1 Motivation

One of the guiding principles of the theory of smooth manifolds is that we can study maps between smooth manifolds by studying what they do locally, i.e. what their differentials do.

Example 1. The inverse function theorem tells us that if we have a map $F : M \to N$ for which $dF : T_pM \to T_{F(p)}N$ is an isomorphism then there exists a neighborhood U of p for which $F\big|_U : U \to F(U)$ is a diffeomorphism.

Before we get into some Lie theory, we'll recall a few ways of thinking about tangent vectors and vector fields.

Philosophy	T_pM	$\mathfrak{X}(M)$
Algebraic	Space of derivations at p, $\delta_p : C^\infty(M) \to \mathbb{R}$	Space of global derivations $C^\infty(M) \to C^\infty(M)$
Dynamical	Equivalence classes of curves $[\gamma]$ at p, $\gamma : (-1, 1) \to M, \gamma(0) = p$	Smooth assignments of classes of curves, i.e. sections of $TM \to M$

We can convert between these notions by using differentiation and local constructions in coordinates. One has the isomorphism $\Psi : (T_pM)_{dyn} \to (T_pM)_{alg}$ by $[\gamma] \mapsto (f \mapsto \frac{d}{dt}\big|_{t=0} f(\gamma(t))$, i.e. we obtain a unique derivation at p by differentiating the composition $f \circ \gamma$ at $t = 0$. We can extend this notion to give an isomorphism $\mathfrak{X}(M)_{dyn} \to \mathfrak{X}(M)_{alg}$ by the vector field $X \in \mathfrak{X}(M)_{dyn}$ is mapped to $(f \mapsto (p \mapsto \Psi(X(p)) \cdot f))$.

These isomorphisms are manifest in the notation often used for the local frame corresponding to a coordinate system. Many times $T_pM = \text{span}\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}$ where we now identify the operator $f \mapsto \frac{\partial f}{\partial x^i}$ with the equivalence class of curves $[t \mapsto p + te_i]$.

Definition 1.1. It is natural to think of vector fields as infinitesimal motions and hence dynamically interesting to ask how well do these infinitesimal motions commute with one another. The Jacobi-Lie bracket $[\cdot, \cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ quantifies this difference and is defined as $[X, Y]f := XY(f) - YX(f)$.

Proposition 1.2. *The Jacobi-Lie bracket, as defined, maps into* $\mathfrak{X}(M)$.

Proof. It is clear that this map is from $\mathfrak{X}(M) \times \mathfrak{X}(M) \to \text{Hom}(C^\infty(M), C^\infty(M))$ so it suffices to show that this yields a derivation.

Given $f, g \in C^\infty(M)$ we have

$$\begin{aligned}
[X,Y](fg) &= XY(fg) - YX(fg) \\
&= X(gY(f) + fY(g)) - Y(gX(f) + fX(g)) \\
&= X(g)Y(f) + gX(Y(f)) + X(f)Y(g) + fX(Y(g)) \\
&\quad - Y(g)X(f) - gY(X(f)) - Y(f)X(g) - fY(X(g)) \\
&= gX(Y(f)) - g(Y(X(f)) + fX(Y(g)) - fY(X(g)) \\
&= g[X,Y]f + f[X,Y]g
\end{aligned}$$

□

2 Mixing Algebra and Differential Geometry

One way to state the mantra is "Studying the global properties of smooth maps is difficult, but the differential allows us to learn a lot about the local properties of a map using linear algebra." In the spirit of Lie theory, we should attempt to combine differential geometry and group theory in order to understand Lie groups. First off, we can use the group action to identify the tangent spaces of the Lie group at every point.

Theorem 2.1. *Let $L_g : G \to G$ be the map of left multiplication by G on itself, i.e. $L_g(g') = gg'$. For each $g \in G$, the differential $d(L_g)_h : T_hG \to T_{gh}G$ is an isomorphism.*

Proof. The map L_g is a smooth bijection $G \to G$, and this map has inverse given by $L_{g^{-1}}$. Hence L_g is a diffeomorphism. □

This then gives us a "canonical" (up to choice between left and right actions) isomorphism $T_eG \to T_gG$ for all $g \in G$, $d(L_g)_e$.

