases is the nature of the fiber. The fiber for a substitution tiling, or a quasiperiodic tiling, will be a Cantor set, while the fiber for a (d-fold) periodic tiling system will be a finite collection of points.

The requirement of polygonal tiles is mostly for convenience. A tiling, such as the Penrose chickens, whose edges follow standard shapes, can be deformed to a tiling system with polygonal tiles, and therefore is a fiber bundles over a torus. The requirement that tiles appear in only a finite number of orientations is more serious. The techniques of this paper do not apply to pinwheel-like tiling spaces [ref].

Our second result is:

Theorem 1.2. *A tiling space P that satisfies the above hypotheses is homeomorphic to a tiling space S whose tiles are marked d-cubes, or equivalently to the d-fold suspension of a $\mathbb{Z}^d$ subshift. The space S is defined by local matching rules if and only if P is.*

Note that this theorem proves the existence of a homeomorphism, not a topological conjugacy. The homeomorphism typically does *not* commute with translations, much less with rotations.

The proofs proceed as follows. We call a tiling space rational (integral) if each edge of each tile is given by a vector with rational (integral) coordinates. In Section 2 we show that every tiling space P can be deformed to a rational tiling space R. This deformation is a homeomorphism of tiling spaces, but not a topological conjugacy. We then show that every rational tiling is a fiber bundle over the torus. This proves Theorem 1.

In Section 3 we prove Theorem 2. We rescale the rational tiling space R into an integer tiling space, and replace the straight edges with zig-zags consisting of unit segments in the several coordinate directions. The faces then become unions of unit squares, the 3-cells become unions of unit cubes, and so on. This gives a "zig-zag" system Z. The tiles of Z may take on odd shapes, and may even be disconnected, but are unions of d-cubes. The space Z is homeomorphic (topologically conjugate, in fact) to the rescaled R. Finally, we consider each constituent d-cube of a tile z in Z to be a tile in a tiling space S, with the matching rule that wherever one such constituent appears, the other consitutents of t also appear nearby. S is a suspension of a subshift, but is also topologically conjugate to Z, and therefore homeomorphic to P.

2. Tiling spaces as fiber bundles

Lemma 2.1. *A tiling space P meeting the above hypotheses is homeomorphic to a rational tiling space R. Furthermore, R has finite type if and only if P does.*

Proof. For greater clarity, we go through the proof in dimension 2 and later indicate how it applies, essentially unchanged, in any dimension. We also illustrate how each step applies to the Penrose system.

Let the tiles of a tiling space P be represented by polygons C_i, $i = 1, \ldots c$ in the plane. If tiles C_j and C_k can meet in a tiling along a common edge, then we identify those edges of C_j and C_k. After these identifications, we have a finite number of directed line intervals $I_1, \ldots, I_n$, and the boundary of each C_j is a sequence of translates of such directed intervals. Let v_i be the vector that gives the displacement of I_i.

In the Penrose "B-tile" system, there are forty triangular tiles, namely those shown in figure 1 and their rotations by multiples of $2\pi/10$. We let t denote the rotation by $2\pi/10$, so $t^5 A$ means tile A rotated by π.

Although A is congruent to B, they are considered separate tiles. Similarly, C and D are considered distinct. Because of the identifications, there are 40 edges, not 120. In the Penrose system, the vectors are given by:

We wish to construct new tiles C_k', $k = 1, \dots, c$ and new intervals I_i', $i = 1, \dots, n$ such that, if the boundary of C_k is $I_{i_1}, \dots, I_{i_k}$, then the boundary of C_k' is $I_{i_1}', \dots, I_{i_k}'$, and such that the coordinates of each v_i' are rational. To do this one must merely solve the linear equations

$$v_{i_1}' + \cdots + v_{i_k}' = 0 \tag{1}$$

for each tile C_k. That is, we have a system of homogeneous linear equations, whose coefficients are integers. One can always find a rational basis for the space of solutions. This space is nonempty, since $v_1, \dots, v_n$ is a solution. One can therefore find rational solutions arbitrarily close to the original solution set $\{v_i\}$. By choosing the v_i''s sufficiently close to the v_i's we can insure that the C_k''s are nonempty, correctly oriented, and homeomorphic to the C_k's.

