PISOT-VIJAYARGHAVAN NUMBERS AND POSITIVE MATRICES

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Abstract. *Branched manifolds*, were introduced in 1969 [W3, W4]. (The 1-dimensional case was introduced in 1967 [W1] and was used to study Anosov diffeomorphisms in 1968 [W2]; this last was popularized under the name "train track" [F-L-P] somewhat later.) Branched manifolds have tangent bundles, so that such concepts as *orientability* and *immersion* make sense. However, the top dimensional homology of our branched manifolds are typically of dimension $k > 1$, so that there are many choices of generators, consistent with the chosen orientation. Thus we say that an oriented branched n-manifold K is *sensed* provided the n dimensional homology is free abelian, and that generators are so chosen that the homomorphisms induced by immersions, relative to the chosen bases, are matrices whose entries are all ≥ 0.(or ≤ 0. Here the inequalities depend upon whether the immersion preserves or reverses the chosen orientations.) Sensed, oriented, branched manifolds are called "SOB's" for short. We use this concept to prove a partial converse to the Perron-Frobenius theorem, and expect to be able to prove that "weak shift equivalence" implies "strong shift equivalence" for certain matrices. Though our theory works, at least in principal, in all dimensions, we have carried out the proofs only in the 3 dimensional case.

Definitions.

The reader is referred to [W4] for foundational material on branched manifolds. However, for completeness, we include enough here for the 2-dimensional case, to satisfy most readers. By a *branched surface* we will mean a metric space K, equipped with local charts U_i and smooth change of coordinate functions $\phi_{(i,j)}$ where the U_i are open subsets of the smooth space sketched in figure 1. See [W4] and [C].

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See figure 1.

Figure 1.

Then one introduces *tangent bundles* as usual, and smooth maps $f : K \to K'$ induce *tangent maps* $Df : TK \to TK'$ of the corresponding tangent bundles. A smooth map is an *immersion* provided its tangent map is an isomorphism at every point. A branched surface is *orientable* provided there is a consistent choice of orientation, and if such a choice is made, we say it is *oriented*.

We are interested in a class of oriented branched n-manifolds K which have the following property:

1. $H_n(K) \approx \mathbb{Z}^k$ and generators $g_1, g_2, ...g_k$ are chosen, so that
2. if $f : K \to K'$ is an orientation preserving (respectively, reversing) immersion, with K, K' in the class, then the matrix of $f_* : H_n(K) \to H_n(K')$ has all entries ≥ 0, (respectively, ≤ 0).

f_* is a matrix as we have chosen generators of $H_n(K)$ and $H_n(K')$, The elements of such a class will be said to be *sensed* oriented branched n-manifolds.

Remark. *Note that oriented (unbranched) manifolds satisfy our definition, where an immersion is a covering of, say, k sheets and the "matrix" involved is* 1×1*, and has k as its only entry.*

We used "sensed, oriented, branched 1 manifolds" in our paper [W2], without introducing this term. In the present paper we deal mostly with "SOB surfaces", that is 2-dimensional sensed, oriented branched manifolds. However, they exist in all dimensions, and we conjecture that our results below have analogues in all dimensions.

We will say a matrix is *nonnegative* if all its entries are ≥ 0, and *positive* if all entries are > 0. A nonnegative matrix is *primitive* provided some power of it is positive. By a PV(Pisot-Vijayarghavan) matrix, is meant a square matrix A with

1. integer entries;
2. determinant $= 1$;
3. one eigenvalue real, > 1; and
4. all other eigenvalues inside the unit circle.

If in addition

5. if μ is a complex eigenvalue of A, then $|\mu| \leq 1/\sqrt{n}$,

we say that A is a *special* PV matrix.

2. Statement of Results

Theorem A. *If A is a 3×3 special PV matrix, then there is an integral matrix P, with P^{-1} integral such that PAP^{-1} is nonnegative.*

For the special case in which A is a companion matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & n & m \end{pmatrix}.$$

this has been verified directly in a few pages, by Dani Berend, [Be]. One needs to do a computer search to find examples not covered by Berend's proof. In addition to Berend, many others have given help and encouragement, including Abe Anderson, Mike Boyle, Cameron Gordon Gary Hamrick, Rafael de la Llave, Frank Raymond, John Tate, and Jeff Vaaler.

