Lie Algebras

Goals:

1) Define Lie algebra & Lie algebra homomorphisms
2) Pull together alot of concepts we have been hinting at for the past few weeks
3) Compute Lie algebras for some Lie groups

Defn: A Lie algebra $\mathfrak{g}$ is a vector space over some field $\mathbb{F}$ with a binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies

1) Bilinearity : $[aX+bY, Z] = a[X,Z] + b[Y,Z]$
2) Alternativity: $[X,X]=0 \quad \forall X \in \mathfrak{g}$
3) Jacobi identity:

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$$

(implies $[X,Y] = -[Y,X]$)

Defn: A Lie algebra homomorphism $\varphi: \mathfrak{g} \to \mathfrak{h}$ is a linear map s.t

$$\varphi[X,Y] = [\varphi(X), \varphi(Y)]$$

Examples:

1) Any vector space, endowed w/ the identically zero Lie bracket.
2) $\mathfrak{gl}(V) = \mathrm{End}(V)$ w/ $[X,Y] = XY - YX$

$\uparrow$
linear endomorphisms of V

If V is finite dim'l $\mathfrak{gl}(V) = \mathfrak{gl}(n, \mathbb{F})$ w/ bracket $[X,Y] = XY - YX$

3) $\mathfrak{sl}_n(\mathbb{F}) = \{ X \in \mathfrak{gl}(n, \mathbb{F}) : Tr(X) = 0 \}$

4) $\mathfrak{u}(n) = \{$ skew-self-adjoint matrices $(X^* = -X)$

6) $\mathfrak{so}(n) = \{$ real skew-symetric matrices $(X^T = -X)$.

we will come back to some of these later in the talk.

Some Motivation for learning about Lie algebras

1) Alluded to but worth saying again, a Lie algebra is from its definition just a flat algebraic object - but in many cases, encoded in this algebraic structure is the geometry of the Lie group

2) one connection being there is a very close relationship between representations of a Lie group & rep'n's of its corresponding Lie algebra.

Recall what we have done up to this point

- seen the tangent space at the identity of a Lie group has a Lie bracket
- seen the exponential map which helps us go between the Lie group & the Lie algebra.

Fact: A map $\rho : G \to H$ (G connected) is uniquely determined by its differential $(d\rho)_e : T_eG \to T_eH$

Question: which maps between $T_eG \to T_eH$ actually arise as differentials of group homomorphisms?

"The answer to this question helps us to see one basic reason we care that T_eG has the added structure of a Lie algebra"

recall: we endowed T_eG w/ the structure of a lie algebra in the following way.

Step 1: consider $\quad \psi_g : G \to G \;,\; h \mapsto ghg^{-1}$

Step 2: Define $\quad Ad(g) := (d\psi_g)_e : T_eG \to T_eG$

this gives us a representation

$$\begin{aligned} Ad : G &\longrightarrow Aut(T_eG) \\ g &\longmapsto Ad(g) \end{aligned}$$

Step 3: Finally we considered the differential of Ad

Define $\quad ad = d(Ad)_e : T_eG \longrightarrow End(T_e(G))$

why do this? This gives us a bilinear map

$$\begin{aligned} T_eG \times T_eG &\longrightarrow T_eG \\ (X, Y) &\longmapsto [X, Y] := ad(X)(Y) \end{aligned}$$

Jacob showed us last time that this bracket operation satisfies the criterion to be a Lie algebra

This also answers are question above: we get the following commutative diagram when $\rho: G \to H$ is a Lie Group morphism.

$$\begin{tikzcd}
T_e G \arrow[r, "(d\rho)_e"] \arrow[d, "ad(v)"'] \arrow[dr, phantom, "\circlearrowright"] & T_e H \arrow[d, "ad(d\rho(v))"] \\
T_e G \arrow[r, "(d\rho)_e"'] & T_e H
\end{tikzcd}$$

In particular:

$$d\rho_e([X,Y]) = [d\rho_e(X), d\rho_e(Y)]$$

Fact #2:

Gives us: A linear map $T_eG \to T_eH$ is the differential of a homomorphism $\rho: G \to H$ iff it preserves the bracket operation. (G connected, simply connected)

OR

(*)

$$\{\text{maps } G \to H\} \xleftrightarrow{1:1} \{\text{maps of Lie algebras } (d\rho)_e: \mathfrak{g} \to \mathfrak{h}\}$$

Recall if we are working w/ a matrix Lie Group, we have

$$[X, Y] = XY - YX$$

Defn: A rep'n of a Lie algebra $\mathfrak{g}$ on a vector space V is a map of Lie algebras

$$\rho: \mathfrak{g} \longrightarrow \mathfrak{gl}(V) = \mathrm{End}(V)$$

$\Leftrightarrow$ an action of $\mathfrak{g}$ on V s.t

$$[X,Y](v) = X(Y(v)) - Y(X(v))$$

what (*) tells us is rep'n of connected, simply connected Lie groups are in 1:1 correspondence w/ rep'n's of its Lie algebra.

Recall some facts about the exponential map:

we have a Lie group & an associated Lie algebra, how do we go back & forth?

