Classification of Denjoy continua

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1 Introduction

Imagine splitting open a line of irrational slope on the torus, splitting less and less as you move out along the line in both directions. The closure, in the torus, of the resulting pair of lines is a *Denjoy* continuum. Such continua also arise in the suspension of a minimal (Denjoy) homeomorphism of a Cantor set and occur as attractors in 'derived from Anosov' (D-A) diffeomorphisms. They are all locally the same (are in fact locally trivial fiber bundles over the circle with fiber a Cantor set) and have the same Čech cohomology. Robbert Fokkink has proved, however, that there are uncountably many distinct Denjoy continua; his theorem is that the Denjoy continua corresponding to irrational slopes α and β are homeomorphic if and only if there is a 2×2 unimodular integer matrix that maps lines of slope α (in the plane) to lines of slope β.

We reprove Fokkink's theorem by an approach that emphasizes the connection between the topology of a Denjoy continuum and the geometry of the continued fraction expansion of its slope. In the process we develop a somewhat novel development continued fractions, via geometric bifurcation theory. This geometric approach may give some insight into higher dimensional analogues to continued fractions, where the theory is far from complete.

2 Denjoy continua

We will use $\mathbb{R}/\mathbb{Z}$ as a model for the circle S^1. Rigid rotation by the real number α is then given by $R_\alpha : S^1 \to S^1,\ R_\alpha(t) = t + \alpha \pmod 1$. Corresponding to the irrational α, the Denjoy homeomorphism $d_\alpha : S^1 \to S^1$ is an orientation preserving homeomorphism of the circle characterized by the properties: the rotation number of d_α is α; there is a Cantor set $C_\alpha \subset S^1$ on which d_α acts minimally; and if u and v are any two components of $S^1 \backslash C_\alpha$ then $d_\alpha^n(u) = v$ for some integer n (see [R]). There is a Cantor function $h_\alpha : S^1 \to S^1$ that semi-conjugates d_α with R_α : h_α is a monotone surjection that collapses the components of $S^1 \backslash C_\alpha$ (and so maps C_α onto S^1) with $R_\alpha \circ h_\alpha = h_\alpha \circ d_\alpha$. The suspension

$$\text{susp}(d_\alpha) \equiv S^1 \times [0,1]/(t,1) \sim (d_\alpha(t), 0)$$

of d_α is homeomorphic with the torus $\mathbb{T}^2$ and h_α induces a semi-conjugacy $H_\alpha : \mathbb{T}^2 \to \mathbb{T}^2$ between the 'Denjoy flow' (the natural flow) on $\text{susp}(d_\alpha)$ and the 'irrational flow' (the natural flow) on the suspension

$$\text{susp}(R_\alpha) \equiv S^1 \times [0,1]/(t,1) \sim (R_\alpha(t), 0)$$

of R_α (which is also homeomorphic with $\mathbb{T}^2$). The *Denjoy continuum*, $\mathbb{D}_\alpha$, is the suspension of $d_\alpha \mid_{C_\alpha}$:

$$\mathbb{D}_\alpha \equiv C_\alpha \times [0,1]/(t,1) \sim (d_\alpha(t), 0) \subseteq \text{susp}(d_\alpha) \simeq \mathbb{T}^2.$$

In what follows, $\mathbb{T}^2$ will stand for the torus gotten by suspension of d_α or R_α or for the torus $\mathbb{R}^2/\mathbb{Z}^2$, as convenient. We will also use the same notation for subsets of $\mathbb{R}^2/\mathbb{Z}^2$ and their analogues under the homeomorphism $(x, y) \to (x - \alpha y, y)$ in $\text{susp}(R_\alpha)$.

Let ℓ be the line $y = \alpha x$ in $\mathbb{R}^2$ and, for each $r > 0$, let Q_r denote the open interval of radius r centered at O in ℓ. Let ℓ' and Q'_r denote the images of ℓ and Q_r in the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ (so in $\text{susp}(R_\alpha)$, ℓ'_r is a flow line). Since for any two components I and J of $S^1 \backslash C_\alpha$ there is an n with $d_\alpha^n(I) = J$, $U \equiv \mathbb{T}^2 \backslash \mathbb{D}_\alpha$ is an open topological disk. Let $0 < r_0 < r_1 < \ldots < r_n \ldots$ be any sequence with $r_n \to \infty$ and let $U_n = U \bigcap H_\alpha^{-1}(Q'_{r_n})$. Then the U_n are open topological disks, $U_n \subset U_{n+1}$, and $\bigcup_{n=0}^\infty U_n = U$. Thus $\mathbb{D}_\alpha = \bigcap_{n=0}^\infty (\mathbb{T}^2 \backslash U_n)$ is a nested intersection and we see that $\mathbb{D}_\alpha$ is homeomorphic with the inverse limit $\varprojlim i_n$, where $i_n : \mathbb{T}^2 \backslash U_n \to \mathbb{T}^2 \backslash U_{n-1}, n = 1, 2, \ldots$, is the inclusion.

Our immediate objective is to refine this inverse limit description of $\mathbb{D}_\alpha$. To this end, let $\mathcal{L}$ be the foliation of $\mathbb{R}^2$ by lines of slope $-1/\alpha$ and let $\mathcal{L}'$ be the image of $\mathcal{L}$ in $\mathbb{T}^2$. The foliation $H_\alpha^{-1}(L')$ of $\mathbb{T}^2 = \text{susp}(d_\alpha)$ is then transverse to the flow lines of the Denjoy flow. Let $\pi_n : \mathbb{T}^2 \backslash U_n \to Y_n$ be the quotient map that collapses components of leaves of $H_\alpha^{-1}(L')$ intersected with $\mathbb{T}^2 \backslash U_n$. That is, π_n identifies points of $\mathbb{T}^2 \backslash U_n$ whose images under H_α lie on a line segment of slope $-1/\alpha$ that misses Q'_{r_n}. There is then a uniquely determined map $g_n : Y_n \to Y_{n-1}$ such that the diagram

$$\begin{array}{ccc}
\mathbb{T}^2\backslash U_n & \xrightarrow{i_n} & \mathbb{T}^2\backslash U_{n-1} \\
\Big\downarrow{\scriptstyle \pi_n} & & \Big\downarrow{\scriptstyle \pi_{n-1}} \\
Y_n & \xrightarrow{g_n} & Y_{n-1}
\end{array}$$

commutes.

