SPACES THAT WON'T SAY NO

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Dedicated to Ricardo Mañé

Abstract. A class of spaces K and their natural maps $f : K \to K'$ were introduced in [W4] which satisfies the defining property that $f_* : H_n(K) \to H_n(K')$ is always a "positive matrix." Here n is the dimension of both K and K'; we follow the usage in symbolic dynamics and say that a matrix is *positive* provided all its entries are ≥ 0. The purpose of this paper is to show how a natural construction, related to the "DA" maps of Smale [W2] often leads to SOB's, and in particular shows that they exist in all dimensions. Here we present the easier conceptual parts of this theory and elicit a geometric property, which when it holds, leads to quite satisfactory proofs.

1. Definitions

This class is determined by the following properties. First of all, K is a branched manifolds of dimension n, say, and is oriented. Secondly, $H_n(K)$ (with integral coefficients) is free abelian of rank k say, and generators $g_1, g_2, ..., g_k$ are chosen for $H_n(K)$. Lastly, the homomorphism $f_* : H_n(K) \to H_n(K')$ induced by any orientation preserving immersion $f : K \to K'$ is a positive $k' \times k$ matrix, relative to the generators chosen for K and K'. Thus, in addition to being an oriented, branched, n-manifold, K is *sensed* by a choice of generators for its top dimensional homology group. Sensed, oriented, branched manifolds are called SOB's for short.

Complete definitions and basic properties of branched manifolds can be found in [W4]; for completeness, we include enough about branched manifolds for most readers. Local charts in dimensions 1 and 2 are (open subsets of) the spaces indicated in figure 1.

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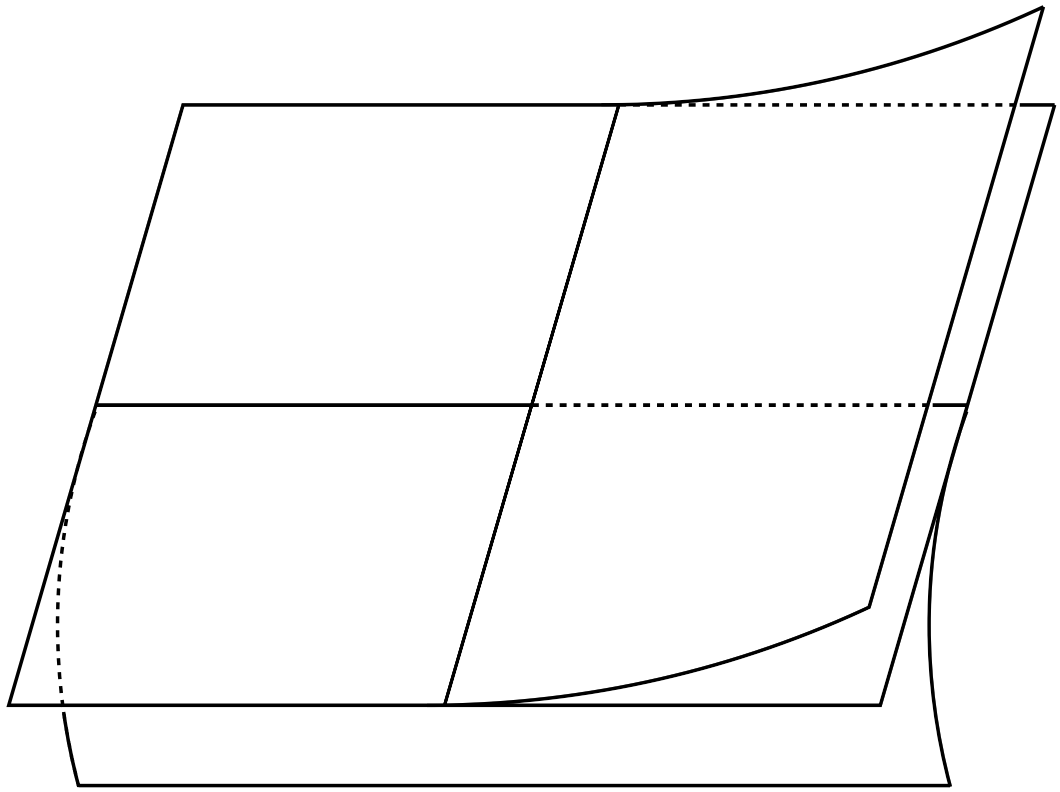