Theorem A'. *If A is 3×3 and satisfies the conditions for special PV matrix with the exception that $detA = -1$, then there is an integral matrix P, with P^{-1} integral such that PAP^{-1} is nonpositive.*

Note that by replacing A with $-A$ this covers 2 more cases.

Conjecture B. *For non-negative, primitive special PV matrices, shift equivalence and strong shift equivalence are the same.*

This conjecture was announced as a theorem; unfortunately our analysis was based on an incomplete understanding of certain other results. The current ideas may apply.

Next we form a quotient space from a 3-dimensional torus, $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$. This construction is carried out in $\mathbb{R}^3$, and makes considerable use of the linear structure. We begin with 2 ingredients, which will be determined by a special PV matrix, below. Given

1. a vector $v \in \mathbb{R}^3$ with all coordinates > 0;
2. a 2 dimensional subspace Π which does not enter the positive orthant; in other words, Π has a normal vector, all of whose coordinates are strictly positive.

Construction. Let $\mathcal{L}$ be the family of all lines L in $\mathbb{R}^3$ parallel to v, let $C_{1/2} = (-1/2, 1/2)^3$, and let $Q_{1/2} = \Pi \cap C_{u0}$. Let Q' and $\mathcal{L}'$ denote the image of Q and $\mathcal{L}$ in $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$. Finally let $\mathcal{C}'$ denote the collection of all connected components C of $L' \cap (T^3 - Q')$, for $L' \in \mathcal{L}'$.

Quotient Proposition. *Let $q' : T^3 - Q' \to K$ be the quotient map and quotient space formed by collapsing the elements of $\mathcal{C}'$ to points. Then K is an orientable, sensible, branched surface. Further A^{-1} induces an orientation preserving immersion $g : K \to K$.*

Proof. The proof of this proposition occupies the bulk of the paper, below. However 2 things should be mentioned now to put the topologists mind at ease: The topology of our spaces may seem unusual, as $T^3 - Q'$ is neither closed nor open in R^3. However, by enlarging the disk to a "lenticular" open set Q^+, we get a compact space $T^3 - Q^+$ and the only effect is that the line intervals C are shortened a bit. Thus the quotients are the same and K has good topology. To see that the map $A : \mathbb{T}^3 \to \mathbb{T}^3$ leaves Q' invariant, note that if the small eigenvalues are real, then this follows by symmetry of the disk about the origin. If they are complex, the special assumption guarantees this, since the ratio of the largest to the smallest radius of $Q' \leq \sqrt{3}$. Thus A^{-1} leaves the complement of Q' invariant. Since it leaves the lines $\mathcal{L}$ invariant as a family and maps each component C into a connected subset of some $L \in \mathcal{L}$, the map $g = qA^{-1}q^{-1}$ is well defined. Finally, A^{-1} preserves or reverses the orientation of Π depending upon whether the determinant of A is 1 or -1 and thus (see below) g has the same effect on the orientation of the branched surface K.

Remark. *We conjecture this propositionis true for a wide class of disks, Q'; in particular for round disks smaller than Q'.*

In section 3 we show how theorems A and A'follow as a corollary to the Quotient Proposition. In section 4 we formulate the proof of the Quotient Proposition as follows: first we give an example of this quotient in its simplest; next we give an algorithm for deducing the structure of the quotient space; and then formulate the 3 basic facts that we must prove, using this algorithm. From these 3 facts, we deduce the Geometric Proposition, a more specific and detailed version of the Quotient Proposition. Section 5 is devoted to the analysis of the simple example. Finally, sections 6, 7, and 8 contain proofs of the 3 basic facts, and thus conclude the paper. In the appendix we give several examples of computer printouts showing how the quotient space can differ from the simplest example.

