NOTES

The Matrix Exponential and the relationship between Lie groups & Lie Algebras

Introduction & History

Sophus Lie lived from 1842-1899 and considered the birth date of what we now call today Lie group as the winter of 1873.

Lie's idée fixe was to develope a theory of Symmetries of Diff eqs that would be analgous to Galois theory.

His Idea was to Construct a theory of Continuous groups that would Complement the theory of discrete groups

NOTES

Though Lie theory would never unify ODE's and continuous groups, Lie is recognized as the creator of Lie theory which has been a fruitful field of study since its creation.

DEF: A Lie group is a differentiable manifold G which is also a group, and such that the group product

$$G \times G \longrightarrow G$$

and the inverse map $g \longrightarrow g^{-1}$ is differentiable.

Rmk. Group objects in the category of smooth manifolds

DEF: A matrix Lie group is any subgroup H of $GL_n(\mathbb{C})$ s.t. if A_n is a sequence of matricies in H, and A_n converges to some matrix A, the A is back in H or not invertible

FACT: Every matrix Lie group is a Lie group.

Note: To prove this you can show that a closed subgroup of a lie group is a Lie group.

NOTES

(say aloud)

~~Prop.~~ Let G ; H be Lie groups, and Φ a group homorphism from G to H. Then if Φ Continuous it is also smooth.

Rmk. ~~Every matrix Lie group is a Smooth manifold~~. As a matrix Lie group is automatically locally path-connected. It follows that a matrix Lie group is path-connected $\Leftrightarrow$ if is connected.

The Matrix Exponential

The e^M plays a Critical role in the theory of Lie groups.

The exponential enters into the def to the Lie algebra of a matrix Lie group and is the passing mechanism for Passing info between groups $\to$ AlGs

NOTES

DEF: Let X be an $M_n(\mathbb{C})$.

The exponential of X is defined by the power series

$$e^{X} = \sum_{m=0}^{\infty} \frac{X^m}{m!} \qquad \boxed{E}$$

Convention: X, Y are the varible in the matrix exponential.

Recall: $\|A\| = \sup \frac{\|Ax\|}{\|x\|}$ for $x \in \mathbb{C}^n$

$\equiv$, $\|A\|$ is the smallest $\lambda \in \mathbb{C}$ s.t

$\|Ax\| \le \lambda \|x\| \quad \forall\, x \in \mathbb{C}^n$.

All $n \times n$ matricies have, finite norm, $\|AB\| \le \|A\|\|B\|$

$\|A+B\| \le \|A\| + \|B\|$; $A_m \to A \iff \|A_m - A\| \to 0$

NOTES

Similarly, A_m is cauchy if $\|A_m - A_n\| \to 0$ as $m,n \to \infty$, every cauchy seq. conv., and if $\sum_{m=0}^{\infty} \|A_m\| < \infty \Rightarrow$ abs. conv.

<u>Prop1</u> For any $n \times n$ $\mathbb{R}$ or $\mathbb{C}$ matrix X, if the series (e^X) converges, then matrix exponential e^X is a continuous function of X.

Proof. $\|X^m\| \le \|X\|^m \Rightarrow \sum_{m=0}^{\infty} \left\| \frac{X^m}{m!} \right\| \le \sum_{m=0}^{\infty} \frac{\|X\|^m}{m!}$

$= e^{\|X\|} < \infty$. Thus the series abs. conv.

Continuity, note since X^m is a cont. funct. of X, the partial sums of e^X are cont.

Note: e^X converges on $\{\|X\| \le R\} \Rightarrow$ Sum is cont.

Prop.

① $e^0 = I$, ② e^X is invertible $(e^X)^{-1} = e^{-X}$

③ $e^{(\alpha+\beta)X} = e^{\alpha X} e^{\beta X}$ $\forall\, \alpha, \beta \in \mathbb{C}$

if $XY = YX \Rightarrow e^{X+Y} = e^X e^Y = e^Y e^X$.

NOTES

⑤ if C invertible $\Rightarrow e^{CXC^{-1}} = Ce^{X}C^{-1}$

⑥ $\|e^{X}\| \leq e^{\|X\|}$.

In general, it is not true that $e^{X+Y} = e^{X}e^{Y}$

and the Lie product Formula is as follows:

$$e^{X+Y} = \lim_{k\to\infty}\left(e^{1/k\,X}e^{1/k\,Y}\right)^{k}$$

In the other direction, if X & Y are sufficiently Small matrices, we have:

$$e^{X}e^{Y} = e^{Z} \quad \text{where,}$$

$$Z = X + Y + \tfrac{1}{2}[X,Y] + \tfrac{1}{12}[X,[X,Y]] - \tfrac{1}{12}[Y,[X,Y]] + \ldots$$

THM: $\det(e^{X}) = e^{\operatorname{trace}(X)}$

Prop. Let X be an $n\times n$ Complex matrix, and view the space of all $n\times n$ Complex matricies as $(\mathbb{C}^{n})^{2}$. Then e^{tX} is a smooth Curve in $\mathbb{C}^{n^2}$, and

$$\frac{d}{dt}e^{tX} = Xe^{tX} = e^{tX}X,$$

in particular $\left.\frac{d}{dt}\right|_{t=0} e^{tX} = X$.

Proof. Differentiate the power series for e^{tX} term by term. (is this valid? don't worry about it - Hall)

$\forall i,j$, $(e^{tX})_{ij}$ is a convergent power series which is known to be differentiable.