Figure 1

Local projections are given as an integral part of the structure of a branched manifold. In detail, each point has a neighborhood U which is the union of finitely many, say m, (smooth) disks D_i, provided with a projection map $\pi : U \to V, V$ open in $\mathbb{R}^n$, where $\pi | D_i$ is a diffeomorphism for each i. This neighborhood is *normally branched* (see [W4] and compare [C]) provided $k \leq n+1$, the disks $D_i = (i, \mathbb{R}^n)$, say $i = 1, ..k$, and $(i, x) \equiv (i+1, x)$, for all $x \in \mathbb{R}^n$ with $x_i \leq 0, i = 1, 2, ..n$. This has further consequences: for example, $(1, x) \equiv (2, x) \equiv (3, x)$, if both $x_1, x_2 \leq 0$. Branch manifolds arize as quotients of foliations, as in figure 2.

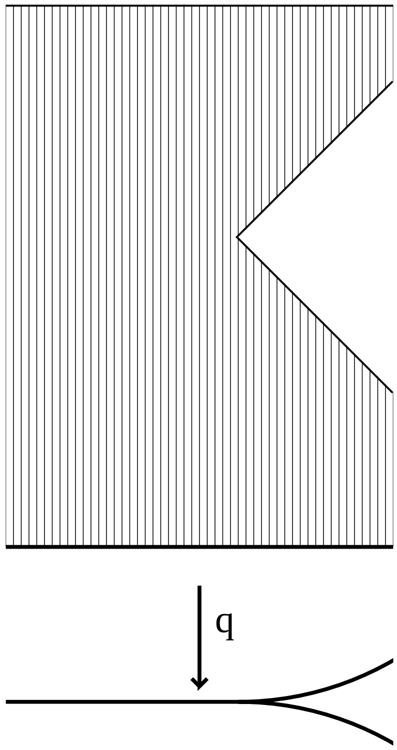

Figure 2

These quotients are as smooth as the foliation one starts with. In this paper they will be quite smooth (analytic) though we will have use only for their C^1 structure. Tangent bundles TK are defined in the usual way and may or may not be orientable. Smooth maps $f : K \to K'$ induce maps $df : TK \to TK'$ of tangent bundles, and as usual, we say f is an *immersion* provided df is an isomorphism at each point. Branched manifolds have been used to study hyperbolic attractors, [W1,W3,W4,C], Anosov diffeomorphisms [W2], pseudo-Anosov diffeomorphisms [T,F-L-P], flows and knots in 3-dimensions, [B-W1,B-W2,F,F-W,G,G-H,H,H-W,S1,S2,W8,W9], and to some extent [W5], symbolic dynamics. The reader is warned that immersions need not be locally 1-to-1; in particular, the projection maps $\pi : U \to \mathbb{R}^n$ are immersions. They *are* locally 1-to-1 on the smooth disks D_i which make up a chart.

2. Basic Examples

Let n be an integer ≥ 3 and let P and N be positive vectors in $\mathbb{R}^n$. That is each coordinate of P, N is positive. Let $\Pi = N^{\perp}$, the plane through the origin, O, perpendicular to N, and let Q be an open, round disk in $\mathbb{R}^n$, of radius r centered at O. Let $\mathcal{L}$ be the family of all lines parallel to the vector P. In the quotient space, $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, let Q_t and $\mathcal{L}_t$ be the images of Q and $\mathcal{L}$ respectively. Then $\mathbb{T}^n - Q_t$ is foliated by the set $\mathcal{C}_t$ of all the connected components C of $L \cap (\mathbb{T}^n - Q_t)$, all $L \in \mathcal{L}_t$.

