1. Introduction

Khovanov homology is an invariant for knots and links smoothly embedded in $S^3$ considered up to smooth isotopy. It was defined by Khovanov in [Kho99] to be a categorification of the Jones polynomial. Khovanov homology and related theories have had numerous topological applications, including a purely combinatorial proof due to Rasmussen of the Milnor conjecture (see [Ras10], see also for instance [Ng05] and [KM11]). In this paper we will consider Khovanov homology calculated with coefficients in $\mathbb{Z}$ unless otherwise stated.

Transverse links in the contact 3-sphere are links that are everywhere transverse to the standard contact structure induced by the contact form $\xi_{st} = dz + r^2 d\theta$. Bennequin proved in [Ben83] that every transverse link is transversely isotopic to the closure of some braid. Furthermore, Orevkov and Shevchisin, and independently Wrinkle, showed that there is a one-to-one correspondence between transverse links (up to transverse isotopy) and braids (up to the braid relations, conjugation, and positive stabilization) (see [OS03], [Wri02]). Hence we can study transverse links by studying braids. Plamenevskaya used this to observe in [Pla06] that Khovanov homology can be used to define an invariant of transverse links. Given a braid $\beta$ whose closure $\hat{\beta}$ is transversely isotopic to a transverse link $K$, she showed that there is a distinguished element $\psi(\beta)$ in the Khovanov chain complex $CKh(\hat{\beta})$ whose homology class $\psi(\beta)$ in the Khovanov homology of $K$ is a transverse invariant that detects the classical self-linking number.

A transverse invariant is called effective if it can distinguish between a pair of smoothly isotopic but not transversely isotopic links with the same self-linking number. It is an open question whether $\psi$ is an effective transverse invariant. Several efforts [LNS15, Wu08, HS16, Col17] have been made to both understand the effectiveness of $\psi$ and to define new invariants related to $\psi$ in the hopes that one of these would be effective. Thus far these efforts have not yielded any transverse invariants arising from Khovanov-type constructions that are known to be effective or not. However, $\psi$ has other applications. For instance, $\psi$ and one of its refinements provide new solutions to the word problem in the braid group [BG15], [HS16].

Another question one can ask about $\psi$ is the following:

**Question 1.1.** Given a transverse knot $K$, what properties of $K$ does $\psi(K)$ detect?

One of the goals in this paper is to explore this question. It is known that $\psi$ does not vanish for transverse links that have a quasi-positive braid representative and that
it does vanish for transverse links with non-right-veering braid representatives\textsuperscript{2} and links with \(n\)-braid representatives that are negative stabilizations of an \((n-1)\)-braid. (Note that quasi-positive braids are all right-veering, see [EHM+15].) Each of these properties – quasi-positive, non-right-veering, and being a negative stabilization – has a straightforward diagrammatic interpretation.

In [Pla15], Plamenevskaya explored Question 1.1 for another transverse invariant, \(\hat{\theta}\), which arises from knot Floer homology [OST08], which she computed using \(\mathbb{Z}/2\mathbb{Z}\) coefficients. In contrast to \(\psi\), \(\hat{\theta}\) is known to be effective [NOT+08]. Plamenevskaya showed that given a transverse link \(K\) with a braid representative \(\beta\), the behavior of \(\hat{\theta}(K)\) is related to dynamical properties of \(\beta\) when \(\beta\) is viewed as acting on the \(n\)-punctured disk \(D_n\).

**Theorem 1.** [Pla15, Theorem 1.2] Suppose \(K\) is a transverse knot that has a 3-braid representative \(\beta\). Every braid representative of \(K\) is right-veering if and only if \(\hat{\theta}(K) \neq 0\).

**Theorem 2.** [Pla15, Theorem 1.3] Suppose \(K\) is a transverse knot that has an \(n\)-braid representative \(\beta\) with fractional Dehn twist coefficient \(\tau(\beta) > 1\). Then \(\hat{\theta}(K) \neq 0\).

Informally, the fractional Dehn twist coefficient of a braid \(\beta\) measures the amount of rotation \(\beta\) effects on the boundary of the punctured disk \(D_n\). The fractional Dehn twist coefficient can be defined in general for elements in the mapping class group of any surface \(\Sigma\) with a single boundary component. As all right-veering braids have fractional Dehn twist coefficient greater than or equal to 0 (see [EHM+15]), Theorem 2 allows us to conclude that, roughly, “most” right-veering braids have non-vanishing \(\hat{\theta}\) (though of course there remain infinitely many braids with fractional Dehn twist coefficient between 0 and 1 inclusive for which the behavior of \(\hat{\theta}\) is not understood). Theorem 2 is similar in flavor to a previous result about contact structures. Work of Honda, Kazez, and Matić in [HKM08], and separately Ozsváth and Szabó in [OS04] proves that a contact structure supported by an open book decomposition with connected binding where the pseudo-Anosov monodromy has fractional Dehn twist coefficient greater than or equal to one has non-vanishing Heegaard Floer twisted contact invariant.

In this paper we first explore the behavior of \(\psi\) with respect to the property of being right-veering and the fractional Dehn twist coefficient. A quick calculation using John Baldwin’s computer program\textsuperscript{3} (available at https://www2.bc.edu/john-baldwin/Programs.html) shows that the statement corresponding to Theorem 1 is not true for \(\psi\), at least over \(\mathbb{Z}/2\mathbb{Z}\) coefficients: there exist right-veering 3-braids (namely the family \(\Delta^k\sigma_2^{-k}\) for sufficiently large \(k \in \mathbb{N}\)) for which \(\psi\) vanishes on their closures. However, for 3-braids the result that corresponds to Theorem 2 does hold:

**Theorem 3.** Suppose \(K\) is a transverse knot that has a 3-braid representative \(\beta\) with fractional Dehn twist coefficient \(\tau(\beta) > 1\). Then \(\psi(K) \neq 0\).

In addition, we show a “stability” property of the homology class of \(\psi\) under adding a sufficient number of full negative twists on two strands to an arbitrary braid word.

\textsuperscript{2}A braid is right-veering if it sends every arc connecting \(\partial D_n\) to one of the punctures to the right of itself.