Lemma 2.1 $\lim\limits_{\leftarrow} g_n \simeq \lim\limits_{\leftarrow} i_n \simeq \mathbb{D}_\alpha$.

Proof The maps π_n induce a continuous surjection $\widehat{\pi} : \lim\limits_{\leftarrow} i_n \to \lim\limits_{\leftarrow} g_n$ by $\widehat{\pi}((x_0, x_1, \dots)) = (\pi_0(x_0), \pi_1(x_1), \dots)$. Since the diameters of the preimages $\pi_n^{-1}(y)$ go to zero uniformly in $y \in Y_n$ as $n \to \infty$, $\widehat{\pi}$ is injective.

In general, the Y_n are one-dimensional branched manifolds that have the homotopy type of the wedge of two circles. We will see in the next section that by choosing the r_n appropriately, the Y_n become homeomorphic with the wedge of two circles.

3 Geometric bifurcation and continued fractions

Let $\alpha > 0$ be irrational and recall that Q_r is the open line interval of radius r centered at the origin O in the line $y = \alpha x$. Also $\mathcal{L}$ is the family of all lines in $\mathbb{R}^2$ having slope $-1/\alpha$. In the quotient space, $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, Q'_r and $\mathcal{L}'$ are the images of Q and $\mathcal{L}$ respectively. Then $\mathbb{T}^2 - Q'_r$ is foliated by the set $\mathcal{C}'_r$ of all the connected components C of $L' \cap (\mathbb{T}^2 - Q'_r)$, all $L' \in \mathcal{L}'$.

Finally, let $K_r = (\mathbb{T}^2 - Q'_r)/\mathcal{C}'{}_r$ be the quotient space in which all the intervals of $\mathcal{C}'_r$ are collapsed to points, and let $q : \mathbb{T}^2 - Q'_r \to K_r$ be the quotient map. Then K_r has the homotopy type of the 1-skeleton of $\mathbb{T}^2$, (which is a figure 8). Furthermore, K_r is an orientable, 1-dimensional branched manifold, or "train track".

For each $z \in \mathbb{Z}^2$, let $Q_r(z) = Q_r + z$, be the translation of Q_r by z. Let

$$\mathcal{C}_r = \tau^{-1}(\mathcal{C}'_r) = \{C : C \text{ is a component of } L \cap (\mathbb{R}^2 - \cup_z Q_r(z)), L \in \mathcal{L}\} .$$

Next let $\mathcal{C}_r(O) = \{C \in \mathcal{C}_r\text{: } C \text{ has its lower end point in } Q_r(O)\}$. Let $D = D_r = \cup \mathcal{C}_r(O)$, the union of all line intervals C which have their lowest point in $Q_r(O)$. Note that D_r is a fundamental domain for the quotient

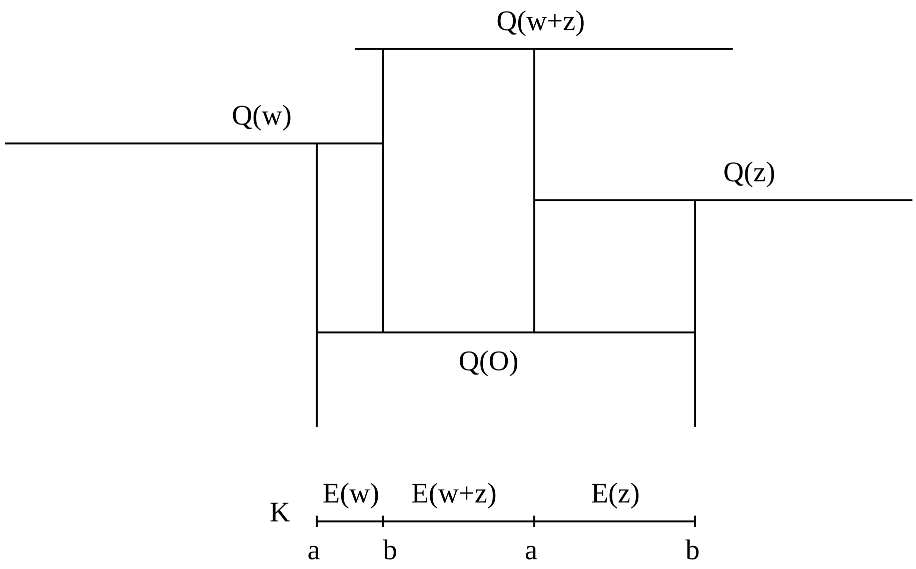

Figure 1

map $q : \mathbb{T}^2 - Q'_r \to K_r$. It follows that the line interval $Q_r = Q_r(O)$ with identifications, is a good model for K_r. We can now define the 1-cells in our complex K_r :

$$E_r(z) = \{x \in Q_r(O) : C = (x, y) \in \mathcal{C}_r \ , \ x \in Q_r(O) \ , \ y \in Q_r(z), z \in \mathbb{Z}^2\}.$$

Note that in this definition, y is above and to the left of x. We usually omit the subscript r in order to simplify the notation.