Assumption. We assume that the elements of $\mathcal{C}_t$ are all line intervals. This is always true if the vector P is irrational; the exceptional case we need to rule out occurs when some of the elements of $\mathcal{C}$ are simple closed curves, which happens if the disk Q is small and the vector P is rational. Finally, let $K = (\mathbb{T}^n - Q_t)/\mathcal{C}_t$ be the quotient space in which all the intervals of $\mathcal{C}_t$ are collapsed to points, and let $q : \mathbb{T}^n - Q_t \to K$ be the quotient map. Then K has the homotopy type of the $(n-1)$-skeleton of $\mathbb{T}^n$, (which consists of n, $(n-1)$-dimensional tori.) Furthermore, K is an orientible, $(n-1)$-dimensional branched manifold. This follows from general results in [W4], or can easily be seen directly. And except for isolated values of the radius of the cell Q, K is normally branched. We prove this below as part of our analysis of the geometric, algebraic, and homological structure of K.

Conjecture. For many choices of the disk Q, $K = (\mathbb{T}^n - Q_t)/\mathcal{C}_t$ is an SOB.

Meanwhile this is known to be true in some special cases:

Proposition 1. *If N is close enough to P, then for Q a round ball of radius $\frac{1}{2}$, the quotient space K is an SOB.*

Proof. For then, the projection $\rho : \mathbb{R}^n \to \Pi$ along P sends each of the standard basis vectors e_i to points within 1 unit of the origin. The projection q is a homotopy equivalence, and the $(n-1)$-dimensional tori $T_1, ..., T_n$, dual to the standard basis $e_1, ..., e_n$, map down to generators $g_1, ..., g_n$ of $H_{n-1}(K)$. Furthermore, the lines L parallel to the vector P are all outwardly transverse to each of the tori T_i, so that the immersion $q|T_i : T_i \to K$ sends T_i in an orientation preserving manner. Thus the cycle g_i is a chain in the $(n-1)$-dimensional cells E_j of K, with all positive coefficients; that is, with orientations agreeing with the chosen orientation of K. As in [W6], one computes that

$$g_i = \sum_i I(z_i, T_i)E(z_i).$$

(We discuss this more fully below.) Thus

$$g_i = E(z_i) + \sum_{j>n} p_{ij}E(z_i),$$

and the theory of [W6] finishes the argument.

Proposition 2. *For $n = 3$, any two positive vectors N, P, and the disk $Q = \Pi \cap [-\frac{1}{2}, \frac{1}{2}]^3$, K is an SOB.*

This is proved in a lengthy preprint [W9]; the paper [W6] explains this argument and contains all details of [W9] except for the rather tedious estimates.

3. Geometry of $(\mathbb{T}^n - Q_t)/\mathcal{C}_t$

We begin with some notation largely in the universal cover $\tau : \mathbb{R}^n \to \mathbb{T}^n$. Recall $Q \subset \Pi \subset \mathbb{R}^n$ is an open $(n-1)$-dimensional disk containing the origin. For each $z \in \mathbb{Z}^n$, let $Q(z) = Q + z$, be the translation of Q by z. Let

$$\mathcal{C} = \tau^{-1}(\mathcal{C}_t) = \{C : C \text{ is a component of } L \cap \mathbb{R}^n - \cup_z Q(z)\} \ .$$

Similarly, define $\mathcal{L} = \tau^{-1}(\mathcal{L}_t)$. Next let $\mathcal{C}_0 = \{C \in \mathcal{C}\colon\ C$ has its lower end point in $Q(O)$. We can now define the $(n-1)$-cells in our complex K :

$$E(z) = \{x \in Q(O) : C = (x, y) \in \mathcal{C}\ ,\ x \in Q(O)\ ,\ y \in Q(z), z \in \mathbb{Z}^n\}.$$

We introduce also the weight $w(E(z)) = \text{length}(C)$. This is independent of the choice of the point $x \in E(z)$.