In the Penrose system, our equations are

$$\begin{aligned} v(t^n a) + v(t^n b) - v(t^n c) &= 0, \quad n = 0, \dots, 9 \\ v(t^{n+6} a) + v(t^{4+n} b) - v(t^n c) &= 0, \quad n = 0, \dots, 9 \\ -v(t^{n+4} a) + v(t^{n+1} b) - v(t^n d) &= 0, \quad n = 0, \dots, 9 \\ -v(t^{n+2} a) + v(t^{n+3} b) - v(t^n d) &= 0, \quad n = 0, \dots, 9 \end{aligned} \tag{2}$$

The following is an integer set of solutions:

$$\begin{aligned} v(a) &= v(t^4 b) &= (1, 4) \\ v(ta) &= v(t^5 b) &= (-1, 4) \\ v(t^2 a) &= v(t^6 b) &= (-3, 2) \\ v(t^3 a) &= v(t^7 b) &= (-4, 0) \\ v(t^4 a) &= v(t^8 b) &= (-3, -2) \\ v(t^5 a) &= v(t^9 b) &= (-1, -4) \\ v(t^6 a) &= v(b) &= (1, -4) \\ v(t^7 a) &= v(tb) &= (3, -2) \\ v(t^8 a) &= v(t^2 b) &= (4, 0) \\ v(t^9 a) &= v(t^3 b) &= (3, 2) \end{aligned}$$

$$\begin{aligned} v(c) &= (2, 0) & \qquad v(d) &= (6, 0) \\ v(tc) &= (2, 2) & v(td) &= (5, 4) \\ v(t^2 c) &= (1, 2) & v(t^2 d) &= (2, 6) \\ v(t^3 c) &= (-1, 2) & v(t^3 d) &= (-2, 6) \\ v(t^4 c) &= (-2, 2) & v(t^4 d) &= (-5, 4) \\ v(t^5 c) &= (-2, 0) & v(t^5 d) &= (-6, 0) \\ v(t^6 c) &= (-2, -2) & v(t^6 d) &= (-5, -4) \end{aligned}$$

$$
\begin{aligned}
v(t^7c) &= (-1,-2) & \qquad v(t^7d) &= (-2,-6)\\
v(t^8c) &= (1,-2) & \qquad v(t^8d) &= (2,-6)\\
v(t^9c) &= (2,-2) & \qquad v(t^9d) &= (5,-4)
\end{aligned} \tag{3}
$$

Note that by picking integer solutions, we have broken the 10-fold rotational symmetry of the Penrose. This is to be expected, as one cannot represent $\mathbb{Z}_{10}$ in $GL(2,\mathbb{Q})$.

Now pick homeomorphisms (say, linear maps) from each I_i to the corresponding I_i', and extend these to homeomorphisms from C_k to C_k'. We now use these homeomorphisms to convert an arbitrary tiling by the tiles $\{C_k\}$ into a tiling by the tiles $\{C_k'\}$. As we shall see, this procedure is continuous and has a continuous inverse, and so defines a homeomorphism between the tiling space P and a rational tiling space R.

Let t be an arbitrary tiling in the tiling space P. We will construct a corresponding tiling $t' \in R$, beginning at the origin. The origin in t sits at a point in a closed tile C_k; we let the origin in t' sit at the corresponding point in C_k'. We then grow outwards, so that the tiling t' is combinatorially identical to t, only with each tile of type C_j replaced with a tile of type C_j', and each edge of type I_j replaced by I_j'. This is shown in figure 2, where a patch of the original Penrose tiling is replaced by a patch of rational Penrose tiling.

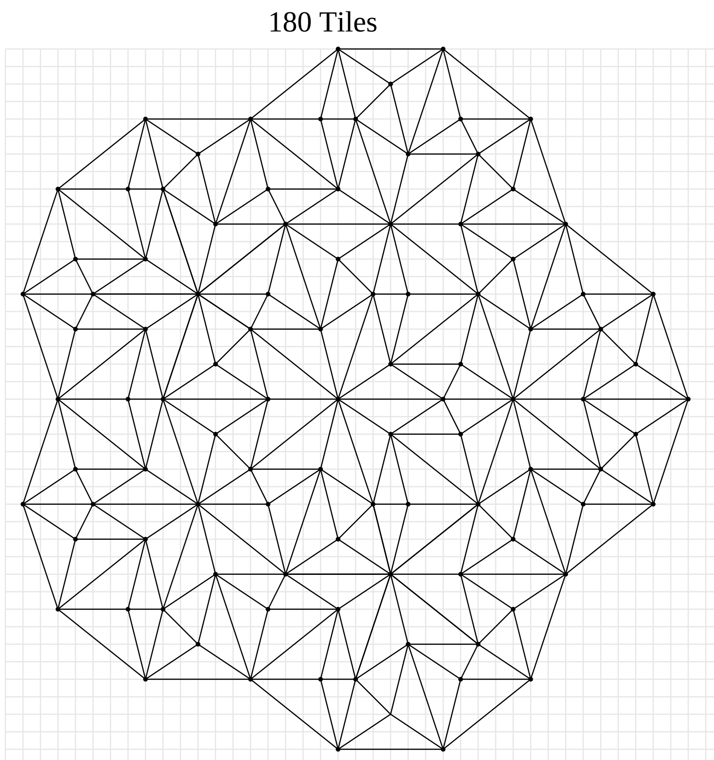

Figure 3

To see that this construction does result is a tiling, we must show that the vertices of t' are well defined. Let x be a vertex of t, and consider two paths from a vertex y of the central seed tile to x. The algebraic difference of these two paths, namely zero, is the boundary of a sum of tiles in t. By equations (1), the algebraic difference of the corresponding sum of vectors v' is also zero. This means that either path can be used to determine the position of x', the vertex in t' that corresponds to x. Once the vertices are defined, the edges and faces follow.