\textsuperscript{3}This computer program determines whether \(\psi\) vanishes or not in reduced Khovanov homology computed over \(\mathbb{Z}/2\mathbb{Z}\) coefficients. Only one of our results (Proposition 5) depends on a calculation using this program. In that Proposition, we use the program to determine that \(\psi\) does not vanish for a certain braid. Notice that non-vanishing in reduced Khovanov homology over \(\mathbb{Z}/2\mathbb{Z}\) coefficients implies non-vanishing in Khovanov homology over \(\mathbb{Z}\) coefficients. Hence all of our theorems and propositions hold for \(\mathbb{Z}\) coefficients.
Theorem 4. Let $D$ be any closed braid $\beta \sigma_2^{-k}$ with braid index $b$. There is some $N$ for which we have that for all $k > N$, $\psi(\beta \sigma_2^{-k}) = \psi(\beta \sigma_2^{-k+1})$.

This echoes the results by [CK05] which demonstrates a stability behavior of the Jones polynomial of a braid under adding full twists on any number of strands, and [Hog] which considers stability in the Khovanov homology of infinite torus braids. We will address the stability of $\psi$ under adding full twists on more than two strands in a future project.

For 4-braids we have:

Proposition 5. The family of 4-braids $\beta_k = \Delta^2 \sigma_3^{-k}$ for all $k \in \mathbb{N}$ satisfy the following properties:

1. $\tau(\beta_k) = 1$.
2. For $k \geq 12$, $\beta_k$ is not quasi-positive.
3. $\psi(\beta_k) \neq 0$.
4. $\hat{\theta}(\beta_k) \neq 0$ (Plamenevskaya, Theorem 2).

Proposition 5 allows us to conclude that a large infinite collection of non-quasi-positive 4-braids with fractional Dehn twist coefficient greater than one have non-vanishing $\psi$. Indeed, using functoriality allows us to conclude that any braid that has a word of the form $\Delta^2 \eta$ where $\eta$ contains only positive powers of $\sigma_1$ and $\sigma_2$ but arbitrarily many negative powers of $\sigma_3$ has non-vanishing $\psi$. Many such braids are not quasi-positive, and thus $\psi$ does not primarily detect quasi-positivity. We remark that in general, it is not known whether sufficiently large fractional Dehn twist coefficient guarantees non-vanishing $\psi$.

A different perspective on Question 1.1 is whether can one understand or characterize smooth link types for which every transverse representative has vanishing $\psi$. In some sense, this question is asking about smooth link types on which $\psi$ has no chance of being effective. Notice that every link type has some transverse representative for which $\psi$ vanishes. Indeed: every link type has infinitely many distinct transverse representatives. For instance, consider a braid word $\beta$ such that $\hat{\beta}$ is a transverse representative of $L$. Then negatively stabilizing $\beta$ yields $\beta'$, another braid representative of $L$, whose closure is transversely non-isotopic to $\hat{\beta}$ as their self-linking numbers differ by two ([OS03], [Wri02]). In addition $\psi(\beta') = 0$.

One way to probe this question is by examining the relationship between the Khovanov homology of a smooth link type $L$ and its maximal self-linking number. The maximal self-linking number of $L$, $\overline{sl}(L)$, is the maximal self-linking number over all transverse representatives of $L$. This quantity is of natural interest since it provides bounds on several topological link invariants, including the slice genus, see [Rud93] and [Ng08]. The distinguished element $\psi$ in the Khovanov chain complex of a transverse representative $\hat{\beta}$ of $L$ lives in homological grading 0 and quantum grading the self-linking number of $\hat{\beta}$. We have the following immediate observation.

Remark 1.2. Suppose the maximal self-linking number of $L$ is $n$. If the every element in homological grading 0 of the Khovanov homology of $L$ has quantum grading strictly greater than $n$, then $\psi$ vanishes for every transverse representative of $L$.

Example 1.3. According to Proposition 4 of [Ng12], the mirror of the knot $11n33$, which we denote $\overline{11n33}$, has maximal self-linking number $-7$. Using the Khovanov polynomial for $11n33$ in KnotInfo [CL], we see that in homological grading 0, the Khovanov homology

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4This is due to a simple calculation of the writhe. It is possible that $\beta_k$ may be not quasi-positive for some values of $k < 12$. 
of $\Pi_{n \geq 3}$ is empty for $q$-grading less than $-3$. We can conclude that every transverse representative of $\Pi_{n \geq 3}$ has vanishing $\psi$.

We describe an infinite family of 3-tangle pretzel knots for which $\psi = 0$ for every transverse representative by Remark 1.2 and give conditions on the parameters of pretzel knots that guarantee that $\psi$ vanishes for every transverse representative of $L$ using the bound on the maximal self-linking number by Franks-William, Morton, and Ng [FW87, Mor86, Ng05] from the HOMFLY-PT polynomial.

**Theorem 6.** Let $K = P(r, -q, -q)$ be a pretzel knot with $q > 0$ odd and $r > 2$ even, then $\psi = 0$ for every transverse link representative of $K$.

Preliminary computational evidence based on the braid representatives of $K$ from the program by Hilary Hunt available at [https://tqft.net/web/research/students/HilaryHunt/][Hun14b] implementing the Yamada-Vogel algorithm indicates that the fractional Dehn twist coefficient of this family may always be less than or equal to one. This is also true for the braid representative stored in KnotInfo for $\Pi_{n \geq 3}$ from Example 1.3 (in fact, it has fractional Dehn twist coefficient 0). It seems possible that this is a characterization of links for which every single braid representative has $\psi = 0$, and we will address this question in the future.

**Organization.** This paper is organized as follows: we give the preliminary background on Khovanov homology and the transverse invariant in Section 3, and we prove Theorem 3 and Theorem 4 in Section 4. Proposition 5 is proven in Section 5 and finally, we prove Theorem 6 in Section 6 where the necessary results on the maximal self-linking number and the HOMFLY-PT polynomial are summarized.

2. Acknowledgments

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3. Background

In this section we will set our conventions and briefly review Khovanov homology, the transverse invariant defined by Plamenevskaya from a braid representative [Pla06], and standard tools used for computing the invariant.