Note that under our assumption, there are only finitely many cells, $E(z)$, because each of the lines L is topologically dense: let $F = \{z \in \mathbb{Z}^n : E(z) \text{ is nonempty}\}$. We define the $(n-2)$ cells as $\alpha = E(z) \cap E(z')$, for all z, z' such that this intersection is $(n-2)$ dimensional. We may and do assume that the weights satisfy $w(E(z)) < w(E(z'))$. This implies, (see figure 3) that $\alpha \subset \rho(\partial Q(z)$, as the line intervals C with lower end point on $E(z)$ near α are shorter than those with lower end point on $E(z')$. We orient the outer boundary of $Q(O)$ as usual by the outward normal, and note, below, that this induces an orientation on the $(n-2)$ cells.

We introduce a relation among our cells $E(z)$ where $E(z_1) < E(z_2)$ provided

a) $\partial E(z_1) \cap \partial E(z_2)$ is $(n-2)$ dimensional (and thus an $(n-2)$ cell common to their boundaries);

b) their weights satisfy $w(E(z_1)) < w(E(z_2))$.

Proposition 3: Stucture Proposition. *The quotient space K is realized as the closure of $Q(O)$ with identifications along the $(n-2)$ cells as follows: for each $z \in F$ and each $x \in \partial E(z)$, x is identified with $x - \rho(z)$.*

Proof. First of all, $\mathcal{C}_0$ covers a fundamental region of $\mathbb{T}^n - Q_t$, since each $C \in \mathcal{C}$ has its lower endpoint on some $Q(z)$. Secondly, $\partial E(z)$ is $(n-2)$ dimensional since P is transverse to the cells $Q(z)$, and thus $\rho | Q(z) : Q(z) \to \Pi$ is a diffeomorphism. For a point $x \notin \cup_F \partial E(z)$, the interval C_x with x as lower end point is the only interval of $\mathcal{C}_0$ containing an integral translate of x. Finally the intervals C_x for $x \in \alpha = E(z) \cap E(z')$, are duplicated in the closure $\bar{Q}(0)$ by the point $T_z(x)$, where $T_z : E(z) \to \mathbb{R}^n$ is defined by $x \to x - \rho(z)$. Note that, as T_z preserves orientation, this orients our $(n-2)$ cells. Also that $T_z : \alpha \to \partial Q(O)$; this follows as for $x \in \partial E(z)$, there exists $y \in \partial Q(z)$ with $\rho(y) = x$. Thus $x - \rho(z) = \rho(y) - \rho(z) = \rho(y - z)$, which is a vector in Π of norm r. This completes the proof of our structure proposition.

We single out some special $(n-1)$ cells $E(z)$ as follows: a cell $E(z)$ is *lenticular*, provided $E(z) = \rho(Q(z)) \cap Q(0)$, or equivalently, provided its boundary $\partial E(z)$ is a subset of $\rho(\partial Q(z) \cup \partial Q(0)$. See figure 3.