This transformation is continuous. If two tilings t and $\tilde{t}$ agree on a large neighborhood of the origin, then t' and $\tilde{t}'$ agree on a large neighborhood of the origin. If t and $\tilde{t}$ differ by a small translation, then t' and $\tilde{t}'$ differ by a small translation, as determined by the homeomorphism between the center tile of t and that of t'. (Since these homeomorphisms are extensions of homeomorphisms between edges I_i and I_i', there is no ambiguity, and no discontinuity, if the origin in t sits on the boundary of a tile.) Similarly, the reverse transformation, from tilings in R to tilings in P, is also continuous. Thus P and R are homeomorphic tiling spaces.

Finally, since each tiling t in P is combinatorially equivalent to a tiling t' in R, any local atlas for the P system can be naturally transformed into a local atlas for the R system, and vice-versa.

In dimension greater than 2, the analysis is essentially unchanged. One obtains an equation of the form (1) for each 2-face of each tile C_k. Once again, there is a rational basis to the space of solutions, and one can find a rational solutiona arbitrarily close to the original vectors. To construct homeomorphisms between tiles C_k and C_k', one must start with homeomorphisms (e.g. linear maps) between edges I_i and I_i', extend these to homeomorphisms of the 2-skeleton, then of the 3-skeleton, and so on. There are no topological obstructions. QED

To complete the proof of Theorem 1.1, we must only prove

Lemma 2.2. *Every rational tiling space is a fiber bundle over the torus.*

Proof. Let R be a rational tiling, and let D be the least common multiple of all the denominators of all the coordinates of displacement vectors v_i for the tiles in R. Rescale R by D, so that all displacement vectors are integers. Then all the vertices in any fixed tiling have the same coordinates (mod $\mathbb{Z}^d$). These coordinates give a natural projection from the space of tilings to the d-torus $\mathbb{R}^d/\mathbb{Z}^d$. QED

3. Square tiling spaces

We have shown that our general tiling space P is homeomorphic to a rational tiling space R that is of finite type if P is (and is not if P is not). By rescaling, we can assume that R is in fact integer. Topological conjugacies preserve finite type [Radin-Sadun]. To complete the proof of Theorem 1.2, it suffices to prove

Lemma 3.1. *Every rational tiling space R is, after rescaling, topologically conjugate to a square-type tiling space S.*

Proof. As before, we work first in 2 dimensions, and then sketch what modifications need to be made in higher dimensions. Also as before, we demonstrate our construction with the Penrose system.

We first rescale R so that R becomes an integer tiling. Furthermore, we assume that each tile contains a circle of radius greater than $\sqrt{2}/2$; this can always be achieved by further scaling. Next we replace each of our straight edges I_i' with zig-zags J_i, that is with sequences of unit displacements in the coordinate directions. We do this in such a way that the maximum distance of a point in J_i from the original edge

I'_i is minimized. In particular, one can always choose J_i such that this distance is no greater than $\sqrt{2}/2$. There is sometimes more than one way to minimize this distance. For example, one could replace a diagonal edge from (0,0) to (1,1) with a zig-zag from (0,0) to (1,0) to (1,1), or with a zig-zag from (0,0) to (1,0) to (1,1). In such a case, one must make a choice and apply it consistently. A possible set of zig-zags for the rational Penrose system is given in figure 3. Under this replacement, the 180-tile patch of figure 2 turns into the patch of figure 4.