Proposition 3.1 *The fundamental domain D takes one of the 2 forms (depending upon r) shown in figures 1 and 2.*

Proof In figures 1 and 2 we indicate lines of slope α as horizontal, and those parallel to the lines in $\mathcal{L}$ as vertical. We also tone-down both scales: whereas the horizontal lines tend to be quite long and vertical segments short, we draw both at a medium scale. (See figure 4, for a realistic sketch.) The irrationality of α means that the vertical lines through the end points of $Q(O)$ will hit in the interiors of some cells, say $Q(w), Q(z)$, so that this part of our picture is just a matter of definition. The interval C_b through the right endpoint of $Q(w)$ hits a third disk, say $Q(v)$, and the interval C_a through the left endpoint of $Q(z)$ hits a possibly different disk, say $Q(v')$. Now consider the translation $T, x \mapsto x - w$, defined on the whole plane. This translation

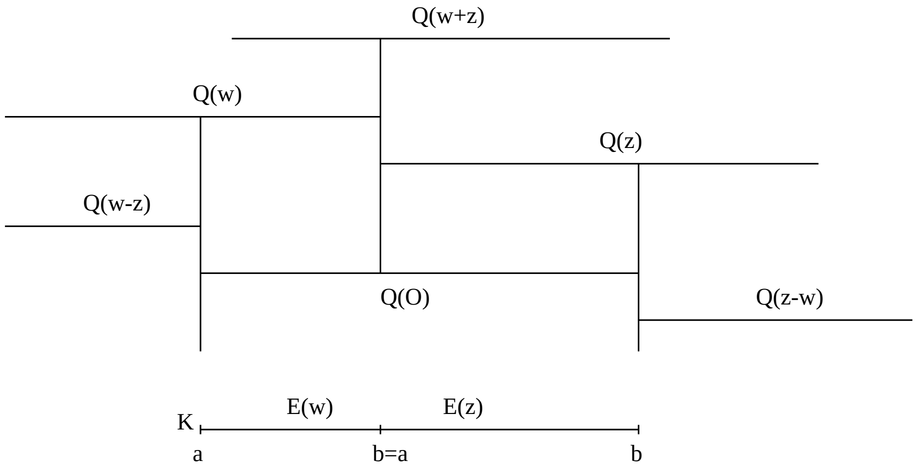

Figure 2

takes $Q(w)$ to $Q(O)$ and the interval C_b into the interval $T(C_b)$ through the left endpoint of $Q(O)$, so that $Q(v)$ is taken into $Q(z)$. Thus $v - w = z$ or $v = w + z$. Similarly the translation $T', x \mapsto x - z$, takes the interval C_a to the left endpoint of $Q(O)$ and one concludes that $v' - z = w$, and thus that $v' = w + z$ as well. This proves that D is as pictured in figure 1 when the intervals C_a and C_b are distinct.

Suppose that for some value of r we have the situation in figure 1. Then, as r increases, the situation in figure 2 occurs for some parameter value, $r = r_0$. We call this a *left bifurcation at* $[w, z]$ in case $Q(w)$ is above $Q(z)$ (as in figure 2). In case $Q(w)$ is above $Q(z)$, we call this a *right bifurcation at* $[w, z]$.

Note that after a left bifurcation at $[w, z]$, the next bifurcation will be at the pair $[w - z, z]$ whereas after a right bifurcation at $[w, z]$ the next bifurcation will be at $[w, z - w]$. In our illustration in figure 2 we see that a right bifurcation is coming up next.

Proposition 3.2 *At a bifurcation value of* r*, say at* $[w, z], K_r$ *is the wedge of 2 circles, with the left circle* $E(w)$ *and the right circle* $E(z)$*.*

Proof First, D is a fundamental domain, as every point of $\mathbb{R}^2 - \cup_z Q(z)$ lies on some element $C \in \mathcal{C}$ and there is a translation taking this interval to an interval with its base on $Q(O)$. Thus the interval $Q(O)$ oriented left to right,

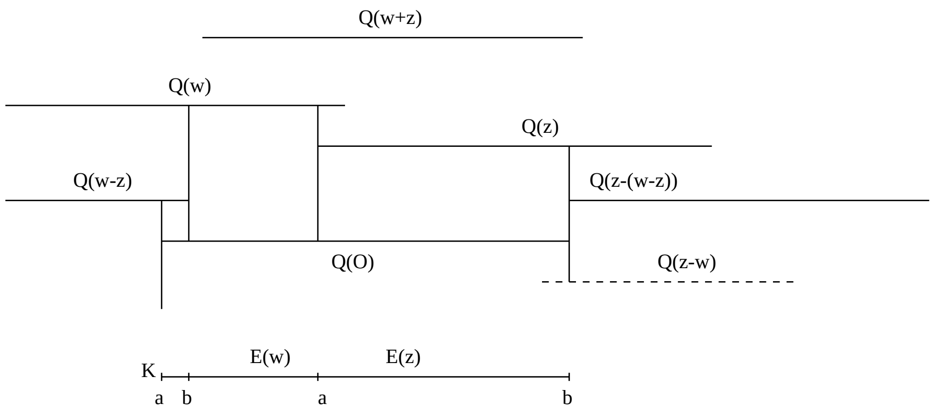

Figure 3. We have illustrated $D = D_r$ just after a left bifurcation at $[w, z]$. Note that, $Q(z - w)$ has no influence upon D since it is below the line $y = \alpha x$. Right bifurcations occur when the top disk on the left is below the top right disk.

with any repeated points identified, gives K_r. And at a bifurcation, the ends of $Q(O)$ are identified with an internal point, yielding the union of 2 circles.

The cells of K_r are labeled $E(w), E(z)$ and are oriented by the left to right orientation of $Q(O)$. As r is increased the bifurcations occur according to a pattern, a_0 of them on the left, then a_1 of them on the right, then a_2 of them on the left, etc..