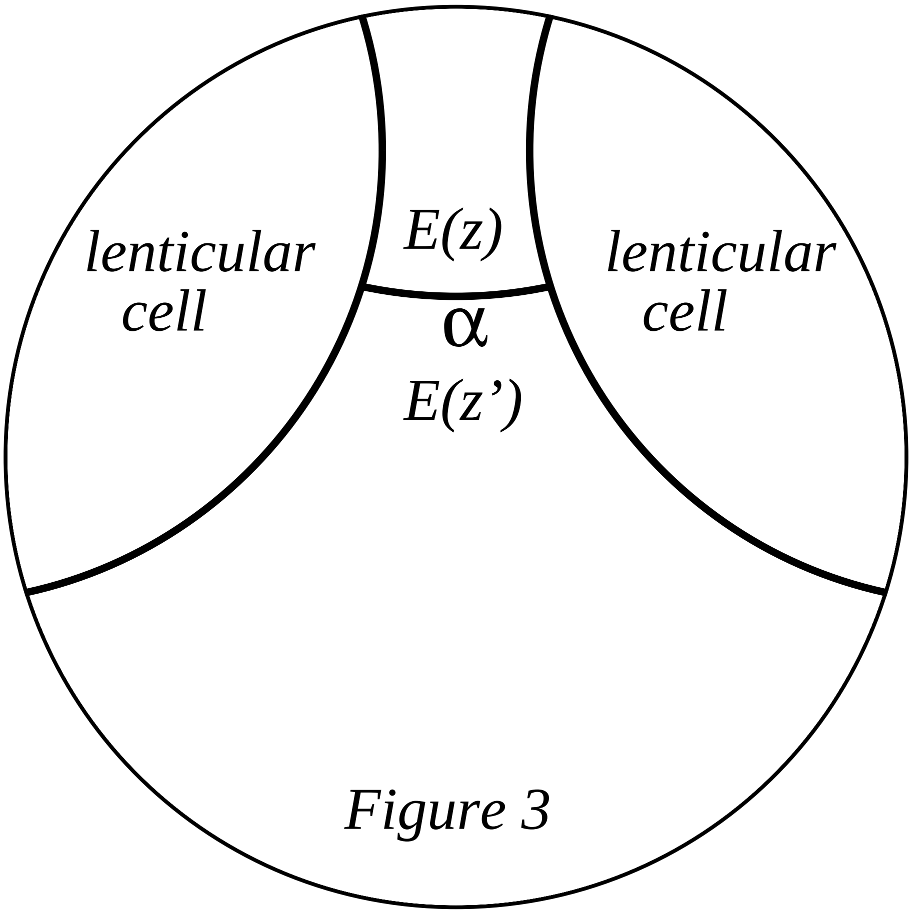

Figure 3

In addition, for each lenticular cell $E(z)$, let $E^*(z)$ be the cochain dual to $E(z)$, that is the cochain which has the value 1 on $E(z)$ and zero on all other cells.

Main theorem.

a. *There are at least n lenticular cells;*

b. $\{q^*(E^*(z)) : E(z)$ *is lenticular*$\}$ *generates* $H^{n-1}(\mathbb{T}^n)$*;*

c. *if there are n and only n lenticular cells, then K is an SOB;*

b$'$ *if* (c) *holds then the map $z \mapsto E(z)$ is an isomorphism $H_1(\mathbb{T}^n) \to H^{n-1}(K)$ and is in fact, $(\tau^*)^{-1} \circ \mathcal{P}$, where $\mathcal{P}$ is Poincaré duality.*

Our proof begins with the following, central Lemma.

Lemma. *For $E(z_1) < E(z_2)$, there is a $z' \in F$ with $E(z') < E(z_2)$ and $z_2 = z_1 + z'$. In addition, there is an $(n-2)$ cell α with $\delta\alpha = E(z_1) + E(z') - E(z_2)$, where δ is* *the coboundary operator.*

Proof. This is very geometric: see figure 4, drawn for $n = 3$.