3.1. Khovanov homology. The readers may refer to [BN02], [Tur17] for excellent introductions to the subject. Given a crossing in an oriented link diagram $D$, a Kauffman state chooses the 0-resolution or the 1-resolution as depicted in the following figure, which replaces the crossing by a set of two arcs. Number the crossings of $D$ from $1, \ldots, n$. Each Kauffman state $\sigma$ on $D$ can be represented by a string of 0 and 1’s in $\{0, 1\}^n$ where 0 at the $i$th position means that the 0-resolution is chosen at the $i$th crossing of $D$ and similarly 1 at the $i$th position means that the 1-resolution is chosen at the $i$th crossing.
The bi-graded chain complex $CKh(D)$ is generated by a direct sum of $\mathbb{Z}$-vector spaces associated to a Kauffman state $\sigma$.

$$CKh(D) := \bigoplus_{\sigma} CKh(D_\sigma),$$

where $CKh(D_\sigma)$ is defined as follows. Let $|s_\sigma(D)|$ be the number of circles in $\sigma$ applied to $D$. Then

$$CKh(D_\sigma) := V^{|s_\sigma(D)|},$$

where $V$ is the free graded $\mathbb{Z}$-module generated by two elements $v_-$ and $v_+$ with grading $p$ such that $p(v_\pm) = \pm 1$. The grading is extended to the tensor product by the rule $p(v \otimes v') = p(v) + p(v')$.

Two gradings $i, j$ are defined on $CKh(D)$ as follows. The homological grading $i$ is defined by $i(v) = \text{sgn}(\sigma) - n_-$, where $\sigma$ is the state giving rise to the vector space $V^{|s_\sigma(D)|}$, while the quantum grading $j$ is defined by $j(v) = p(v) + i(v) + n_+ - n_-$. We shall indicate the $\mathbb{Z}$-vector space with bigrading $(i, j)$ in $CKh(D)$ as $CKh_{ij}(D)$. For the differential of the chain complex, we first define a map $d_c$ on $CKh(D)$ from $\sigma$ to $\sigma_c$:

$$d_c : \sigma \rightarrow \sigma_c,$$

where $\sigma$ and $\sigma_c$ differ in their resolution at exactly one crossing $c$ where $\sigma$ chooses the 0-resolution and $\sigma_c$ chooses the 1-resolution. From $\sigma$ to $\sigma_c$, either two circles merge into one or a circle splits into two. In the first case, the map $d_c$ contracts $V \otimes V$, representing the pair of circles in $s_\sigma(D)$, to $V$, representing the resulting circle in $s_{\sigma_c}(D)$ by the merging map $m$ as defined below.

$$m(v_+ \otimes v_+) = v_+$$
$$m(v_+ \otimes v_-) = (v_- \otimes v_+) = v_-$$
$$m(v_- \otimes v_-) = 0.$$

In the second case where a circle splits, $d_c$ is giving by the splitting map $\triangle$ taking $V \rightarrow V \otimes V$ as follows.

$$\triangle(u_+) = u_+ \otimes u_- + u_- \otimes u_+$$
$$\triangle(u_-) = u_- \otimes u_-.$$

Now on $CKh^i(D_\sigma)$ the differential $d$ is defined by

$$d = \sum_{c \text{ crossing in } D \text{ on which } \sigma \text{ chooses the 0-resolution}} (-1)^{\text{sgn}(\sigma, \sigma_c)} d_c.$$
The sign $\text{sgn}(\sigma, \sigma_c)$ is chosen so that $d \circ d = 0$, say $\text{sgn}(\sigma, \sigma_c)$ is the number of 1’s in the string representing $\sigma$ before $c$. The resulting homology groups $Kh(D)$ are independent of this choice.

Khovanov [Kho00] defined and showed that $Kh(D)$ is independent of the diagram chosen for the link $L$, so $Kh(L) = Kh(D)$ is a link invariant.

3.2. The transverse element. Here we follow the convention of Plamenevskaya [Pla06] except for a minor change in notation. In her paper the bi-grading is indicated as $Kh_{i,j}$, whereas in this paper the homological grading is placed on top as $Kh_i j$.

Let $\beta$ be a braid representative of a link $L$ giving a closed braid diagram $\hat{\beta}$ of $L$. Consider the oriented resolution $\sigma_\beta$ of $\hat{\beta}$ where we take the 0-resolution for each positive crossing and the 1-resolution for each negative crossing.

Definition 3.1. The transverse invariant of a closed braid representative $\hat{\beta}$ of $L$, denoted by $\psi(\beta)$, is the homology class in $Kh(L)$ of following element in the vector space associated to $\sigma_\beta$.

$$\psi(\beta) := v_- \otimes v_- \otimes \cdots \otimes v_- \in V^{\otimes |\sigma_\beta(\hat{\beta})|} = CKh(\hat{\beta}_{\sigma_\beta})$$

Plamenevskaya has shown that this is, up to sign, a well-defined homology class in $Kh(L)$ [Pla06, Proposition 1] under transverse link isotopy, and that $\psi(\beta)$ is a transverse link invariant which lies in $Kh_{0}^{sl(\beta)}$, where $sl(\beta)$ is the self-linking number of the transverse link $\hat{\beta}$ given by

$$sl(\beta) = -b + n_+(\hat{\beta}) - n - (\hat{\beta}).$$

Note $b$ is the braid index of $\beta$.

We will omit $\beta$ from $\psi(\beta)$ and simply write $\psi$ whenever it is clear what the closed braid representative $\beta$ is.

3.3. Functoriality and properties of $\psi$. Using the map on Khovanov homology induced by a cobordism between a pair of links, Plamenevskaya proved the following useful result for computing $\psi$.

Theorem 7. [Pla06 Theorem 4] Suppose that the transverse braid $\hat{\beta}^-$ is obtained from the transverse braid $\hat{\beta}$ by resolving a positive crossing (note that it has to be the 0-resolution). Let $S$ be the resolution cobordism, and $f_S : Kh(\hat{\beta}) \rightarrow Kh(\hat{\beta}^-)$ be the associated map on homology, then

$$f_S(\psi(\beta)) = \pm \psi(\beta^-).$$

A consequence of this is that if $\psi(\beta) = 0$ then $\psi(\beta^-) = 0$. Similarly, suppose that $\hat{\beta}^+$ is obtained from $\hat{\beta}$ by resolving a negative crossing, then $\psi(\beta) \neq 0$ implies that $\psi(\beta^+) \neq 0$. When we use this property in our computations, we will often cite it as “functoriality”.