There is, of course, an arithmetic algorithm for computing the continued fraction expansion of α. We recall here the geometric definition of continued fractions, such as in Stark, ([S], pp 187-8). Let ℓ be the line $y = \alpha x, V_{-2} = (1, 0), V_{-1} = (0, 1)$, and

$$V_0 = V_{-2} + a_0 V_{-1},$$

where a_0 is the unique integer such that $V_{-2} + a_0 V_{-1}$ is either on ℓ or on the same side of ℓ as V_{-2} but $V_{-2} + (a_0 + 1)V_{-1}$, is on the opposite side of ℓ as V_{-2}. In general, if V_{n-2} and V_{n-1} have been defined, we let

$$V_n = V_{n-2} + a_n V_{n-1},$$

where a_n is the unique integer such that $V_{n-2} + a_n V_{n-1}$ is either on ℓ or on the same side of ℓ as V_{n-2} but $V_{n-2} + (a_n + 1)V_{n-1}$ is on the opposite side of ℓ as V_{n-2}.

Proposition 3.3 *The sequence $a_0, a_1, a_2, \ldots$ is identical to the sequence occurring in the continued fraction expansion of α. That is*

$$\alpha = \langle a_0, a_1, a_2, \ldots \rangle = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}.$$

The proof of Proposition 3.3 follows immediately from Lemma 3.4.

Lemma 3.4 *Let $\alpha = \langle a_0, a_1, a_2, \ldots \rangle$ be a positive irrational. As r increases beyond $\max\{1/2\sqrt{1+\alpha^2}, 1/2\sqrt{1+\alpha^{-2}}\}$ bifurcations occur at each of the following pairs of points in the order listed: The first a_0 bifurcations are left bifurcations at*

$$[-V_{-2} - iV_{-1}, V_{-1}], \qquad i = 0, \ldots, a_0 - 1 \ (a_0 \textit{ may be } 0);$$

then a_1 right bifurcations at

$$[-V_0, V_{-1} + iV_0], \qquad i = 1, 2, \ldots a_1 - 1;$$

then a_2 left bifurcations at

$$[-V_0 - iV_1, V_1], \qquad i = 1, 2, \ldots a_2;$$

then a_3 right bifurcations at

$$[-V_2, V_1 + iV_2], \qquad i = 1, 2, \ldots a_3;$$

$$\ldots$$

Proof We start the induction with the simple picture in figure 4.

Thus the first bifurcations will be either a_0 left bifurcations at $[-V_{-2} - iV_{-1}, V_{-1}]$, for $i = 0, \ldots, a_0 - 1$ in case $a_0 > 0$, or a_1 right bifurcations at $[-V_0, V_{-1} + iV_0], i = 0, \ldots a_1 - 1$. (Recall that for $i > 0, a_i$ is always positive.) Suppose that at the inductive step we have found that there is a right bifurcation at $[-V_n, V_{n-1} + iV_n] = [w, v]$ for some $i \leq a_{n+1} - 1$, where n is even. There are 2 possible pairs at which the next bifurcation may occur.

As previously noted, if there is a right bifurcation at $[w, v]$, the next bifurcation is at $[w, v - w]$. Thus here, the next bifurcation is at $[-V_n, V_{n-1} + (i+1)V_n]$. If $i + 1 < a_{n+1}$ then $V_{n-1} + (i+1)V_n$ and $V_{n-1} + (i+2)V_n$ are on

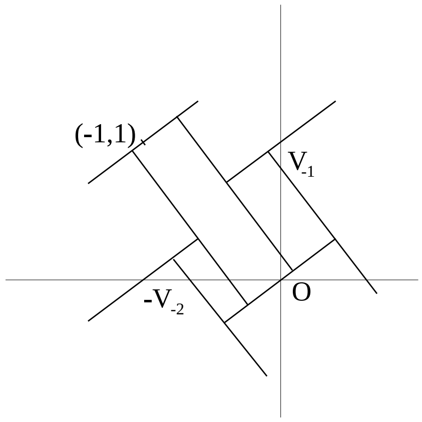

Figure 4

the same side of ℓ, (that is, both above). Hence $-V_n$ is closer to ℓ than is $V_{n-1}+(i+1)V_n$ so that the next bifurcation at $[-V_n, V_{n-1}+(i+1)V_n]$ is also a right bifurcation. If $i+1=a_{n+1}$, then $V_{n-1}+(i+1)V_n$ and $V_{n-1}+(i+2)V_n$ are on opposite sides of ℓ. Thus $V_{n-1}+(i+1)V_n$ is closer to ℓ than is $-V_n$, so that the next bifurcation, at $[-V_n, V_{n-1}+a_{n+1}V_n]$ is a left bifurcation. This completes the inductive step in case n is even. The n odd case is similar.

Remark Lemma 3.4 implies that the geometry of the continued fraction expansion of α can be recovered from the bifurcation sequence.

Next, let $r < s$ be 2 successive bifurcation values. We proceed to compute the maps $i_{rs} : K_s \to K_r$, induced by the inclusion $\mathbb{T}^2 - Q'_s \to \mathbb{T}^2 - Q'_r$.

Recall that we use the left to right orientation for the cells $E(w), E(z)$. These oriented cells are the generators for the 1-dimensional homology, taken in order.

Lemma 3.5 *If the bifurcation at r is a left bifurcation, then*

$$i_{rs\#} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

whereas, if the bifurcation at S is a right bifurcation, then

$$i_{rs\#} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

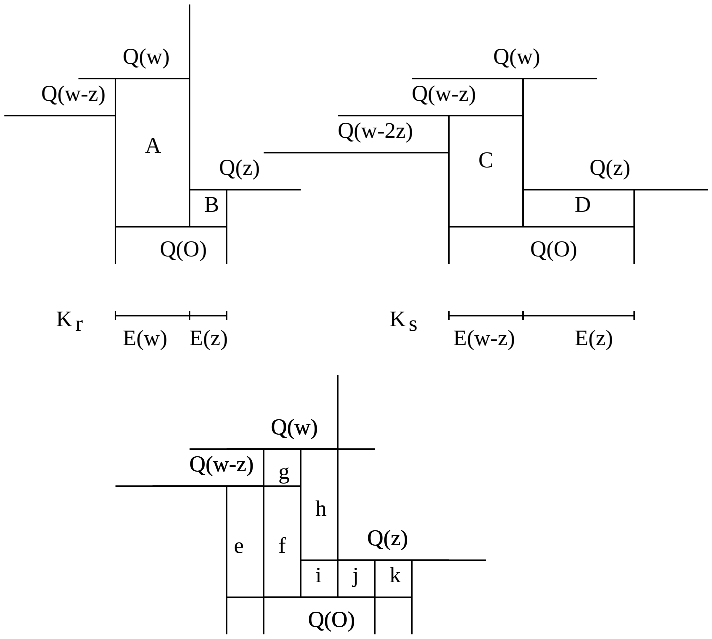

Figure 5

Proof In figure 5 we have pictured D_r and D_s separately on top, and below we have them together as they lie in $\mathbb{R}^2$.