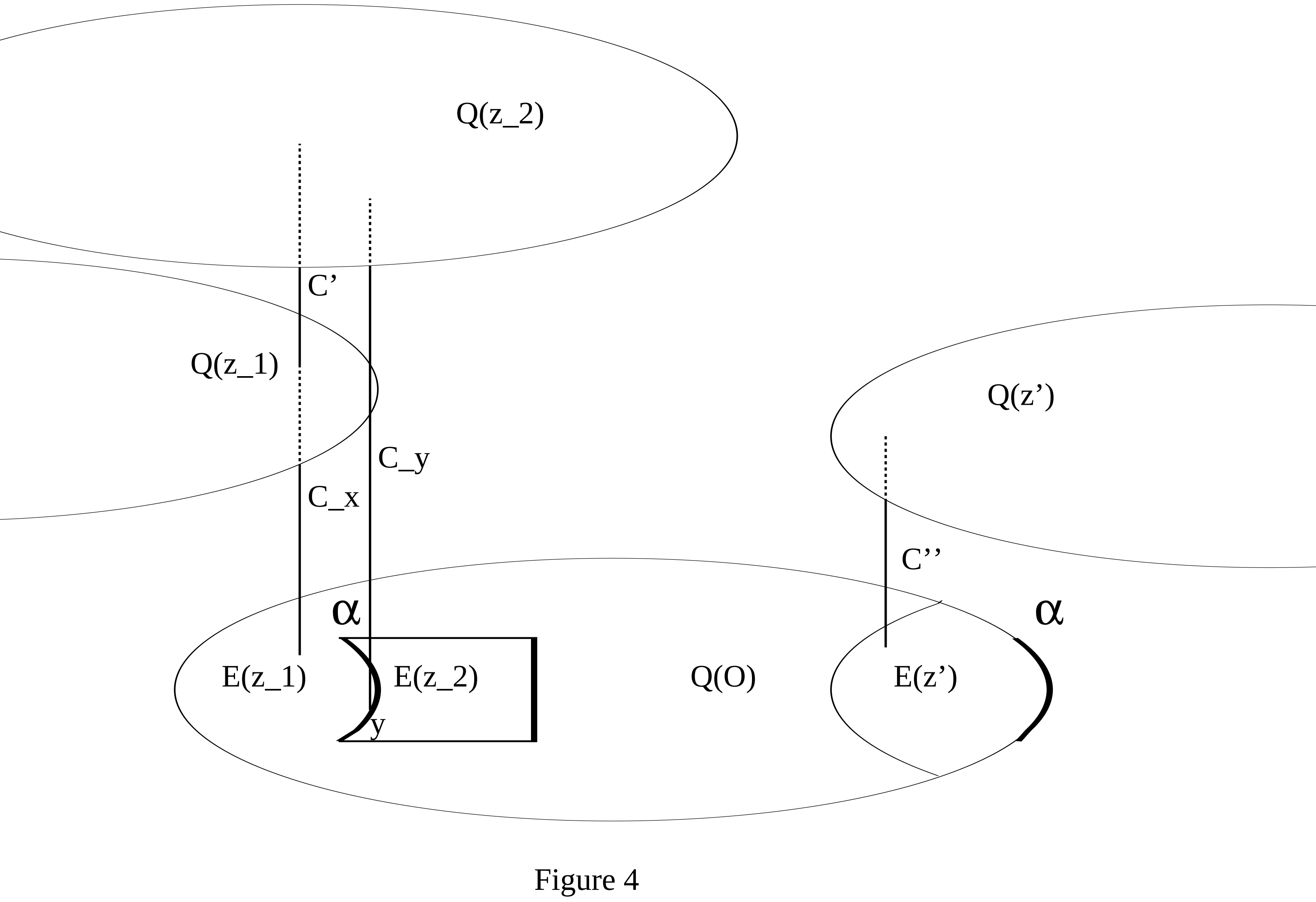

Figure 4

Note that directly above an interval C_x, with $x \in E(z_1)$ near $y \in \alpha = \partial E(z_1) \cap \partial E(z_2)$, there is an interval $C' \in \mathcal{C}$ (though not in $\mathcal{C}_0$) with C' cut off by $Q(z_1)$ and $Q(z_2)$. Thus length(C_x) + length(C') = length(C_y). Also, the translation by $-z_1$ moves C' to C'', say, and $C'' \in \mathcal{C}_0$, as its base is in $Q(O)$. As C'' was chosen to be near α but in the interior of $Q(z_1)$, it follows that the bottom point of C'' is in a unique cell, say $E(z')$; this shows that $z_2 - z_1 = z'$. Finally, the translation

$\alpha - z_1 \subset \partial Q(O)$, so that in K, it is identified with α. Thus the coboundary $\delta\alpha = E(z_1) + E(z') - E(z_2)$. This completes the proof of the Lemma.