3. Proof of Theorems A and A'

First, it is a result of Handelman (and others, [H], [Br]) that there is a basis for $\mathbb{Z}^3$ such that with respect to this basis, the operator A has its Perron-Frobenius eigen vector v in the positive orthant and the eigen plane of the other eigen values fails to intersect this orthant. Thus we may as well assume that A has these properties, with respect to the standard basis $\mathbb{Z}^3$. Thought of as the standard basis for $\mathbb{R}^3$ this basis induces a basis for $\mathbb{T}^3$, and with respect to this basis, $A = A_{1*} : H_1(\mathbb{T}^3) \to H_1(\mathbb{T}^3)$. Thus we have the commutative diagram:

$$\begin{array}{ccc}
H_1(\mathbb{T}^3) & \xrightarrow{A_*} & H_1(\mathbb{T}^3) \\
\downarrow\approx & & \downarrow\approx \\
H^2(\mathbb{T}^3) & \xrightarrow{A^{-1*}} & H^2(\mathbb{T}^3) \\
\approx\downarrow i_* & & \approx\downarrow i_* \\
H^2(\mathbb{T}^3 - Q') & \xrightarrow{A^{-1*}} & H^2(\mathbb{T}^3 - Q') \\
q\uparrow\approx & & \approx\uparrow q \\
H^2(K) & \xrightarrow{g^*} & H^2(K) \\
\downarrow\approx & & \downarrow\approx \\
H_2(K) & \xrightarrow{g^t_*} & H_2(K)
\end{array}$$

In this diagram, the top vertical arrow is Poincaré duality, and the next is that induced by the inclusion $i : \mathbb{T}^3 - Q' \to \mathbb{T}^3$. This is an isomorphism, as Q' is acyclic.

The next vertical map is that induced by the quotient, q' which is an isomorphism by the Vietoris- Beagle-Smale mapping theorem. As mentioned above, A^{-1} induces the map $g : K \to K$. The bottom isomorphism is vector-space duality, valid as there is no torsion. Thus, with P defined as the composition of these vertical maps, we have, $A = PBP^{-1}$, has all entries ≥ 0, or all entries ≤ 0, depending as to whether the map on K induced by A^{-1} preserves or reverses the orientation of K. This sign depends upon whether A or equivalently, A^{-1} preserves or reverses the orientation of the eigenplane, Π, and thus upon the determinant of A, This completes the proof of Theorems A and A'.

4. The Algorithm and organization of the proof of the Geometric Proposition

That K is a branched surface follows from a rather general proposition which was proved in our paper [W4]; alternatively, this can be verified directly and rather easily in this linear situation. Locally the "disk structure" of K (and hence the "local orientation") of K is determined by planes which are transverse to the foliation $\mathcal{L}$. (This can be thought of as in either $\mathbb{T}^3$ or $\mathbb{R}^3$, as it is a local property.) And since we can choose these transverse planes coherently, e.g. all planes parallel to Π, K is thereby orientable. The smoothness of such quotients depends upon that of the foliation $\mathcal{L}$ which is analytic in our case.

That K can be "sensed" is new and occupies the bulk of the paper. One would like to have conceptual proofs of these results, especially as they are likely (or at least possibly) true in higher dimensions.[1]

The easy part of our task:

Remark. *K has the homotopy type of the 2-skeleton of T^3 and thus $H_2(K)$ is free abelian of rank 3.*

Proof. The map $\mathbb{T}^3 - Q' \to K$ is a Vietoris map since the inverse image of each point is a line interval and thus acyclic. Then by the Vietoris-Beagle-Smale theorem [Smale], the quotient map is a homotopy equivalence. It follows that $H_2(K)$ is free abelian of rank 3.

[1] In sharp contrast, several of our steps involve detailed case-by-case geometric visualization.

The remainder of the proof, and the paper, consists of three main steps, which together establish the nature of the cell complex that the branch set of K partitions it into. We learn a great deal about the structure of this complex, using the linear geometry of $\mathbb{R}^3$ and thus the topology of $\mathbb{T}^3$. As we do most of the construction in $\mathbb{R}^3$, we introduce the following notation there:

Notation. Recall that $\mathcal{L}$ is the family of all lines L in $\mathbb{R}^3$ parallel to v, and that V is the "constant" vector field in $\mathbb{R}^3$ parallel to v. For convenience we use prefer our cubes to have lattice points as vertices. This is a change from the notation introduced in artical 2, but does not affect the quotient space K at all. Thus let $C_{u0} = (0,1)^3$, let Π' be the plane parallel to Π and passing through $(1/2, 1/2, 1/2)$, let $Q_0 = \Pi' \cap C_{u0}$, and for each $z \in \mathbb{Z}^3$, let $C_u(z)$ and $Q(z)$ be C_{u0} and Q_0 translated by z. Finally, let $\mathcal{C}$ denote the union of all connected components C of $L \cap (\mathbb{R}^3 - \cup_z Q(z))$, such that $L \in \mathcal{L}$. We use x, y, z as coordinates and assume that the notation has been chosen so that the Perron-Frobenius eigen vector v has coordinates $(1, v_2, v_3)$, where $1 \geq v_2 \geq v_3$. It seems advisable to start with an example.