Definition 2.2. Let H and G be Lie groups. A Lie group homomorphism is a smooth map $\varphi : G \to H$ which is a group homomorphism. This means that $\varphi \circ L_g = L_{\varphi(g)}\varphi$.

One of the most powerful facts in Lie theory is that the local information of the differential at the identity actually gives all the information we need to know about the differential of a Lie group homomorphism.

Theorem 2.3. *Let* $\varphi : G \to H$ *be a Lie group homomorphism, then for any* $g \in G$ $d\varphi_g = d(L_{\varphi(g)})_e \circ d\varphi_e \circ d(L_{g^{-1}})$.

Proof. We have $\varphi = L_{\varphi(g)} \circ \varphi \circ (L_g)^{-1} = L_{\varphi(g)} \circ \varphi \circ L_{g^{-1}}$. The chain rule then gives

$$d\varphi_g = d(L_{\varphi(g)})_e \circ d\varphi_e \circ d(L_{g^{-1}})_g.$$

□

This means that the differential of a Lie group homomorphism "looks the same" everywhere and if we know the behavior of the map at the identity, we know its behavior everywhere. For instance, if $\varphi : G \to H$ a Lie group homomorphism and $d\varphi : T_eG \to T_eH$ is an isomorphism, then φ is a local diffeomorphism.

A group acts naturally on itself in another way apart from left and right multiplication, in particular through conjugation. Denote the conjugation map $h \mapsto ghg^{-1}$ by $\mathrm{Ad}_g h$. This is the adjoint representation of G on itself. This tells us something about how the group elements commute with one another. Since $Ad_g(e) = e$, $d(Ad_g)_e : T_e(G) \to T_e(G)$ we can study the action of G on itself by understanding this family of isomorphisms. By abuse of notation we write $d(\mathrm{Ad}_g)_e = \mathrm{Ad}_g$. This abuse is somewhat justified by the case of matrix groups. If G is a matrix Lie group and $\xi \in T_eG = \mathfrak{g}$ we have $Ad_g(\xi) = \frac{d}{dt}\big|_{t=0} Ad_g(\exp(t\xi)) = \frac{d}{dt}\big|_{t=0} g\exp(t\xi)g^{-1} = \frac{d}{dt}\big|_{t=0} \exp(gt\xi g) = g\xi g^{-1}$.

Now, we have constructed a family of linear maps $T_eG \to T_eG$ parametrized by G. Taking our habit of differentiating all maps involving G to get local information and thinking of the map for fixed ξ as $\mathrm{Ad}.\xi : G \to T_eG$ we get a family of maps $d(\mathrm{Ad}.\xi) : T_eG \to T_eG$, or now a map $T_eG \times T_eG \to T_eG$. This map is bilinear and is quite familiar in the case of a matrix group. We have

$$\begin{aligned}
d(Ad.\xi)_e(\eta) &= \frac{d}{dt}\bigg|_{t=0} Ad_{\exp(t\eta)}\xi \\
&= \frac{d}{dt}\bigg|_{t=0} \exp(t\eta)\xi\exp(t\eta)^{-1} \\
&= \frac{d}{dt}\bigg|_{t=0} \exp(t\eta)\xi\exp(-t\eta) \\
&= \left(\frac{d}{dt}\bigg|_{t=0} \exp(t\eta)\right)\xi + \xi\left(\frac{d}{dt}\bigg|_{t=0} \exp(-t\eta)\right) \\
&= \eta\xi - \xi\eta
\end{aligned}$$

This is the commutator bracket. In essence, the commutator bracket tells us, near the identity, how much elements commute with one another. The Baker-Campbell-Hausdorff formula hints that the "local behavior" of multiplication is controlled entirely by the commutator bracket, and we might expect this intuition to extend for more general Lie groups.

There is a more global way in which we can define the Lie bracket using the idea of Left invariant vector fields.

Definition 2.4. Let $\mathfrak{X}(G)$ denote the collection of smooth vector fields on G a Lie group. The action of G on itself by left multiplication gives an action of G on $\mathfrak{X}(G)$ by the pushforward, more explicitly $((L_g)_*X)(h) = d(L_g)_{g^{-1}h}(X(g^{-1}h))$. A vector field X on G is called left invariant if it is fixed under the action of G, that is $X(h) = d(L_g)_{g^{-1}h}X(g^{-1}h)$. Denote the collection of left invariant vector fields on G by $\mathfrak{g}$.