3.4. Skein exact sequence. Let $D$ be a link diagram and let $D_0$ and $D_1$ be link diagrams differing locally in the 0-resolution and the 1-resolution, respectively, at a negative crossing $c$. The skein exact sequence of homology comes from the short exact sequence given by the following maps.

$$\alpha : CKh^i_{j+1}(D_1) \rightarrow CKh^i_j(D),$$

induced by inclusion, and

$$\gamma : CKh^i_j(D) \rightarrow CKh^{i-u}_{j-3u-1}(D_0),$$
induced by the quotient map. Let \( u = n_-(D_0) - n_-(D) \), where we pick an orientation on \( D_0 \). Then we have the long exact sequence below \([Wat07]\). See \([Ras05]\) for example for an alternate formulation of the skein exact sequence using the oriented skein relation for the Jones polynomial.

\[
\cdots \to Kh^{ij+1}_j(D_1) \xrightarrow{\Delta} Kh^{ij}_j(D) \xrightarrow{\sim} Kh^{ij-1}_{j-3u-1}(D_0) \to Kh^{ij+1}_{j+1}(D_1) \to \cdots.
\]

4. 3-BRAIDS

In this section we prove a series of lemmas leading up to the proof of Theorem 1 using the classification of 3-braids by Murasugi \([Mur74]\). This depends on understanding \( \psi \) for closed 3-braids of the form \( \Delta^2\sigma_1\sigma_2^{-k} \) for \( k > 0 \). We also give an alternate proof using the skein exact sequence \((1)\) and use it to prove Theorem 4 concerning the stability of the homology class of \( \psi \) under adding full twists on 2 strands to a braid in Section 4.2. The reader may skip to Section 4.4 to first see how this closed 3-braid comes up in the proof of Theorem 1.

4.1. The transverse element for the braid \( \Delta^2\sigma_1\sigma_2^{-k} \).

**Theorem 8.** Working in the 3-braid setting, for \( k \in \mathbb{Z} \), \( k \geq 0 \), and \( k \) odd,

\[ \psi(\Delta^2\sigma_1\sigma_2^{-k}) \neq 0 \]

**Proof.** Notice first that for \( k \) odd, the 3-braid \( \Delta^2\sigma_1\sigma_2^{-k} \) closes to a knot rather than to a link. In \([Bal08]\), Baldwin showed that the family of 3-braids

\[ \Delta^2d\sigma_1\sigma_2^{-a_1}\sigma_1\sigma_2^{-a_2}\cdots\sigma_1\sigma_2^{-a_n} \]

where the \( a_i \geq 0 \) and some \( a_j \neq 0 \) is quasi-alternating if and only if \( d \in \{-1, 0, 1\} \). By work of Manolescu and Ozsváth in \([MO07]\), quasi-alternating links are Khovanov homologically \( \sigma \)-thin. This means that the Khovanov homology over \( \mathbb{Z} \) takes a particularly simple form: supported in only one \( \delta = j - i \) grading where \( \delta = \sigma \), the signature of the link. Recall also that Rasmussen’s s-invariant is defined to be the maximum \( j \)-grading (inherited from the Khovanov complex) of the element in the Lee complex that contributes to Lee homology. Lee homology for knots is particularly simple, and is only supported in \( i \)-grading 0. Hence for Khovanov \( \sigma \)-thin links, Rasmussen’s s-invariant is defined to be the signature \( \sigma \).

Next, notice that for these knots, \( sl = \sigma - 1 \) (Remark 7.6, \([BP10]\)). Thus \( sl = s - 1 \). Then work of Baldwin and Plamenevskaya in \([BP10]\) (Theorem 1.2) implies that \( \psi \neq 0 \).

**Corollary 9.** Working in the 3-braid setting, for \( k \in \mathbb{Z} \), \( k \geq 0 \),

\[ \psi(\Delta^2\sigma_1\sigma_2^{-k}) \neq 0 \]

**Proof.** If \( k \) is odd, Theorem \([8]\) proves the result. Suppose that \( k \) is even. Then by Theorem \([8]\), \( \psi(\Delta^2\sigma_1\sigma_2^{-k-1}) \neq 0 \). Then by functoriality, \( \psi(\Delta^2\sigma_1\sigma_2^{-k}) \neq 0 \).

4.2. An alternate proof of Theorem \([8]\) and stability of \( \psi \).

**Proof.** A direct proof of Theorem \([8]\) may be obtained by considering the long exact sequence \((1)\) for \( \Delta^2\sigma_1\sigma_2^{-k} \) at a negative crossing \( c \). In fact, we will consider a simplified braid
word $\sigma_1^2\sigma_2^2\sigma_1^{-k+2} \sim \Delta^2\sigma_1^{-k}$ by the following computation. The subwords which are underlined indicates a change from $\sigma_i\sigma_j\sigma_{i}$ to $\sigma_j\sigma_i\sigma_{j}$.

$$\begin{align*}
\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1^{-k} & = \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1^{-k+1} \\
& = \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1^{-k+1} \\
& = \sigma_1^2\sigma_2^2\sigma_1^{-k+2}
\end{align*}$$

See Figure 1 below.

![Diagram](image)

**Figure 1.** A diagram for the closed braid $\sigma_1^2\sigma_2^2\sigma_1^{-k+2}$ with $-k+2 = -3$ and the last negative crossing marked $c$.

Note that $n_-(D) = k - 2$ and $n_-(D_0) = 2$ is equal to the number of negative crossings in $D_0$ with the $k - 1$ negative kinks removed, and $D_1 = \Delta^2\sigma_1^{-k+1}$. Thus $u = n_-(D_0) - n_-(D) = 2 - (k - 2) = 4 - k$, and we have the long exact sequence

$$\cdots \to \text{Kh}^{i+k-5}_{j+3k-6}(D_0) \to \text{Kh}^{i+1}_{j+1}(D_1) \to \text{Kh}^{i}_{j}(D) \to \text{Kh}^{i+k-4}_{j+3k-5}(D_0) \to \text{Kh}^{i+1}_{j+1}(D_1) \to \cdots$$

Since $D_0$ is the unknot, $\text{Kh}^{i}_{j}(D_0)$ is non-trivial only when $i = 0$. From this we can see that there is an isomorphism

$$\text{Kh}^{0}_{j}(D) \overset{\beta}{\cong} \text{Kh}^{0}_{j+1}(D_1)$$

when $k - 5 \neq 0$. This is satisfied when $k \geq 6$. Let $\psi_1$ denote $\psi(\sigma_1^2\sigma_2^2\sigma_1^{-k+3})$ of $\text{Kh}(D_1)$. When $k \geq 6$, if we have $\psi_1 \neq 0 \in \text{Kh}^{0}_{j+1}(D_1)$, then $\beta(\psi) = \psi_1$ which implies that $\psi$ is nonzero. Computation for $\psi_1$ where $k = 6$ then completes the proof. 