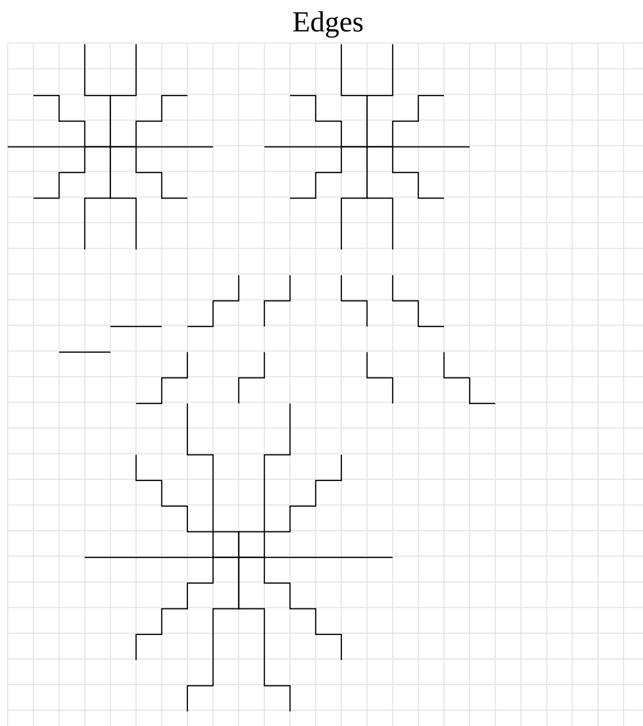

Figure 3

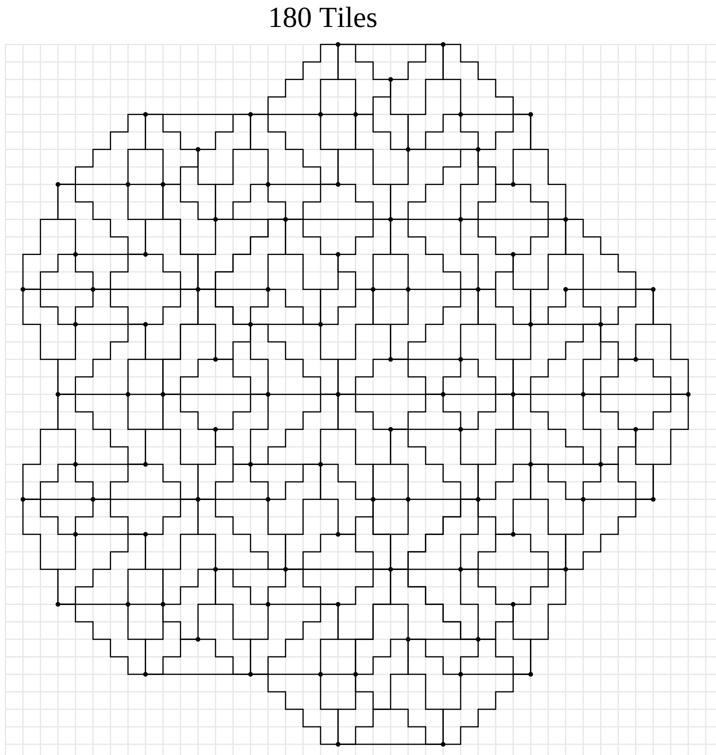

Figure 4

This defines a space Z of tilings whose edges are zig-zags. To each tiling t' in R we generate a tiling z in Z by replacing each straight edge in t' with its corresponding zig-zag. If a tile type C'_k in the R system is bounded by several straight edges I'_i, then the tile type D_k in the Z system is defined to be the region bounded by the corresponding zig-zags J_i's. The condition that C'_k contains a circle of radius greater that $\sqrt{2}/2$ ensures that D_k is nonempty. (It may, however, be disconnected). It may happen that geometrically non-congruent tile types C'_k generate congruent tile types D_k; however, as marked tiles, these D_k's should be considered distinct.

The operation of replacing straight edges with zig-zags is reversible and does not require a choice of origin. It therefore commutes with translation and defines a topological conjugacy between R and Z.

In the tiling system Z, the basic tiles are irregularly shaped regions D_k bounded by zig-zags, and we have already seen that each D_k is nonempty. Suppose that the tile D_k has area n. Then D_k can be decomposed as the union of n unit squares $D_k^1, \dots, D_k^n$. In the tiling system S, the basic tiles are the squares D_k^i, and we apply a matching rule that says that wherever one of the D_k^i squares is found, the other $n-1$ squares that make up D_k are also found nearby, arranged to form the larger region D_k. A tiling in S can therefore be amalgamated into a tiling by tiles D_k. We allow in S those tilings, and only those tilings, that amalgamate into tilings in Z. In this way, the tiling system S is naturally conjugate to Z.

The proof in higher dimensions is almost identical. In dimension 3, one must pick zig-zags J_i to replace the straight edges I_i'. If several edges I_i' bound a 2-face of a tile C_k', we must find a union of unit squares (oriented in the coordinate directions), bounded by the appropriate zig-zags J_i, that approximates this face. The tile D_k is then the solid region bounded by these zig-zag faces. In dimension $d > 3$, one works recursively, replacing edges I_i' with zig-zags J_i, then replacing 2-cells with unions of squares, 3-cells with unions of cubes, and so on up through dimension $d-1$. The tiles D_k are the d-cells bounded by the $d-1$ cells constructed in this manner. One can compute a universal bound for each dimension, so that the $d-1$ dimensional zig-zags are within that universal bound of the original faces of the C_k''s. As long as the C_k''s contain a sphere of radius greater than that bound, the resulting D_k will be nonempty.

QED

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