Mixture of Continuous and Discrete Distributions

Consider a cumulative distribution function $F_X(x)$ which we assume is a mixture of continuous and discrete distributions. We want to show how to write $F_X(x)$ as a two-point mixture of a continuous distribution and a discrete distribution.

Suppose that $F_X(x)$ has jump discontinuities of size J_i at x_i, $i \in I$, some index set. Then define $G_D(x) = \sum_{x_i \leq x} J_i$. Note that $G_D(x)$ is not a cumulative distribution function. Let $J = \sum_{i \in I} J_i$. Note that $J \leq 1$. Then $F_D(x) = \frac{1}{J} G_D(x)$ is a cumulative distribution function. The random variable associated with $F_D(x)$ takes discrete values at x_i, $i \in I$, with probabilities J_i/J.

Let $G_C(x) = F_X(x) - G_D(x)$. Note that $G_C(x)$ is a continuous function. Again $G_C(x)$ is not a cumulative distribution function, but $F_C(x) = \frac{1}{1-J} G_C(x)$ is a cumulative distribution function.

Thus, we have written $F_X(x) = G_C(x) + G_D(x) = (1-J)F_C(x) + JF_D(x)$ as a two-point mixture of a continuous cdf and a discrete cdf.

Now we can solve the question on the practice midterm. The function $F_T(x)$ has only one jump discontinuity of size $1/2$ at $x = 1$. Therefore,

$$F_D(x) = \begin{cases} 0, & x < 1 \\ 1, & x \geq 1. \end{cases}$$

and

$$F_C(x) = \begin{cases} 0, & x < 1 \\ \frac{x^2 - 2x + 1}{2}, & 1 \leq x < 2 \\ 1, & x > 1. \end{cases}$$

The remainder of the question is solved in the answer sheet. Note there was no error on the answer sheet.