Example. Let $v = (1,1,1)$ and Π be the plane through the origin which is normal to v. For further simplicity, we let Q be the *round* disk (again open) of maximal diameter lying in $\Pi \cap C_u(0,0,0)$.

Claim. In this case, K is the quotient space of a round disk D as indicated in figure 2. There are 6 arcs, $A_1, ...A_6$, in D which are sub arcs of the same radius as D. For each i there is a euclidian translation $\tau_i : A_i \rightarrow A'_i$, where the A'_i form a partition of the boundary of D. (These translations are unique by the geometry.)

Figure 2.

We prove this claim in 2 steps: first a general algorithm, then a computation.

Notation. Recall that we have the quotient map $q : \mathbb{T}^3 - Q' \to K$, which we may think of as

$$q : \mathbb{R}^3 - \cup_z Q(z) \to K.$$

Recall that $\mathcal{C}$ denotes the collection of all connected components C of $L \cap (\mathbb{R}^3 - \cup_z Q(z))$, and let $\mathcal{C}_0$ denote set of all $C \in \mathcal{C}$ which have their lower endpoint on $Q(0,0,0)$. Note that $\cup \mathcal{C}_0$ is a fundamental domain for the quotient K since each of the intervals C has its lower endpoint on some Q. Let D denote the closure of $Q(0,0,0)$. For $x \in D$, there is a unique line L of our foliation through x which in turn is either the lower endpoint of an interval $C_x \in \mathcal{C}_0$, or an internal point of C_x if x is in the boundary of D. This determines a map $f : D \to K$ by $f(x) = q(C_x)$. f is clearly continuous and is onto, in fact $f(Q) = K$, as pointed out above.

Introduce the set $z_1, ..., z_k$, of all lattice points z such that some $C \in \mathcal{C}_0$ has its upper end in $Q(z)$. (This is finite by the almost periodicity of our lines.) Set

$$E(z_i) = \cup\{f(x) : \text{ the upper endpoint of } C_x \in Q(z_i)\}.$$

Note that f fails to be a local homeomorphism only at points x at which the upper end point is on the boundary of some other Q : these points form the branch points—and 1-dimensional skeleton—of K.

Set $A(z_i) = \{x : C_x \text{ contains a boundary point of } Q(z_i)\}$. Convexity properties of the Q's imply the A's are connected and thus arcs in this example. In general some A's may have finitely many components; we use the term *arcunion* for these. Each arc A_i has a unique representative $A_i' \subset bdryD$, and the translation $\tau_i : A_i \to A_i'$ is the composition of the euclidian translation taking $z_i \to (0,0,0)$. with the projection onto D along V. Two A_i' might overlap or coincide in the periodic case, though by taking intersections, we get that the A_i' partition the boundary of D.

Summary. K is the identification space obtained from D, the closure of $Q(0,0,0)$, by identifying finitely many arcunions A_i in the interior of D, with, respectively corresponding arcunions A_i', lying on the boundary of D, via euclidian translations $\tau_i, i = 1, ...k$. The cells $E(z_i)$ that these arcunions divide K into, consist of the lower end points of the subcollection $\mathcal{C}_0$ of arcs C which have their lower end point on $Q(0,0,0)$ and their upper end point on $Q(z_i), i = 1, ..., k$. Some of the $E(z_i$ are actually the union of finitely many cells, but this causes no problems.

5. Analysis of the example

For the computation in the special easy case (figure 2) of our example, one can recognize the second drawing as the projection of a cube along its diagonal which is parallel to v,, and thus *see* the picture. Alternatively, one sees that the 3 vectors $(a, -a, 0), (b, b, -2b), (c, c, c)$ form an orthonormal basis if $2a^2 = 6b^2 = 3c^2 = 1$. The first 2 span the plane Π and the other is parallel to v. Hence the matrix of the projection $p : \mathbb{R}^3 \to \Pi$ is

$$\begin{pmatrix} a & -a & 0 \\ b & b & -2b \end{pmatrix}.$$

Also, for each z_i, the inner product $w_i = (z_i, (c, c, c))$, is the length of those intervals $C \in \mathcal{C}_0$ which have their upper endpoint in $Q(z_i)$. In the third sketch, we have drawn the circles centered on the hexagonal lattice which is the image of $\mathbb{Z}^3$ under the matrix above. We have labeled the regions cut out with the "weights" w_i. Note that if a point lies in the interior of several circles, it belongs only to the cell $E(z_i)$ for which the weight w_i is minimum, for the other Q's are above $Q(z_i)$ relative to V.