The proof using the skein exact sequence demonstrates a stability behavior in the homology class of $\psi$ under adding full twists on two strands to any braid $\beta$. We restate the theorem here for convenience.

**Theorem 4.** Let $D$ be any closed braid $\widehat{\beta}\sigma_2^{-k}$ with braid index $b$. There is some $N$ for which we have that for all $k > N$, $\psi(\beta\sigma_2^{-k}) = \psi(\beta\sigma_2^{-k+1})$.

**Proof.** We again consider the skein exact sequence (1) at a negative crossing $c$ of $\sigma^{-k}$, and consider $D$, $D_0$, where we take the 0-resolution at $c$, and $D_1$, where we take the 1-resolution.
Let $D'_0$ be the diagram $D_0$ with the $k - 1$ kinks removed under Reidemeister moves I, and let $D' = \hat{\beta}$, then

$$u = n_-(D_0) - n_-(D) = n_-(D'_0) - (n_-(D') + k),$$

since $n_-(D_0) = n_-(D'_0)$ as the crossings in $(k - 1)$ kinks in $D_0$ are positive with any chosen orientation on $D_0$. The long exact sequence is then

$$\cdots \to Kh_{j-3u-1}^i(D_0) \xrightarrow{\alpha} Kh_{j+1}^i(D_1) \xrightarrow{\beta} Kh_j^i(D) \to Kh_{j-3u-1}^i(D_0) \to \cdots,$$

as before. The bigrading of nontrivial homology groups of a link with diagram $D$ are bounded by the span of the Jones polynomial. In particular, let $s := \max_{\sigma} |s_{\sigma}(D)|$. We have that if $Kh_j^i(D) \neq 0$, then

$$-n_- \leq i \leq n_+,$$

$$-n_- - s + n_+ - 2n_- \leq j \leq n_+ + s + n_+ - 2n_-.$$

Now for any $n > 0$ it is possible to find a $k > 0$ such that $u > n$ by (2). We choose $k$ large enough such that $u > n_+(D_0) + 1$ and

$$3u > -(b - 1) - n_+(D_0) + 3n_-(D_0) + s - 1.$$  

This will force for $i = 0$ and $j = sl(\hat{\beta})$, the two Khovanov homology groups $Kh_{j-3u-1}^i(D_0)$ and $Kh_{j-3u-1}^{i-u}(D_0)$ to be trivial, so then we have

$$Kh_j^0(D) \cong Kh_{j-1}^0(D_1)$$

for all such $k$. This completes the proof. \hfill \Box

### 4.3. Fractional Dehn twist coefficient

If $h$ is any element of the mapping class group of $\Sigma$ a surface with one boundary component, we denote its fractional Dehn twist coefficient by $\tau(h)$. Here are a few short observations about the fractional Dehn twist coefficient:

**Proposition 10.** [Malyutin, Mal04, Lemma 5.4] Suppose a braid word $\beta \in B_n$ is $\sigma_i$-free for some $i \in \{1, \ldots, n - 1\}$. Then $\tau(\beta)$ is 0.
Proof. We provide a proof here that is different from Malyutin’s. Suppose $D_n$ is the $n$-punctured disk and view $B_n$ as its mapping class group, that is, there is a natural isomorphism between $B_n$ and $\text{Homeo}^+(D_n, \partial D_n)$ modulo isotopy. Since $\beta$ is missing the Artin generator $i$, the $i$’th arc (counting from the top) in Figure 2 is fixed by $\beta$. Since a properly embedded essential arc is fixed by $\beta$, the fractional Dehn twist coefficient of $\beta$ is 0 (see [KR13]).

\[ \square \]

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{figure2.png}
\caption{Invariant arcs}
\end{figure}

Proposition 11. Suppose an $n$-braid $\beta_{n,k}$ has a word of the form $\Delta^{2n}(\alpha)^k$ for $\alpha$ any $\sigma_i$-free word, $n$ and $k$ integers. Then $\tau(\beta_{n,k})$ is $n$.

Proof. Kazez and Roberts in [KR13] give us the following useful fact (Proposition 2.7): suppose $S$ is a surface with boundary $C$. Denote a Dehn twist about $C$ by $D_C$. Then if $g = D^{-p}(h^q)$, then $\tau(g) = q\tau(h) - p$.

In the braid setting, $D_{\partial D_n} = \Delta^2$. So we have, for $p = -n$, $q = k$, and $h = \alpha$, $\tau(\beta_{n,k}) = q\tau(\alpha) - p = (q \cdot 0) - p = n$ (using Proposition 10).

\[ \square \]

The following method for bounding the fractional Dehn twist coefficient of a braid is due to Malyutin:

Proposition 12. [Malyutin, [Mal04], Proposition 13.1] If a braid $\beta \in B_n$ is represented by a word that contains $r$ occurrences of $\sigma_i$ and $s$ occurrences of $\sigma_i^{-1}$ for some $i \in \{1, \ldots, n-1\}$, then $-s \leq \tau(\beta) \leq r$.

4.4. Proof of Theorem 1

Proposition 13. For any $\beta \in B_3$ whose fractional Dehn twist coefficient is strictly greater than one, $\psi(\beta) \neq 0$.

Proof. According to Murasugi’s classification of 3-braids, every $\sigma \in B_3$ comes in the following types up to conjugation ([Mur74]):

a) $\Delta^{2d} \sigma_1^{a_1} \sigma_2^{-a_1} \sigma_1^{a_2} \cdots \sigma_1^{a_n} \sigma_2^{-a_n}$ where $a_i \geq 0$ for all $i$ and some $a_i > 0$

b) $\Delta^{2d} \sigma_2^m$ where $m \in \mathbb{Z}$

c) $\Delta^{2d} \sigma_1^{-1}$ where $m = -1, -2, -3$

where $d$ can take on any integer value.

The fractional Dehn twist coefficient is invariant under conjugation, and hence whichever of these conjugacy classes a 3-braid belongs to determines that braid’s fractional Dehn twist coefficient.