We see that K_r has two 1-cells, $E(w), E(z)$. As we are at a bifurcation, the 3 vertical lines are equivalent in $\mathbb{T}^2$, because they differ by integer translations. Then the next bifurcation occurs when the radius has increased to the point where $Q(w-z)$ and $Q(z)$ extend to the same vertical segment, halfway between the sides of A. (Of course, all the Qs are growing longer.) It is helpful to note that the left cell, $E(w-z)$ is the same length as $E(w)$, the one it replaces, whereas $E(z)$, the right cell is now longer. In fact, its length is the sum of the previous 2 lengths.

To simplify notation, we drop the rs, refer to the map induced by the inclusion just as i, and proceed to compute $i : K_s \to K_r$. Recall that $q : D_r \to K_r$ is the quotient map. First, note that $e \sim h$, as they sit above the left end of $Q(O)$ and $Q(z)$ respectively, and thus differ by an integral translation. Similarly, $g \sim k$. Then

$$i(E(w-z)) = i(q(C)) = i(q(e \cup f)) = i(q(h \cup f)) = q(A) = E(w).$$

On the other hand

$$\begin{aligned} i(E(z)) &= i(q(i \cup j \cup k)) = i(q(i \cup j \cup g)) = i(q(g \cup i)) + i(q(j)) \\ &= q(A) + q(B) = E(w) + E(z) \ . \end{aligned}$$

That is to say, if the bifurcation at s is a left bifurcation, then

$$i_{rs\#} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

If the bifurcation at s is a right bifurcation, then the computation is much the same, differing only in exchanging right and left. This completes the proof and the section.

4 Classification of Denjoy continua

We return now to the description of Denjoy continua. Let $\alpha > 0$ be irrational, let $\mathbb{D}_\alpha$ be the associated Denjoy continuum, and let the maps $f_n \equiv i_{r_n}, r_{n-1} \colon K_{r_n} \to K_{r_{n-1}}$ be as in the previous section where $r_0 < r_1 < \dots$ are the bifurcation values for α.

Lemma 4.1 $\mathbb{T}_\alpha \simeq \lim\limits_{\leftarrow} f_n$.

Proof From Lemma 2.1, $\mathbb{D}_\alpha \simeq \lim\limits_{\leftarrow} g_n$. The map H_α, restricted to $\mathbb{T}^2 \backslash U_n = \mathbb{T}^2 \backslash (U \cap H_\alpha^{-1}(Q'_{r_n}))$, takes leaves of the foliation $H_\alpha^{-1}(\mathcal{L}')$ intersected with $\mathbb{T}^2 \backslash U_n$ to leaves of the foliation $\mathcal{L}'$ intersected with $\mathbb{T}^2 \backslash Q'_{r_n}$. This induces a homeomorphism $h_n \colon Y_n \to K_n$ with $f_n \circ h_n = g_n \circ h_{n-1}, n = 1, 2, \dots$. The h_n in turn induce a homeomorphism from $\lim\limits_{\leftarrow} g_n$ to $\lim\limits_{\leftarrow} f_n$.

An element of the group $SL(2, \mathbb{Z})$ of 2×2 matrices with integer coefficients and determinant ± 1 will be called nonnegative if each of its entries is nonnegative.

Lemma 4.2 *Let* $L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *and let* $A \in SL(2, \mathbb{Z})$ *be nonnegative. Then* A *has a unique factorization* $A = A_0 A_1 \dots A_n$ *with* $A_0 \in \{I, J\}$ *and* $A_i \in \{L, R\}$ *for* $i = 1, 2, \dots, n$.

Proof The array

$$\begin{array}{ccccccccc}
\frac{0}{1} & & & & & & & & \frac{1}{1} \\[2ex]
\frac{0}{1} & & & & \frac{1}{2} & & & & \frac{1}{1} \\[2ex]
\frac{0}{1} & & \frac{1}{3} & & \frac{1}{2} & & \frac{2}{3} & & \frac{1}{1} \\[2ex]
\frac{0}{1} & \frac{1}{4} & \frac{1}{3} & \frac{2}{5} & \frac{1}{2} & \frac{3}{5} & \frac{2}{3} & \frac{3}{4} & \frac{1}{1} \\
 & & & & \vdots & & & &
\end{array}$$

is called the Farey array. If $\frac{e}{g}$, $\frac{f}{h}$ are consecutive fractions in some row of this array then, in the next row, appearing between $\frac{e}{g}$ and $\frac{f}{h}$ is $\frac{l}{k} = \frac{e+f}{g+h}$. Then $\begin{pmatrix} e & l \\ g & k \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} L$ and $\begin{pmatrix} l & f \\ k & h \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} R$. That is, moving down and to the left in the array corresponds to multiplication (on the right) by L and moving down and to the right corresponds to multiplication (on the right) by R. It is an easily established fact (see, for example, [HW]) that if a, b, c, d are any integers satisfying $0 \leq a \leq c$, $\ 0 \leq b \leq d$, and $ad - bc = -1$, then the fractions $\frac{a}{c}$, $\frac{b}{d}$ occur consecutively in some row of the Farey array, and this occurrence is unique. Thus, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ is nonnegative with $\det(A) = -1$ and $a \leq c$, $b \leq d$, $A \neq J$, then A can be uniquely factored in the form $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} A_2 A_3 \dots A_n$ with $A_i \in \{L, R\}, i = 2, \dots, n$. Since $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = JL$, we have the desired factorization and its uniqueness. If $0 \leq c \leq a$, $0 \leq d \leq b$, and $\det(A) = -1$, then $\bar{A} = JAJ = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$ is as above so $\bar{A} = JL\bar{A}_2 \dots \bar{A}_n$, uniquely, with $\bar{A}_i \in \{L, R\}, i = 2, \dots, n$. Then