Proposition 2.5. *The set* $\mathfrak{g}$ *is a vector subspace of* $\mathfrak{X}(G)$*, is closed under the Jacobi-Lie bracket* $(X, Y) \mapsto [X, Y] = XY - YX$ *(one can naturally define this bracket when we consider* $\mathfrak{X}(G)$ *as the space of derivations of* $C^\infty(G)$*) and satisfies the following Jacobi identity for all* $X, Y, Z \in \mathfrak{g}$

$$[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0$$

Proof. The first statement follows from the fact that $(L_g)_*$ acts linearly.

For the second statement, we have $(L_g)_*(XY) = ((L_g)_*X)((L_g)_*Y)$ and hence

$$(L_g)_*[X, Y] = ((L_g)_*X)((L_g)_*Y) - ((L_g)_*Y)((L_g)_*X) = XY - YX$$

The final statement is a standard computation using the commutator bracket and application of associativity. □

A priori the space of left invariant vector fields could be some very large collection of vector fields that captures too much information, but the following proposition shows that this space is finite dimensional and isomorphic to T_eG

Proposition 2.6. *Let* G *be a Lie group. The evaluation map* $\psi : \mathfrak{g} \to T_eG$ *is a linear isomorphism.*

Proof. The map $\psi : \mathfrak{g} \to T_eG$ is injective since $\psi(X) = 0$ then $X(g) = d(L_g)_eX(e) = d(L_g\psi(X) = 0$ for all g and $X = 0$. We can define an inverse by taking $\xi \in T_eG$ and $X_\xi(h) := d(L_h)_e\xi$. We must show that X_ξ is smooth. Let U be a coordinate neighborood of e for which $U \cap gU = \varnothing$. This exists by the following argument. Since G is Hausdorff there are neighborhoods $\tilde{U}$ and $\tilde{V}$ of e and g respectively for which $\tilde{U} \cap \tilde{V} = \varnothing$. Since left multiplication is a diffeomorphism $g^{-1}\tilde{V}$ is a neighborhood of e. Then $\tilde{U} \cap g^{-1}\tilde{V}$ is a neighborhood of e and $g(\tilde{U} \cap g^{-}1\tilde{V}) \subset V$. Then take $U = \tilde{U} \cap g^{-1}\tilde{V}$. Intersecting U with a coordinate neighborhood of e gives a new coordinate neighborhood. We want to show that if X_ξ is smooth on U then it is smooth everywhere. Since $(L_g)_* X_\xi = X_\xi$ then $(L_g)_*(X_\xi|_U) = X_\xi|_{gU}$. Let $\Phi : U \to \mathbb{R}^n$ be a chart. our map X_ξ can be thought of as $\psi : T_eG \times U \to TU$ gotten by $(\xi, g) \mapsto (T_eL_g\xi)$. The coordinate chart gives an isomorphism $T\varphi : TU \to U \times \mathbb{R}^n$ when then have $T\varphi \circ \psi(\xi, g) = (T_g\varphi \circ T_eL_g(\xi), g)$. Since the group action is smooth this map is smooth and X_ξ is smooth. □

This gives our first interesting constraint on the differential topology of Lie groups, namely they must be parallelizable:

Theorem 2.7. *If* G *is a Lie group,* G *is parallelizable.*

Proof. Let $e_1, \dots, e_n$ be a basis for T_eG. Under ψ this gives a set of n vector fields $\{X_{e_1}, \dots, X_{e_n}\}$. We see that $X_{e_i}(g) = T_eL_ge_i$ and hence this set gives a global frame on G and G is parallelizable. □

3 Lie Algebras

The map ψ actually interweaves the two bilinear operations we have defined, i.e. for all $\xi, \eta \in \mathfrak{g}$, $[\xi, \eta] = \mathrm{ad}_\xi \eta$. This tells us that what seemed like global information, namely that of the commutator bracket of vector fields is actually a local thing gotten by taking a derivative of the conjugation near the identity.