All braids in classes (a) and (b) have fractional Dehn twist coefficient $d$, since

$$\tau(\sigma_1^{-a_1} \sigma_2^{-a_2} \cdots \sigma_1 \sigma_2^{-a_n}) = 0$$
by Proposition 12 $\tau(\sigma_2^m) = 0$ by Proposition 10 and for any braid $\beta$, $\tau(\Delta^2\beta) = 1 + \tau(\beta)$. All braids in class (c) have fractional Dehn twist coefficient less than or equal to $d$, since by Proposition 12 $\tau(\sigma_1^m\sigma_2^{-1}) \leq 0$ for negative values of $m$.

Hence for each of the classes, we need only consider $d > 1$. Since by Proposition 9 we know that the model braid $\psi(\Delta^2\sigma_1\sigma_2^{-k}) \neq 0$ for all positive $k$ then every other braid in (a) with $d > 1$ has $\psi \neq 0$ by functoriality. Indeed: by making $k$ possibly quite large, we can achieve every other braid in (a) with $d > 1$ by inserting positive crossings. Now let’s consider the braids in (b). Observe that again here if we know that our model braid $\psi(\Delta^2\sigma_1\sigma_2^{-k}) \neq 0$ for all $k$ then functoriality guarantees us that $\psi$ does not die. Indeed, we can get to $(\Delta^2)^m d$ for $d > 1$, $m$ any integer, from $\Delta^2\sigma_1\sigma_2^{-k}$ by inserting positive crossings.

Finally, a straightforward manipulation of the braid words yields that the braids in (c) with $d > 1$ are all quasi-positive. Hence for the braids in (c) with $d > 1$, $\psi \neq 0$ (Pla06).

Remark 4.1. Using John Baldwin’s computer program for computing $\psi$ and functoriality, we determined that for $k \geq 5$, $\psi(\Delta^2\sigma_2^{-k}) = 0$. This is in contrast to the situation for the transverse invariant in knot Floer homology (Pla15).

5. A four-braid example

Proposition 5 follows from the following theorem.

Theorem 14. The 4-braid family

$$\beta_k = \Delta^2\sigma_3^{-k}$$

where $k \in \mathbb{N}$ satisfies

$$\psi(\beta_k) \neq 0.$$ 

Notice that for $k > 12$, $\beta_k$ is not quasi-positive since the writhe is $12 - k < 0$. (It may be that $\beta_k$ is not quasi-positive for some values of $k \leq 12$ as well; we are simply guaranteed that it is not quasi-positive for $k > 12$). So Theorem 14 gives an infinite family of non-quasi-positive braids with non-vanishing $\psi$.

Proof. First, using John Baldwin’s computer program, we determined that

$$\psi(\beta_9) \neq 0.$$ 

By functoriality (adding in positive crossings) this guarantees that $\psi(\beta_k) \neq 0$ for all $1 \leq k < 9$. We will construct an inductive argument for all $k > 9$. Before proceeding with the inductive argument, we introduce some notation and make some general observations. Let $D_0(\beta_k)$ and $D_1(\beta_k)$ be the knots or links that are obtained by replacing the last crossing of $\beta_k$ with its 0- and 1-resolutions, respectively, and taking the closure (see Figure 5). Notice that $D_1(\beta_k) = \hat{\beta}_{k-1}$. We orient $D_1(\beta_k)$ with the same orientation as $\hat{\beta}_{k-1}$ (that is: all strands oriented downwards). We orient $D_0(\beta_k)$ so that all the three outer strands are oriented downwards above the braid word.

We first observe that for all $k > 1$, $D_0(\beta_k)$ is isotopic to the disjoint union of the unknot, oriented counter-clockwise, and the Hopf link $\sigma_2^2$. So the Khovanov homology of $D_0(\beta_k)$ is (where $j$ represents the quantum grading and $i$ represents the homological grading):
In addition, the number of negative crossings in $D_0(\beta_k)$ is 6 for all $k > 1$. Finally, the long exact sequence [\(\mathcal{L}\)] for Khovanov homology corresponding to taking the resolution of a distinguished negative crossing, here the last negative crossing in the word, takes the following form

\[
\cdots \to Kh_{j-3u-1}^i(D_0(\beta_k)) \to Kh_{j+1}^i(D_1(\beta_k)) \to Kh_j^i(\hat{\beta}_k) \to Kh_{j-3u-1}^{i-u}(D_0(\beta_k)) \to \cdots
\]

where $u$ is the difference in the number of negative crossings between $D_0(\beta_k)$ and $\hat{\beta}_k$, that is, $u = n_-(D_0(\beta_k)) - n_-(\hat{\beta}_k) = 6 - k$. We are interested in $Kh(\hat{\beta}_k)$ in homological grading 0 and $q$-grading the self-linking number of $\beta_k$, so $u = 0$ and $q = -4 + 12 - k = 8 - k$. So we have:

\[
\cdots \to Kh_{2k-11}^{k-7}(D_0(\beta_k)) \to Kh_{9-k}^0(D_1(\beta_k)) \to Kh_{8-k}^0(\hat{\beta}_k) \to Kh_{2k-11}^{k-6}(D_0(\beta_k)) \to \cdots
\]

For the base case of our induction, $k = 10$, this becomes:

\[
\cdots \to 0 \to Kh_{-1}^0(D_1(\beta_{10})) \to Kh_{-2}^0(\hat{\beta}_{10}) \to 0 \to \cdots
\]
and hence $Kh_{-1}^0(K_1(β_{10})) = Kh_{-2}^0(β_{10})$ is isomorphic to $Kh_{-2}^0(β_{10})$. Furthermore, this isomorphism is induced by the map that naturally sends $ψ(β_0)$ to $ψ(β_{10})$. Hence since $ψ ∈ Kh_{-1}^0(β_{0})$ is non-zero, the isomorphism implies that $ψ ∈ Kh_{-2}^0(β_{10})$ is non-zero.