NOTES

(May skip for more interesting results)

Computing the Matrix exponential

Case 1 : X is diagonalizable. This $\Rightarrow$ that $\exists$ an invertible $\mathbb{C}$ matrix C s.t. $X = CDC^{-1}$ where,

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}$$

we know $e^D = \begin{bmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{bmatrix}$ so by prop

$$e^X = CDC^{-1}$$

Thus if you can diagonalize X, you can explicitly Compute e^X

$\boxed{E}$ $\quad X = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}$ e-vectors $\begin{pmatrix} 1 \\ i \end{pmatrix}$ & $\begin{pmatrix} i \\ 1 \end{pmatrix}$

e-values $-ia$ & ia,

Thus, $C = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ maps basis vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to e-vectors of X

$\Rightarrow C^{-1}XC = D \Rightarrow X = CDC^{-1}$

hence $e^X = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} e^{-ia} & 0 \\ 0 & e^{ia} \end{pmatrix} \begin{pmatrix} 1/2 & -i/2 \\ -i/2 & 1/2 \end{pmatrix} = \begin{bmatrix} \cos(a) & -\sin(a) \\ \sin(a) & \cos(a) \end{bmatrix}$

if $X \in GL_2(\mathbb{R})$ then $e^X \in GL_2(\mathbb{R})$

NOTES

Case 2: X is nilpotent. ($X^m = 0$ for some $m \in \mathbb{N}$)

everything terminates after m term so it's super nice!

$\boxed{E}$ $X = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}$, $X^2 = \begin{bmatrix} 0 & 0 & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$X^3 = 0$

$$\Rightarrow e^{tX} = \begin{pmatrix} 1 & ta & tb + \frac{1}{2}t^2 ac \\ 0 & 1 & tc \\ 0 & 0 & 1 \end{pmatrix}$$

Case 3 X arbitrary

if X neither diagonalizable nor nil potent

X can be written in the form $X = S + N$ by the Jordan Canonical form, N-nil, S-diag and $SN = NS$. $\Rightarrow e^X = e^{S+N} = e^S e^N$.

NOTES

The matrix Logarithm

This should be the inverse function to e^x.
we look to the $\mathbb{C}$-log for guidance. we cannot define a $\operatorname{Log} X$ for all M or $M : \det(M) \neq 0$.
we will define $\operatorname{Log} X$ in a nbhd of the I.

Recall: $\log z = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(z-1)^m}{m}$
is defined and analytic in a circle of radius 1 about $z=1$. $e^{\log z} = z \quad \forall z$ with $|z-1|<1$
$\forall u$ with $|u| < \log 2$, $|e^u - 1| < 1$ and $\log e^u = u$

THM: The function $\log A = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(A-I)^m}{m}$
is defined and continuous on the set of all $n \times n$ $\mathbb{C}$-M, A with $\|A-I\| < 1$, and $\log A$ is $\mathbb{R}$ if A is $\mathbb{R}$.

$\forall A$ with $\|A-I\| < 1$ $\quad e^{\log A} = A$.

$\forall X$ with $\|X\| < \log 2$, $\|e^X - I\| < 1$ and $\log e^X = X$

(Proof.) $\log A$ converges w/ given condition. Continuity is the same as e^X. So show $e^{\log A} = A$ case wise is the same as ...

when A is diag and when it is not (if not use JCF to make $A_m \to A$ where A_m diag.

NOTES

Prop : $\exists$ a constant c s.t. $\forall$ $n \times n$ matrices C with $\|B\| < 1/2$,

$$\|\log(I+B) - B\| \leq c\|B\|^2$$

Proof.

$$\log(I+B) - B = \sum_{m=2}^{\infty} (-1)^m \frac{B^m}{m} = B^2 \sum_{m=2}^{\infty} (-1)^m \frac{B^{m-2}}{m}$$

so that

$$\|\log(I+B) - B\| \leq \|B\|^2 \sum_{m=2}^{\infty} \frac{(1/2)^m}{m}$$

FACT: Let X be $n \times n$ complex matrix, and let C_m be a sequence of matricies s.t. $\|C_m\| \leq \frac{\text{constant}}{m^2}$.

Then,

$$\lim_{m \to \infty} \left[I + \frac{X}{m} + C_m\right] = e^X$$

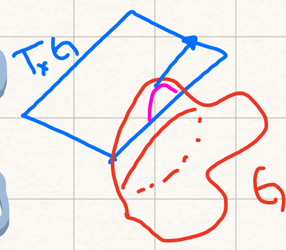
Def: A function $A: \mathbb{R} \to GL_n(\mathbb{C})$ is called a one-parameter group if

1. A is continuous
2. $A(0) = I$
3. $A(t+s) = A(t)A(s)$ $\forall$ $t \in \mathbb{R}$

THM: If A is a one parameter group in $GL_n(\mathbb{C})$ then $\exists!$ ($n \times n$ - $\mathbb{C}$ - M) X s.t. $A(t) = e^{tX}$.

$n = 1 \Rightarrow GL(1, \mathbb{C}) \cong \mathbb{C}^*$

NOTES



Let G be a matrix Lie group. Then the Lie algebra of G denoted $\mathfrak{g}$, is the set of all matricies s.t. e^{tX} is in G for all $t \in \mathbb{R}$.

DEF: if G is a matrix Lie group with Lie alg. $\mathfrak{g}$, then the exponential mapping for G is the map

$$\exp : \mathfrak{g} \longrightarrow G$$

$\boxed{E}$ Show Lie algebras $su(2) \cong so(3)$.

$\boxed{E}$ Pass from $S^1 \xrightarrow{e^X} SO(2)$