$$A = J\bar{A}J = JRA_2 \dots A_n \text{ where } A_i = \bar{\bar{A}}_i = \begin{cases} L, & \text{if } \bar{\mathrm{A}}_\mathrm{i} = \mathrm{R} \\ R, & \text{if } \bar{\mathrm{A}}_\mathrm{i} = \mathrm{L} \end{cases}.$$

In case $\det(A) = 1$ then $\det(JA) = -1$ so, from above $JA = JA_1 \dots A_n$ and $A = IA_1 \dots A_n$ with $A_i \in \{L, R\}$, uniquely.

Let $\alpha, \alpha' > 0$ be irrationals. As in Lemma 2.1, $\mathbb{D}_\alpha$ and $\mathbb{D}_{\alpha'}$ are homeomorphic with $\varprojlim f_n$ and $\varprojlim f'_n$, respectively where $f_n \colon K_n \to K_{n-1}$, $f'_n \colon K'_n \to K'_{n-1}$ and the spaces K_n, K'_n are homeomorphic with a wedge of two circles. Let $S_{l,n}$, $S_{r,n}$ and $S'_{l,n}$, $S'_{r,n}$ denote the "left" and "right" circles of K_n, K'_n, respectively, (the left and right circles were denoted by $E(w)$ and $E(z)$ in the previous section), and let $\{p_n\} = S_{l,n} \cap S_{r,n}$ and $\{p'_n\} = S'_{l,n} \cap S'_{r,n}$ be the branch points. Let $\mathring{S}_{l,n} = S_{l,n} \backslash \{p_n\}$, etc. From the proof of Lemma 3.5, there are embeddings $f^{-1}_{l,n} \colon \mathring{S}_{l,n-1} \to \mathring{S}_{r,n}$ and $f^{-1}_{r,n} \colon \mathring{S}_{r,n-1} \to \mathring{S}_{l,n}$ with $f_n \circ f^{-1}_{l,n} = id$ and $f_n \circ f^{-1}_{r,n} = id$. Let $f'^{-1}_{r,n}$ and $f'^{-1}_{l,n}$ be similarly defined. Now let i_n embedd $K_n \backslash \{p_n\}$ into $\lim\limits_{\leftarrow} f_n$ by

$$i_n(t) = \begin{cases} (f_1 \circ \ldots \circ f_n(t), \ldots, t, f^{-1}_{l,n+1}(t), f^{-1}_{r,n+2}(f^{-1}_{l,n+1}(t)), \ldots), & t \in \mathring{S}_{l,n} \\ (f_1 \circ \ldots \circ f_n(t), \ldots, t, f^{-1}_{r,n+1}(t), f^{-1}_{l,n+2}(f^{-1}_{r,n+1}(t)), \ldots), & t \in \mathring{S}_{r,n} \end{cases}.$$

Let i'_n embedd $K'_n \backslash \{p'_n\}$ into $\lim\limits_{\leftarrow} f'_n$ analogously.

The continua $\mathbb{D}_\alpha$ and $\mathbb{D}'_{\alpha'}$ are orientable in a natural way — let the positive flow direction be the positive orientation — and this gives rise to positive orientations on $\lim\limits_{\leftarrow} f_n$ and $\lim\limits_{\leftarrow} f'_n$ by means of the homeomorphism of Lemma 2.1. If we give the arcs $\mathring{S}_{l,n}$, $\mathring{S}_{r,n}$, $\mathring{S}'_{l,n}$, and $\mathring{S}'_{r,n}$ the left-to-right orientation (these arcs lie in the left-to-right oriented cells $E(w)$ and $E(z)$ of the previous section) then the embeddings i_n and i'_n are orientation preserving, as are the projections $\pi_n \colon \lim\limits_{\leftarrow} f_n \to K_n$ and $\pi'_n \colon \lim\limits_{\leftarrow} f'_n \to K'_n$.

Suppose now that $\psi \colon \mathbb{D}_\alpha \to \mathbb{D}'_\alpha$ is a homeomorphism. Then ψ either preserves or reverses the positive orientation. We will assume that ψ preserves positive orientation (otherwise, use negative orientation on $\mathbb{D}'_\alpha$ and right-to-left orientation on the $S'_{l,n}$ and $S'_{r,n}$). ψ then induces an orientation preserving homeomorphism, which we also denote by ψ, from $\lim\limits_{\leftarrow} f_n$ to $\lim\limits_{\leftarrow} f'_n$, and the maps $\psi_{k,n} \equiv \pi'_k \circ \psi \circ i_n \colon K_n \backslash \{p_n\} \to K'_k$ and $\psi^{-1}_{n,k} \equiv \pi_n \circ \psi \circ i_k \colon K'_k \backslash \{p_k\} \to K_n$ are orientation preserving.

For definiteness, we will measure distance in K_n and K'_n using arc length along the cells $E(w), E(z)$. For $\epsilon > 0$, let X_ϵ (X'_ϵ) be the open ϵ-ball in K_n (K'_n) centered at p_n (p'_n). So, for small $\epsilon > 0, X_\epsilon$ and X'_ϵ are homeomorphic with the letter X. Let $K_{n,\epsilon} = K_n \backslash X_\epsilon$, $K'_{n,\epsilon} = K'_n \backslash X'_\epsilon$, and let $\psi_{k,n,\epsilon} = \psi_{k,n} |_{K_{n,\epsilon}}$, $\psi^{-1}_{n,k,\epsilon} = \psi^{-1}_{n,k} |_{K'_{k,\epsilon}}$.