We proceed to the inductive step. Suppose $ψ ∈ Kh_{9-k}^0(β_{k-1}) = Kh_{9-k}^0(D_1(β_k))$ is non-zero for some $k > 10$. Then notice that both $k - 7$ and $k - 6$ will be greater than 3, and hence $Kh_{9-k}^{k-7}(D_0(β_k)) = Kh_{9-k}^{k-6}(D_0(β_k)) = 0$. The map on chain complexes yields an isomorphism

$$Kh_{9-k}^0(D_1(β_k)) → Kh_{9-k}^0(β_k)$$

and so $ψ ∈ Kh_{9-k}^0(β_k)$ is non-zero as well. □

6. Bennequin-type inequalities and the maximum self-linking number

In this section we prove Theorem [16]. We will first give the necessary backgrounds on the maximal self-linking number and results which bound the maximal self-linking number using the HOMFLY-PT polynomial of the link. We follow the conventions of KnotAtlas [KAT] for the HOMFLY-PT polynomial. The explicit computations for the HOMFLY-PT polynomials and the Poincare polynomials are done using the KnotTheory package.

**Definition 6.1.** The maximal self-linking number, $sl(L)$ of a smooth link $L$ is the maximum of $sl(L)$ taken over all transverse link representatives $L$ of $L$.

Let $P_L(a, z)$ be the HOMFLY-PT polynomial of $L$, normalized so that $P = 1$ for the unknot and defined by the following skein relation.

$$aP \leftrightarrow \bar{a}^{-1}P \leftrightarrow = zP \leftrightarrow$$

(3)

**Figure 4.**

By [FW87] and [Mor86], we have the following inequality.

**Theorem 15.** ([FW87], [Mor86])

$$\bar{sl}(L) ≤ − \deg_a(P_L(a, z)) − 1,$$

where $\deg_a(P_L(a, z))$ is the maximum degree in $a$ of $P_L(a, z)$.

Ng also provides a skein-theoretic proof that unifies several similiar inequalities in [Ng08]. The transverse element $ψ(β)$ for a braid representative $β$ of $L$ is always supported in the grading $i = 0$ and $j = sl(L)$ in $Kh(L)$ [Pla06, Proposition 2]. Recall that by Remark 1.2, this means that if $Kh_j^0(L) = 0$ for all $j ≤ sl(L)$, then $ψ(β) = 0$ for every braid representative $β$ of $L$.

Consider 3-tangle pretzel knots $K = P(r, −s, −t)$ where $r > 0$ is even and $s, t > 0$ are odd. Our convention is illustrated in Figure 5 below. Since $K$ is a negative knot, there is a single state, the all-1 state, which chooses the 1-resolution on all the crossings of $D$ and gives the generators for the chain complex at homological grading $i = 0$. We will also denote by $|s_1(D)|$ the number of state circles in the all-1 state. It has been shown [Kho03, Proposition 5.1] that $Kh(K)$ is thin at this grading so there are only two possible nontrivial homology groups at $j$-grading equal to $|s_1(D)| − n_−$ and $j$-grading equal to $|s_1(D)| − n_− − 2$.
Note however that the Khovanov homology of an adequate knot is thick by the same Proposition.

Now

\[ |s_1(D)| = r + 1. \]

Thus

\[ |s_1(D)| - n_- = 1 - s - t, \]

and \( Kh(K) \) can only have nontrivial homology groups for \( i = 0 \) at \( j = 1 - s - t \) and \( j = 1 - s - t - 2 \).

**Example 6.2.** The pretzel knot \( K = P(2, -5, -5) \). \( Kh(K) \) has nontrivial homology groups supported in \( i = 0, j = -11 \) and \( i = 0, j = -9 \), and trivial homology groups for all other \( j \) when \( i = 0 \). Its Poincare polynomial \([\text{KAT}]\) is

\[
\frac{1}{q^{11}} + \frac{1}{q^{9}} + \frac{1}{q^{29}t^{10}} + \frac{1}{q^{27}t^{9}} + \frac{1}{q^{25}t^{9}} + \frac{2}{q^{23}t^{7}} + \frac{2}{q^{19}t^{6}} + \frac{1}{q^{17}t^{5}} + \frac{1}{q^{15}t^{4}} + \frac{1}{q^{17}t^{3}} + \frac{1}{q^{13}t^{2}}.
\]

It has HOMFLY-PT polynomial \([\text{KAT}]\)

\[
P_K(a, z) = 10a^{10} - 13a^{12} + 4a^{14} + 39a^{10}z^2 - 32a^{12}z^2 + 4a^{14}z^2 + 57a^{10}z^4 - 27a^{12}z^4 + a^{14}z^4 + 36a^{10}z^6 - 9a^{12}z^6 + 10a^{10}z^8 - a^{12}z^8 + a^{10}z^{10}.
\]

A braid representative for \( K \) via the Yamada-Vogel algorithm \([\text{Yam87}], [\text{Vog90}]\), implemented in the program \([\text{Hum14a}]\) is

\[
\sigma_1^{-1} \sigma_2^{-5} \sigma_1^{-1} \sigma_2^{-5}.
\]

**Lemma 6.3.** Let \( K = P(r, -5, -5) \) be a pretzel knot such that \( r > 2 \) is even. Then

\[
\deg_a(P_K(a, z)) = 2 + r + 10,
\]

while it possibly has nonzero Khovanov homology groups in grading \((0, -9)\) and \((0, -11)\).

Note that for \( K = P(2, -5, -5) \) in the example above, \( \max_a(P_K(a, z)) = 14 = 10 + 2 + 2 \) as predicted.

**Proof.** We prove this by induction on \( r \). Note that the base case is done by Example 6.2. Assume that for \( r - 2 \), \( \deg_a(P((r - 2), -5, -5)) = 2 + (r - 2) + 10 \). Apply relation (4) to the top negative crossing of \( P(r, -5, -5) \). Denote the diagram obtained by switching the crossing by \( D_+ \) and the diagram obtained by resolving the crossing following the orientation by \( D_0 \). Then we have

\[
a^{-1}(P_D(a, z)) = a(P_{D_+}(a, z)) - z(P_{D_0}(a, z)).
\]
Note that a Reidemeister move reduces the diagram $D_+$ to that of $P(p-2,-5,-5)$ and $D_0$ is just the $(2,10)$-torus link with the orientation as indicated in Figure 6.