Remark. Given a homomorphism of Lie groups $\Phi : G \to H$ there is a corresponding map $\Phi_* : \mathfrak{g} \to \mathfrak{h}$ given by the composition $\mathfrak{g} \to T_eG \to T_eH \to \mathfrak{h}$ and this map intertwines the bracket operation on $\mathfrak{g}$ with that on $\mathfrak{h}$, i.e. $[\Phi_*(\xi), \Phi_*(\eta)] = \Phi_*[\xi, \eta]$.

The object $\mathfrak{g}$ allows us to study G via linear algebra and begs to be studied independently without reference to the underlying group G.

Definition 3.1. A Lie algebra for $k = \mathbb{R}$ or $\mathbb{C}$ is a k-vector space $\mathfrak{g}$ and a bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is anti-commutative: $[\xi, \eta] = -[\eta, \xi]$ for all $\eta, \xi \in \mathfrak{g}$ and satisfies the Jacobi identity:

$$[[\xi, \eta], \zeta] + [[\zeta, \xi], \eta] + [[\eta, \zeta], \xi] = 0$$

Remark. The assignment $G \mapsto \mathfrak{g}$ and $\Phi \mapsto \Phi_*$ gives a functor LieGp $\to$ LieAl. Is this functor an equivalence of categories?

Remark. When a Lie group G acts on a smooth manifold M by diffeomorphisms it gives a map $\mathfrak{g} \to \mathfrak{X}(M)$ by associating $\xi \mapsto (p \mapsto \frac{d}{dt}\big|_{t=0}\gamma(t) \cdot p)$ in this way if we think of $\mathfrak{X}(M)$ as $T_e\mathrm{Diff}(M)$ this map is the induced map from $G \to \mathrm{Diff}(M)$.

We can use vector calculus to find the Lie algebra of various matrix groups: Recall that $\det(\exp(t\xi)) = \exp(\mathrm{tr}(t\xi))$ and for $A(t)$ and $B(t)$ smooth curves in $M_n(k)$ $\frac{d}{dt}|_{t=0}(A(t)B(t)) = \dot{A}(0)B(0) + A(0)\dot{B}(0)$.

The special linear group is the space of volume preserving linear maps on a space, this is defined as $SL_n(k) := \{A \in M_n(K) | \det(A) = 1\}$. Multiplicativity of det shows that $SL_n(k)$ is a group and we can see that $\mathfrak{sl}_n(k) = \ker \mathrm{tr}$ through the following argument. $\det : GL_n(k) \to k^*$ is a Lie group homomorphism and as such if $\det_* : \mathfrak{gl}_n(k) \to k$ is non-zero, the map is a submersion. We have $\det_*(\xi) = \frac{d}{dt}|_{t=0}\det(\exp(t\xi)) = \frac{d}{dt}|_{t=0}\exp\mathrm{tr}(t\xi)) = \mathrm{tr}(\xi)$. Since the matrix $\mathrm{diag}(1, 0, \ldots, 0)$ has non-zero trace, det is a submersion and $\det^{-1}(1)$ is a Lie subgroup of $GL_n(k)$. The tangent space to the level set of a submersion is given by the kernel of its differential so $\mathfrak{sl}_n(k) \cong T_eSL_n(k) = \ker \mathrm{tr}$ the trace free $n \times n$ matrices.

If B is a matrix the map $(u, v) \mapsto u^tBv$ defines a bilinear function on k^n. The space of automorphisms of k^n which preserve B form a Lie group in many circumstances by the following: Automorphisms are linear maps A with $A^tBA = B$. Define a map $F_B : GL_n(k) \to M_n(k)$ by A^tBA meaning that $\mathrm{Aut}(k^n, B) = F_B^{-1}(B)$. At $h \in \mathrm{Aut}(k^n, B)$ we have

$$\begin{aligned} T_hF_B(\xi) &= \frac{d}{dt}|_{t=0}(h + t\xi)^tB(h + t\xi) && (1) \\ &= \frac{d}{dt}|_{t=0}(h^tBh + t\xi^tBh + th^tB\xi + t^2\xi^tB\xi) && (2) \\ &= \xi^tBh + h^tB\xi && (3) \end{aligned}$$

Since h is always invertible, this map is of constant rank and hence $\mathrm{Aut}(k^n, B)$ is a Lie group with Lie algebra given by $\ker T_eF_B = \{\xi \in M_n(k) | \xi^tB + B\xi = 0\}$.