Proof of the Main Theorem. First, by repeated application of our Lemma we may write, for any $z \in F$,

$$E(z) = \sum_{i=1}^{k} p_i E(z_i) \quad , \quad z = \sum_i p_i z_i, p_i \geq 0$$

where the $E(z_i)$ are of minimal weight and thus lenticular. It follows that the cohomology classes of the lenticular cells, $E^*(z)$, generate $H^{n-1}(K)$. As $H^{n-1}(K) \approx H^{n-1}(\mathbb{T}^n)$, is n-dimensional, we see that there are at least n lenticular cells. If there are *only* n of these, then the set $\{E^*(z)\}, E(z)$ lenticular, forms a basis of $H^{n-1}(K)$.

We now specialize to this case, and number the $z \in F$ so that $E(z_1), ..., E(z_n)$ are lenticular, and $E(z_j), j > n$ are the remaining cells. Then for $j > n$,

$$E^*(z_j) = \sum_{i=1}^{n} a_{ij} E^*(z_i) \quad , \quad z_j = \sum_{i=1}^{n} a_{ij} z_j \ , \quad a_{ij} \geq 0 \ .$$

These coefficients are unique, of course. Let $Z_1, ...Z_n$ be $n-1$ chains "vector space dual" to the $E^*(Z_i)$. That is

$$Z_i = E(z_i) + \sum_{j>n} b_{ij} E(z_j) \ ,$$

so that $E^*(z_i)$ evaluated on Z_j, $E^*(z_i) \cdot Z_j = \delta_{ij}$. We can compute b_{ij} by evaluation,

$$b_{ij} = E^*(z_i) \cdot E(z_j) = E^*(z_i) \cdot a_{ij} E(z_j) = a_{ij} \ ,$$

which we know is positive. Then $Z_1, ...Z_n$ forms a basis for $H_{n-1}(K)$, as there is no torsion.

One can describe specific $n-1$ chains $Y_i = q_*^{-1}(Z_i)$, whose classes form a basis of $H_{n-1}(K)$, as the situation is so simple; however, we know they exist, as q is a homotopy equivalence. To compute the coefficients a_{ij}, consider an interval $C = (x, y)$ which has its end points on $Q(O)$ and $Q(z_j)$ respectively. Then C together with short arcs joining x to O and y to z_j defines a 1-cycle, w_i, homologous to z_j, considered as a 1-cycle on $\mathbb{T}^n$. Then the intersections of Y_i and w_j occur only along the arc C and thus $I(z_i, Y_j)$ is the multiplicity of the map q at x. It follows that $I(z_i, Y_j)$ is the coefficient in the sum,

$$q_* Y_i = Z_i = \sum_i I(z_i, Y_j) E(z_j) \ .$$

In particular, $I(z_i, Y_j) = \delta_{ij}$, so that $z_1, ..., z_n$ are the Poincaré duals of $Y_1, ..., Y_n$. Therefore $F_l = \{z_1, ..., z_n\}$ forms a basis of $H_1(\mathbb{T}^n)$, when the z_i are considered as 1 dimensional cycles, and in turn, a basis of $\mathbb{Z}^n$, when they are considered as elements of $\mathbb{R}^n$.

It remains only to verify that K is an SOB. This follows quickly from the equation

$$Z_i = E(z_i) + \sum_{j>n} a_{ij} E(z_j) \ , \qquad a_{ij} > 0 \ ,$$

as shown in [W6]. The crucial fact is that $E(z_i), i = 1, ...n$, occurs in the cycle Z_j, only for $i = j$. (these cells are *unshared*. We repeat this short argument: now let K' be another member of this class with notation as above, decorated with "primes," and let $f : K \to K'$ be an orientation preserving immersion. Then the induced map on the chain level sends each $n-1$-cell into K', preserving orientation, so that