$D_0$ has the HOMFLY-PT polynomial $[\text{KAT}]$

$$P_{T(2,10)}(a,z) = -\frac{a^9}{z} + \frac{a^{11}}{z} - 15a^9z + 10a^{11}z - 35a^9z^3 + 15a^{11}z^3 - 28a^9z^5 + 7a^{11}z^5 - 9a^9z^7 + a^{11}z^7 - a^9z^9.$$  

Note that $\deg_a(zP_{D_0}) = \deg_a(P_{D_0})$, which satisfies

$$\deg_a(P_{D_0})(a,z) < \deg_a P_{P(2,-5,-5)}(a,z) < \deg_a P_{P(r-2,-5,-5)}(a,z),$$

for all even $r > 2$. Hence

$$\deg_a a^{-1}P_D(a,z) = \deg_a aP_{D_+}(a,z),$$

so

$$\deg_a P_D(a,z) = 2 + \deg_a P_{D_+}(a,z).$$

Using Lemma 6.3 and Theorem 15 gives that

$$\mathfrak{sl}(P(r,-5,-5)) < -(2 + r + 10) < -11.$$  

Therefore, $\psi = 0$ for every braid representative of $P(r,-5,-5)$ for all $r > 2$.

**Example 6.4.** Here is a table of some pretzel knots in Lemma 6.3 and their braid representatives.
$P(4, -5, -5)$ has braid representative
\[ \sigma_1^{-1}\sigma_2^{-5}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_1\sigma_2\sigma_3^{-1}\sigma_4\sigma_5^{-3}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}. \]

$P(6, -5, -5)$ has braid representative
\[ \sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_6^{-1}\sigma_7^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_6^{-1}\sigma_7^{-1}\sigma_4^{-1}\sigma_5^{-1}. \]

$P(8, -5, -5)$ has braid representative
\[
\begin{align*}
&\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_6^{-1}\sigma_7^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_6^{-1}\sigma_7^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_6^{-1}\sigma_7^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_6^{-1}\sigma_7^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_6^{-1}\sigma_7^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_6^{-1}\sigma_7^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_6^{-1}\sigma_7^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_6^{-1}\sigma_7^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_6^{-1}\sigma_7^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_6^{-1}\sigma_7^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_6^{-1}\sigma_7^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_6^{-1}\sigma_7^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_6^{-1}\sigma_7^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_6^{-1}\sigma_7^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_6^{-1}\sigma_7^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_6^{-1}\sigma_7^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_6^{-1}\sigma_7^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_6^{-1}\sigma_7^{-1}. 
\end{align*}
\]

Remark 6.5. Each of these braid representatives has a single $\sigma_1$ and a single $\sigma_1^{-1}$. By Proposition [12] this implies that each of their fractional Dehn twist coefficients lies in the interval $[-1, 1]$. Notice that every transverse link has some braid representative with fractional Dehn twist coefficient in $[-1, 1]$, since any $n$-braid that is a positive stabilization of some $(n-1)$-braid has fractional Dehn twist coefficient lying in $[0, 1]$.

Question 6.6. Suppose $K$ is a smooth link type such that $\psi(\beta) = 0$ for all braid representatives $\beta$ of $K$. Then is the fractional Dehn twist coefficient of each braid representative $\beta$ of $K$ less than or equal to one?

Notice that if the answer to Question 6.6 is yes, the contrapositive would be a statement similar in flavor to, but different from, Theorem 2 in the setting of Khovanov homology: that if a smooth link type $K$ has some braid representative whose fractional Dehn twist coefficient is strictly greater than one, then it has some transverse representative for which $\psi$ does not vanish.

It is not hard to generalize the above examples using the computation for the HOMFLY-PT polynomial for torus knots by Jones [Jon87]. For our purposes it is enough to have the following Lemma which we prove here by inducting on the skein relation definition the HOMFLY-PT polynomial.

Lemma 6.7. If $q > 1$ is odd, then
\[ \deg_a(T_{2,q}) = q + 1, \]
with negative coefficients. If $q > 1$ is even, then (with the orientation given)
\[ \deg_a(T_{2,q}) = q + 1, \]
with positive coefficients.

Proof. We give a proof here by induction. The base cases are $q = 2$ and $q = 3$. We see respectively [KAT] that the term with the highest $a$-degree in $P_{2,2}$ is
\[ \frac{a^3}{z}. \]
For $P_{T_{2,3}}$, this is
\[-a^4.\]
For $T_{2,q}$ where $q > 3$, we expand a single crossing by (3). This gives that
\[
P_{T_{2,q}} = a^2 P_{T_{2,q-2}} - a z P_{T_{2,q-1}}.
\] Assuming the induction hypothesis, we have
\[
\deg_a P_{T_{2,q-2}} = q - 1
\]
with positive/negative coefficients. Similarly, we have
\[
\deg_a P_{T_{2,q-1}} = q
\]
with negative/positive leading coefficients. Plugging this into (3) gives that there is no cancellation between the terms with the maximal degree $q + 1$, and the coefficients are either all positive when $q$ is even, or all negative when $q$ is odd.

**Theorem 16.** Let $K = P(r, -q, -q)$ be a pretzel knot with $q > 0$ odd and $r > 2$ even, then $\psi = 0$ for every transverse link representative of $K$.

*Proof.* Similar to the proof of Lemma 6.3, we apply relation (3) to the top negative crossing of $P(r, -q, -q)$. Denote the diagram obtained by switching the crossing by $D_+$ and the diagram obtained by resolving the crossing following the orientation by $D_0$. Then we have
\[
a^{-1}(P_D(a, z)) = a(P_{D_+}(a, z)) - z(P_{D_0}(a, z)),
\]
and $D_0$ is the diagram of the $T_{2,2q}$ with the orientation as in Figure 6. By Lemma 6.7 we know that $\deg_a P_{T_{2,2q}} = 2q + 1$.

We first consider $P(2, -q, -q)$. Switching the top negative crossing results in $D_+$ being a connected sum of $2 T_{2,q}$'s, and $D_0$ is $T_{2,2q}$. Therefore $\deg_a P_{D_+} = q + 1 + q + 1 = 2q + 2$ and $\deg_a P_{D_0} = 2q + 1$. This clearly shows
\[
\deg_a P_D(a, z) = 2 + \deg_a P_{D_+}(a, z) = 2q + 4.
\]

For $r > 2$ we may induct on $r$ and obtain that
\[
\deg_a P_D(a, z) = 2 + r + 2q.
\]
This gives
\[
\overline{sl}(P(r, q, q)) \leq -2 - r - 2q - 1.
\]
On the other hand, for $Kh(P(r, -q, -q))$ there are only two possible nontrivial homology groups for $i = 0$ with $j$-grading equal to $|s_1(D) - n_-| = 1 - 2q$ and $-1 - 2q$.

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