Lemma 4.3 *Given $\delta > 0$, k, there are $\epsilon > 0$, N such that for $n \geq N$ the endpoints of $K_{n,\epsilon}$ are taken within δ of each other by $\psi_{n,k,\epsilon}$. (Similarly, for $K'_{k,\epsilon}$, $\psi^{-1}_{k,n,\epsilon}$.)*

Proof As $\epsilon \to 0$ and n gets large, i_n takes the endpoints of $K_{n,\epsilon}$ closer and closer together.

Thus, for small $\delta > 0$, ϵ sufficiently small, and $n(k)$ sufficiently large, the maps $\psi_{n,k,\epsilon}(\psi^{-1}_{k,n,\epsilon})$ can be extended "linearly" to $\tilde{\psi}_{n,k,\epsilon}\colon K_n \to K'_k$ $(\tilde{\psi}^{-1}_{k,n,\epsilon}\colon K'_k \to K_n)$ with $\operatorname{diam}(\tilde{\psi}_{n,k,\epsilon}(X_\epsilon)) < \delta$ $(\operatorname{diam}(\tilde{\psi}^{-1}_{k,n,\epsilon}(X'_\epsilon)) < \delta)$.

Lemma 4.4 *Given $\epsilon > 0$ and n, there are $\delta_1 > 0$ and $\delta_2 > 0$ and L, M so that if $\epsilon_1 < \delta_1$, $\epsilon_2 < \delta_2$, $l \geq L$, and $m \geq M$ then the diagram*

$$
\begin{tikzcd}
K_n & & K_m \arrow[ll, "f_{n+1,m}"'] \arrow[dl, "\tilde{\psi}_{m,l,\epsilon_1}"] \\
& K'_l \arrow[ul, "\tilde{\psi}^{-1}_{l,n,\epsilon_2}"] &
\end{tikzcd}
$$

ϵ - commutes. (Here $f_{n+1,m} \equiv f_{n+1} \circ f_{n+2} \circ \ldots \circ f_m$.)

Proof Off $X_\epsilon \cup (\tilde{\psi}_{m,l,\epsilon_1})^{-1}(X'_{\epsilon_2})$, $\tilde{\psi}^{-1}_{l,n,\epsilon_2} \circ \tilde{\psi}_{m,l,\epsilon_1} = \pi_n \circ \psi^{-1} \circ i'_l \circ \pi'_l \circ \psi \circ i_m$. Since $i'_l \circ \pi'_l \to id$ uniformly as $l \to \infty$, $\pi_n \circ \psi^{-1} \circ i'_l \circ \pi'_l \circ i_m \simeq \pi_n \circ i_m = f_{n+1,m}$ (for $m > n$). The previous lemma provides the M, δ_1, δ_2 so that the diagram ϵ-commutes on $X_\epsilon \cup (\tilde{\psi}_{m,l,\epsilon_1})^{-1}(X'_{\epsilon_2})$.

Of course Lemma 4.4 holds with the roles of f and f', K and K', $\tilde{\psi}^{-1}$ and $\tilde{\psi}$ interchanged.

The group $SL(2,\mathbb{Z})$ acts on the irrationals by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\alpha = \frac{a+b\alpha}{c+d\alpha}$ and two irrationals are *equivalent* if they are in the same orbit of this action. Recall from section 2 that the continued fraction expansion of α is denoted by

$$\alpha = \langle a_0, a_1, a_2, \ldots \rangle = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}.$$

Theorem 4.5 (Serret) *The irrationals $\alpha = \langle a_0, a_1, \ldots \rangle$ and $\alpha' = \langle a'_0, a'_1, \ldots \rangle$ are equivalent if and only if there are integers N and k with $a'_{n+k} = a_n$ for all $n \geq N$.*

Proof See [L].

We are now ready to prove the theorem of Fokkink ([F]).

Theorem 4.6 (Fokkink) *Let α and α' be two irrationals with associated Denjoy continua $\mathbb{D}_\alpha$ and $\mathbb{D}_{\alpha'}$. Then $\mathbb{D}_\alpha$ and $\mathbb{D}'_\alpha$ are homeomorphic if and only if α and α' are equivalent.*

Proof Suppose that α and α' are equivalent and let $\sigma \in SL(2,\mathbb{Z})$ be such that $\sigma\alpha = \alpha'$. As a linear transformation of $\mathbb{R}^2$, σ takes lines of slope α to lines of slope α' and, since σ and σ^{-1} preserve the integer lattice, σ descends to a homeomorphism $\tilde{\sigma}$ of $\mathbb{T}^2$. Viewing the domain and range of $\tilde{\sigma}$ as $\mathrm{susp}(R_\alpha)$ and $\mathrm{susp}(R_{\alpha'})$, respectively, $\tilde{\sigma}$ takes the horizontal foliation to the horizontal foliation. By extending linearly across $U = \mathbb{T}^2 \backslash \mathbb{D}_\alpha$ in $\mathrm{susp}(d_\alpha)$, $\tilde{\sigma}$ lifts to a homeomorphism $\tilde{\tilde{\sigma}}$: $\mathrm{susp}(d_\alpha) \to \mathrm{susp}(d_{\alpha'})$ with $H_{\alpha'} \circ \tilde{\tilde{\sigma}} = \tilde{\tilde{\sigma}} \circ H_\alpha$ that takes $\mathbb{D}_\alpha$ homeomorphically onto $\mathbb{D}'_\alpha$.