The reader may note that that

$$Z_1 = E(1,0,0) \cup E(1,1,0) \cup E(1,0,1) \cup E(1,1,1)$$

is a 2 dimensional torus, being a hexagon with opposite sides identified (the so-called "twisted torus"). By permuting the bases, one gets 2 more tori, Z_2, Z_3, which are the chosen basis which in effect "senses" the branched surface K. The crucial fact here about this specific choice of these cycles is that the three cells, $E(1,0,0), E(0,1,0), E(0,0,1)$ each occur *only* in their respective cycles. We call such cells *unshared cells*.

For computations with the polygonal cells used in the rest of this paper, a computer program was written, with the aid of my colleague Rafael de la Llave. It has proved a great aid. In fact, without it there would be no paper at all. Some of its figures are including in the appendix. (Abe Anderson has written another with round disks with radius a parameter—it looks like small round disks would work, so as to prove the result of this paper without the $\sqrt{3}$ factor!)

We abstract this last example as the

Standard Model. K is an oriented branched manifold of dimension n with branch lines dividing K into cells E_j such that:

1. K is a cell complex with cells $E_1, ..., E_k, ..., E_r$ oriented compatibly with K;
2. $H_n(K)$ is free abelian of rank k;
3. there are cycles
$$Z_i = E_i + \sum_{j=k+1}^{r} p_{i,j} E_j, i = 1, ..., k$$
whose homology classes generate $H_n(K)$;
4. $p_{i,j} \geq 0$.

Definition. The cells $E_i, i = 1, 2, 3$ are said to be *unshared* cells.

Standard Model Proposition. *With the generators of $H_2(K)$ chosen to be the cycles $[Z_i]$, these "standard models" form a class of sensed oriented branched manifolds.*

Proof. Now let K' be another member of this class with notation as above, decorated with "primes", and let $f : K \to K'$ be an orientation preserving immersion. Then the induced map on the chain level sends each n-cell into K', preserving orientation, so that

$$f_*(Z_i) = \sum_{j=1}^{m'} p'_{i,j} E'_j, i = 1, ...k, p'_{i,j} \geq 0,$$

$$= \sum_{j=1}^{k'} A_{i,j} Z'_j,$$

$$= \sum_{j=1}^{k'} A_{i,j} E'_j + \sum_{j=k'+1}^{m'} B_{i,j} E'_j,$$

for some $B_{i,j}$, where the last equation uses the fact that the first k' n-cells of K' are "unshared". The second equality uses the fact that there are no boundaries, as we are in the top dimension. Comparing the first and third equation, we see that

$A_{i,j} = p'_{i,j}$ and thus $A_{i,j} \geq 0, i = 1, ..., k; j = 1, ..., k'$. The reader may note that the occurrence of a positive integral matrix is reminiscent of Markov partitions; but this is just on the *cycle* level, as opposed to the *cell* level, as in a Markov partition. On the cell level the map may (and must) map more haphazardly, as we know by a theorem of Franks [F]. However all the p'_{ij} are integral, because the image of a cycle must contain a cell if it hits the interior of this cell.

Geometric Proposition. *The quotient space K is a smooth oriented branched surface, the branch lines are smooth and partition K into a cell complex with positively oriented 2-dimensional cells $E(z_i)$ corresponding to lattice points $z_i \in \mathbb{Z}^3, i =$ 1...k, such that:*

1. z_1, z_2, z_3 *form a basis for* $\mathbb{Z}^3$.
2. *The Perror-Frobenius vector v is given by $v = \alpha z_1 + \beta z_2 + \gamma z_3$, where $\alpha, \beta, \gamma > 0$.*
3. *There are 2-cycles Z_1, Z_2, Z_3 whose classes form a basis of $H_2(K)$.*
4. $Z_i = E(z_i) + \sum_{j=4}^{k} p_{ij} E(z_j), i = 1, 2, 3.$
5. $z_j = \sum_{i=1}^{3} p_{ji} z_i$. *where the* $p_{ji} \geq 0$.