$$\begin{aligned} f_*(Z_i) &= \sum_{j=1}^{m'} p'_{i,j} E'(z_j) \ , \qquad i = 1, ...k, p'_{i,j} \geq 0 \ , \\ &= \sum_{j=1}^{k'} A_{i,j} Z'_j \ , \\ &= \sum_{j=1}^{k'} A_{i,j} E'(z_j) + \sum_{j=k'+1}^{m'} B_{i,j} E'(z_j) \ , \end{aligned}$$

for some $B_{i,j}$, where the last equation uses the fact that the first $k', n-1$-cells of K' are "unshared." The second equality uses the fact that there are no boundaries, as we are in the top dimension. Comparing the first and third equation, we see that $A_{i,j} = p'_{i,j}$ and thus $A_{i,j} \geq 0$, $i = 1, ..., k$; $j = 1, ..., k'$.

Remark. There *are* examples, with $n = 3$, for which there are more than 3 lenticular cells. In every case known to the author, one can, by varying the radius of Q, obtain the appropriate number, that is 3. Finally, in these "excess" cases, we believe that the set $F_l = \{z : E(z) \text{is lenticular}\}$ generates $\mathbb{Z}^n$.

Proposition. *The branched manifold is normally branched except for bifurcation values of the radius r. For a range of r of the form $[a, b]$, $a > 0$, there are only finitely many bifurcation points.*

Proof. First of all, note that it suffices to prove this for an open interval J. Note that the set $F = F_r \subset \mathbb{Z}^n$ is finite for each $r > 0$. By a covering argument, $G = \cup_{r \in J} F_r$ is likewise finite. Now let $G' = \{\rho(z) : r \in J, \text{ and } z \in F_r\}$. Each subset $g \subset G'$ of n distinct elements, determines a unique $n-1$ dimensional sphere, of radius, say, $r(g)$. The set of all these, $B = \{r(g) : g \subset G' \text{ a subset of } n \text{ elements}\}$ is finite. But then B contains all the bifurcation points in J. To see this, note that at most $n-1$ of the $n-2$ dimensional spheres $\partial Q(z)$, $z \in F_r$, could contain a point in common. For otherwise, their centers would be such a set $g \subset G'$ considered above, and their common point, the center of the $n-2$ sphere determined by g. This would contradict the definition of B. That is, for $r \in J - B$, the cells $Q(z)$, $z \in F_r$, are in general position. It follows that K is normally branched, as this property is stable and generic, as proved in [W4].

We also include a direct proof of this last. Let y be a point of K which we assume to be a branch point, and say there are k values of $z \in F$ with $y \in \partial E(z)$. By the above, $k \leq n-1$. We order the k z's by weight, $z_1, ..., z_k$ and add $z_0 = O$. Enlarge $Q(z_i)$ to $Q^+(z_i)$, with a slightly larger radius. Let $y_0 = y$ and set

$$y_i = (\rho | Q^+(z_i))^{-1}(y_0) \ , \qquad i = 1, ...k \ .$$

Choose a neighborhood U_0 of y_0 in $Q(O)$, and let $U_i = (\rho | Q^+(z_i))^{-1}(y_0)$. If our enlargement is sufficiently small, the U_i are diffeomorphic to U_0 via ρ, and thus copies of $\mathbb{R}^{n-1}$. Then the map $\rho : \cup U_i \to U_0$ serves as the map π in our definition of normally branched, after the appropriate identifications are made. These identifications are as follows. Let v_i be the inward normal to $\partial Q(z_i)$ near y_i. Set

$$f_{ji} = (\rho|Q^+(z_i))^{-1} \circ \rho|Q^+(z_j) \ , \qquad df_{ji}(v_i) = v_{ij} \ .$$

Then $\{v_{ji}\}_{j=1}^n$ leads to a coordinate system $\{x_{ji}\}$ for U_j. Then the part of U_{j+1} which is not in $Q(z_{j+1})$ satisfies the equation $x_{j+1,j} \leq 0$. This is exactly the portion of U_{j+1} that needs to be identified with U_j. Thus with these identifications, we see that U is normally branched. This completes the proof of our proposition.

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