Recall the exponential map

$$\exp : \mathfrak{g} \to G$$

(which can be defined in several ways)

"one way is through 1-parameter subgroups:

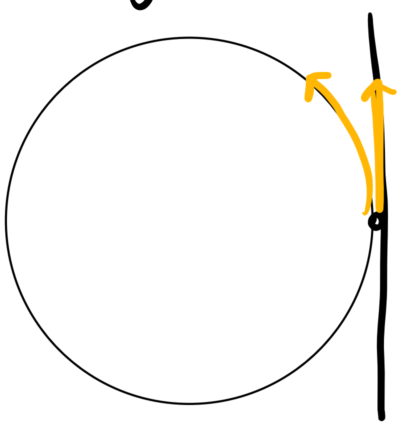
$$\exp(X) = \gamma(1)$$

verified through fact 2
$\mathbb{R} \to G$

where $\gamma : \mathbb{R} \to G$ is the unique one-parameter subgroup of G whose tangent vector at I is X"

For matrices remember $\exp(X) = e^X$ & if $A(t)$ is some 1 parameter subgroup in $GL_n(\mathbb{C})$, $\exists!\, X$ s.t $A(t) = e^{tX}$

Ex:



$\leftarrow$ tangent line $= \{it : t \in \mathbb{R}\}$

$\exp(it) = e^{it} = \cos(t) + i\sin(t)$

This also shows the group structure is encoded in the lie algebra:

This is because

1) exp(x) generate G — since the differential of the exponential map at origin in $\mathfrak{g}$ is an isomorphism, im exp contains a nbhd of the identity in G. G connected: this will generate all of G

2) For X & Y in a sufficiently small nbhd of the origin in $\mathfrak{g}$, we can write

$$\exp(X)\exp(Y) = \exp(Z)$$

Example computations..

we have a few ways to determine the lie algebra of a lie group (all matrix examples)

1) Brute force look at tangent space

$$\{\gamma : (-\varepsilon, \varepsilon) \to G : \gamma(0) = I, \gamma'(0) = A\}$$

OR

2) (For matrices) $\{X : e^{tX} \in G \quad \forall t \in \mathbb{R}\}$

Examples:

1) $G = SL_n(\mathbb{F}) = \{X \in Gl_n(\mathbb{F}) : \det(X) = 1\}$ ($\mathbb{R}$ | $\mathbb{C}$)

consider $\mathfrak{sl}_n(\mathbb{R}) = \{X \in \mathfrak{gl}_n(\mathbb{F}) : \text{trace}(X) = 0\}$

consider $\gamma : (-\varepsilon, \varepsilon) \to SL_n(\mathbb{R})$
w/ $\gamma(0) = I$ & $\gamma'(0) = A$

what can we say about A?

claim: $\frac{d}{dt}\Big|_{t=0} \det(\gamma(t)) = \mathrm{trace}(\gamma'(0))$

→ pretty easy to see for 2x2 matrices ←
not a very enlightening computation though...

we see $\gamma'(0) = A$ must have trace 0 since

$$\frac{d}{dt}\Big|_{t=0} \det(\gamma(t)) = 0$$

certainly $\mathfrak{g} \subseteq \mathfrak{sl}_n(\mathbb{F})$

take any traceless matrix A then consider

$$\gamma(t) = e^{tA}$$

Also illustrating the exponential map.

then $\det(e^{tA}) = e^{tr(tA)} = e^0 = I$.

^ recall

$$\Rightarrow \quad \mathfrak{g} = \mathfrak{sl}_n(\mathbb{H})$$

2) $SO(3) = \{ A \in Gl_3(\mathbb{R}) : AA^T = I \}$

$\mathfrak{so}(3) = \{ A \in \mathfrak{gl}_3(\mathbb{R}) : X^T = -X \}$

(skew-symmetric matrices)

take $\gamma : (-\varepsilon, \varepsilon) \to SO(3)$ w/ $\gamma(0) = I$, $\gamma'(0) = A$

then, $\forall t \in (-\varepsilon, \varepsilon)$

$$\gamma(t)\gamma(t)^T = I$$

$$\Rightarrow \quad \frac{d}{dt}\Big|_{t=0} \gamma(t)\gamma(t)^T = 0$$

$$= \gamma'(0)\gamma(0)^T + \gamma(0)(\gamma'(0))^T = 0$$

$$A + A^T = 0$$

$$A^T = -A$$

Take M, skew-symmetric matrix, define

$$\gamma(t) = e^{tM}$$

then wts: $e^{tM}(e^{tM})^T = I$

$$= e^{tM} e^{tM^T}$$

$$= e^{tM + tM^T}$$

$$= e^{t(M + M^T)}$$

$$= I$$

3) $SU(2) = \{ M \in GL_2(\mathbb{C}) : M^* M = I, \det M = 1 \}$

$$\mathfrak{su}(2) = \{ M \in \mathfrak{gl}_2(\mathbb{C}) : M^* = -M \}$$

$$\boxed{\mathfrak{su}(2) \cong \mathfrak{so}(3)}$$

upshot: A connected Lie group is wholly specified by the Lie algebra together w/ the fundamental group.

Final Remark: Every f.d Lie algebra is the Lie algebra of a Lie group.