Conversely, suppose that ψ is a homeomorphism from $\mathbb{D}_\alpha$ onto $\mathbb{D}'_\alpha$. We assume that ψ preserves positive orientation and we assume that $\alpha > 0$, $\alpha' > 0$ (β and $-\beta$ are equivalent). Let $\epsilon > 0$ be given. Using Lemma 4.4 repeatedly, there are integers $0 = n_0 < n_1 < n_2 < \ldots$ and $0 < m_1 < m_2 < \ldots$ and maps s_i, t_i, $i = 1, 2, \ldots$, such that all triangles in the diagram

$(*)$

$$\begin{tikzcd}
K_{n_0} & & K_{n_1} \arrow[ll, "f_{n_1,n_0+1}"'] \arrow[dl, "t_1"] & & K_{n_2} \arrow[ll, "f_{n_2,n_1+1}"'] \arrow[dl, "t_2"] & & \arrow[ll] & \cdots \\
& K'_{m_1} \arrow[ul, "s_1"] & & K'_{m_2} \arrow[ll, "f'_{m_2,m_1+1}"'] \arrow[ul, "s_2"'] & & K'_{m_3} \arrow[ll, "f'_{m_3,m_2+1}"'] \arrow[ul, "s_3"'] & & \arrow[ll]
\end{tikzcd}$$

ϵ- commute. Here $s_i = \tilde{\psi}^{-1}_{m_i,n_{i-1},\epsilon_i}$, $t_i = \tilde{\psi}_{n_i,m_i,\delta_i}$ for some small ϵ_i, δ_i, and $f_{i,j} \equiv f_{i+1} \circ \ldots \circ f_j$, etc. For $\epsilon > 0$ sufficiently small, application of first homology (with $\mathbb{Z}$ coefficients) to $(*)$ yields a commuting diagram of linear maps

$$
(**)\qquad
\begin{tikzcd}
\mathbb{Z}\oplus\mathbb{Z} & & \mathbb{Z}\oplus\mathbb{Z} \arrow[ll, "(f_{n_1,n_0+1})_\#"'] \arrow[dl, "T_1"] & & \mathbb{Z}\oplus\mathbb{Z} \arrow[ll, "(f_{n_2,n_1+1})_\#"'] \arrow[dl, "T_2"] & {} \arrow[l] & \cdots \\
& \mathbb{Z}\oplus\mathbb{Z} \arrow[ul, "S_1"] & & \mathbb{Z}\oplus\mathbb{Z} \arrow[ul, "S_2"] \arrow[ll, "(f'_{m_2,m_1+1})_\#"'] & & \mathbb{Z}\oplus\mathbb{Z} \arrow[ul, "S_3"] \arrow[ll, "(f'_{m_3,m_2+1})_\#"'] & {} \arrow[l]
\end{tikzcd}
$$

Since ψ is orientation preserving, so are $\psi^{-1}_{m_i,n_{i-1}}$ and ψ_{n_i,m_i}. It follows that for small $\epsilon > 0$, the maps S_i and T_i take positively oriented cycles ($E(w)$ and $E(z)$) to nonnegative linear combinations of positively oriented cycles. That is, if we choose the ordered basis $\langle E(w), E(z)\rangle$, $E(w)$ and $E(z)$ left-to-right oriented, for each of the groups $H_1(K_{n_i})$ and $H_1(K'_{m_i})$, then we may interpret $(**)$ as a commuting diagram of nonnegative elements of $SL(2,\mathbb{Z})$. Also each $(f_i)_\#$ and $(f'_j)_\#$ is either an $L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ or $R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, depending on i and j and the continued fraction expansions $\alpha = \langle a_0, a_1, \dots\rangle$ and $\alpha' = \langle a'_0, a'_1, \dots\rangle$ (from Lemmas 3.4 and 3.5): $(f_1)_\#, \dots, (f_{a_0})_\#$ are $L's$, $(f_{a_0+1})_\#, \dots, (f_{a_1})_\#$ are $R's, \dots$; and $(f'_1)_\#, \dots, (f'_{a'_0})_\#$ are $L's$, $(f'_{a'_0+1})_\#, \dots, (f'_{a'_1})_\#$ are $R's, \dots$.

Now since $\det(L) = 1 = \det(R)$, either $\det(S_i) = 1 = \det(T_i)$ for all i or $\det(S_i) = -1 = \det(T_i)$ for all i. Assume the former. Then, from Lemma 4.2, the S_i and T_i can be expressed as products of $R's$ and $L's$. By uniqueness of such factorizations, the pattern of $R's$ and $L's$ at the end of the factorization of $(f_{n_1,n_0+1})_\# = (f_{n_0+1})_\# \cdot (f_{n_0+2})_\# \cdot \dots \cdot (f_{n_1})_\#$ must be the same that in T_1, which in turn, must be the same as the beginning pattern of $R's$ and $L's$ in $(f'_{m_2,m_1+1})_\# = (f'_{m_1+1})_\# \cdot \dots \cdot (f'_{m_2})_\#$. The rest of the pattern in $(f'_{m_2,m_1+1})_\#$ is that of S_2 which must be the beginning pattern in $(f_{m_2,m_1+1})_\#$, etc. These patterns are determined by the continued fraction expansions of α and α', as above, so that for some N and k, $a'_{n+k} = a_n$ for all $n \geq N$, and it follows from Theorem 4.5 that α and α' are equivalent.

In case $\det(S_i) = -1 = \det(T_i)$ for all i, the factorizations of S_i and T_i given by Lemma 4.2 all begin with $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. For $C \in SL(2,\mathbb{Z})$ let $\bar{C} = JCJ$ and let $S_i = JS'_i$, $T_i = JT'_i$. Then, from $(**)$,

$$(f_{n_k,n_0+1})_\# = JS'_1(f'_{m_k,m_1+1})_\# JT'_k = \bar{S}'_1\left(\overline{f'_{m_k,m_1+1}}\right)_\# T'_k\ .$$

So again we see that the pattern of $R's$ and $L's$ in $(f'_{m_1+1})_\# \cdot (f'_{m_1+2})_\# \cdot \ldots$ agrees with that in $(f_N)_\#(f_{N+1})_\# \ldots$, some N, only with the roles of R and L interchanged. Thus, in any case, α is equivalent to α'.

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