The crux of our argument is to find as basis of $H_2(K)$ for which there are 3 "unshared" cells, $E(z_i), i = 1, 2, 3$. This comes down to proving 3 propositions:

The Basis Proposition. *There is a basis z_1, z_2, z_3 of $\mathbb{Z}^3$ such that*

e. *some of the intervals of $\mathcal{C}_0$ have their upper end points on each of the cells $Q(z_i), i = 1, 2, 3$. and*

u. *no interval of $\mathcal{C}_0$ which terminates on a different $Q(i, j, k)$, will hit any of these $Q(z_i)$ when prolonged.*

The P-F Proposition. *The new basis, $\{z_1, z_2, z_3\}$, has the vector v in its positive orthant.*

Tori Proposition. *We can choose a tori $\{T_{23}, T_{31}, T_{12}\}$ forming a basis, Poincaré dual to $\{z_1, z_2, z_3\}$, such that*

1. *the vector field V is outwardly transverse to each T_i;*
2. *T_i intersects no Q.*

The second part (u) of the Basis Proposition is not used below. We include it because (i) it is essentially obvious in the present setting and (ii) we feel it should play a role. In particular, it is how we recognize the unshared cells, for example in computer printouts. We end this section with a proof that these propositions together that our quotient space K can be sensed, just as in the standard model; each of the remaining sections is devoted to proving one of these 3 propositions.

Proof of the Geometric Proposition

Parts (1) and (2) are direct consequences of the basis and P-F propositions. Let $Z_i = q_*(T_i), i = 1, 2, 3$; then condition (3) follows from the fact that the quotient map is a Vietoris map. To prove the remaining, we establish some conventions, linking the geometry and topology of $\mathbb{T}^3$. First, if $z \in \mathbb{Z}^3$ then the line interval $[O, z]$, where O denotes the origin, maps down to a simple closed curve in $\mathbb{T}^3$, and thus defines a 1-dimensional cycle; by abuse of notation, we refer to this 1-cycle as "the 1-cycle z." If z is a multiple of another lattice point, this 1-cycle will have an integral coefficient equal to the multiplicity of the quotient map of $[O, z]$ onto this simple closed curve. For 2-dimensional cycles, we only need consider our basis z_1, z_2, z_3 : the parallelogram of $z_i, z_j, i \neq j$, maps down to a 2-dimensional torus $T^2_{i,j}$, in $\mathbb{T}^3$, which we call $z_i \times z_j$, and think of it as a 2-dimensional cycle as well. Orientations are all positive if we stick to the 3 2-cycles, $z_1 \times z_2, z_2 \times z_3, z_3 \times z_1$. Now consider one of the lattice points, $z_m, m = 1, ...k$. We choose $C \in \mathcal{C}_0$, which has its upper end point in $Q(z_j)$. Then by adding short intervals to C lying respectively on the disks $Q(0, 0, 0),\ Q(z_j)$, we get an arc C^+ which represents the 1-cycle z_j in the quotient torus, $\mathbb{T}^3$. Then the intersection number $I(T_{ij}, z_m) = I(z_i \times z_j, z_m)$ is exactly the number of times C^+ or C intersects the torus T_i, since each of these intersections is outward and thus positive and $T_i \cap C^+ = T_i \cap C$. Note that this is the determinant of the matrix with rows z_i, z_j, z_m.

Applying this fact, we see that $Z_i = \sum_{j=1}^k I(T_i, z_j)E(z_j)$. For example, consider T_1. Since $I(T_1, z_j) = I(z_2 \times z_3, z_j)$, we have that the coefficients of $E(z_1), E(z_2), E(z_3)$ in Z_1 are $1, 0, 0$, respectively, by properties of matrices. Similarly p_{1j} is an intersection number and hence positive. This proves condition (4) for $i = 1$. Similar equations hold by permuting $1, 2, 3$, which proves the rest of (4). Condition (5) is duality again. Thus the fact that our quotient K is a standard model follows from the 3 propositions above; these are in turn proved in the last 3 sections.

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