Root systems and Dynkin diagrams

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Chapter 1

Introduction

1.1 Recap

- Given a Lie algebra, it splits into solvable and semisimple part via the radical.
- Venn diagram + arrow diagram from Amy L.
- Solvable stuff is “easy” for representation theory purposes.
- Semisimple = no non-zero solvable/abelian ideals. I.e. $\mathrm{rad}(\mathfrak{g}) = 0$
- Simple = no non-zero proper ideals.
- Simple complex Lie algebras are classified.
- Ask Questions! If bored count the vertices in the bounty picture.

1.2 Overview

I will talk a bit about this classification, how simple and semisimple are related. Root decompositions, root systems, their classifications and ((Dynkin diagrams)). Hand write the intro once the talk is written.

Advantages and disadvantages of not giving proofs.

Chapter 2

Structure of semisimple Lie algebras

2.1 The Killing Form

Let $\mathfrak{g}$ be a Lie algebra (real of complex), recall the representation:

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), X \mapsto [X, \cdot] = \mathrm{ad}X$$

Definition 2.1. The Killing form on $\mathfrak{g}$ is defined as:

$$B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}, B(X, Y) = \mathrm{Tr}(\mathrm{ad}X \circ \mathrm{ad}Y) = \mathrm{Tr}([X, [Y, \cdot]])$$

Remark 2.2. This is NOT the trace of the product XY.

Theorem 2.3. *[KK96, Theorem 1.42] $\mathfrak{g}$ is semisimple $\iff$ The killing form B is non-degenerate.*

Theorem 2.4. *[KK96, Thoerem 1.54] $\mathfrak{g}$ semisimple, then $\mathfrak{g}$ decomposes as a direct sum of simple Lie algebras.*

Proof. Proof uses the Killing form. □

2.2 Root Space Decomposition

Let $\mathfrak{g}$ be a Lie algebra **over** $\mathbb{C}$.

Analogy 2.5. $T : V \to V$ diagonaliseable, then $V = \bigoplus_{\lambda \in \Lambda \subset \mathbb{C}} V_\lambda$, where $V_\lambda = \{v \in V : Tv = \lambda v\}$.

We need some magic...

Definition 2.6. A Cartan subalgebra of a complex Lie group is:

- A Nilpotent subalgebra $\mathfrak{h}$ such that $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.
- (For $\mathfrak{g}$ semisimple) a maximal abelian subalgebra $\mathfrak{h}$ such that $\forall H \in \mathfrak{h}$ we have that the $\mathrm{ad}H \in \mathrm{End}(\mathfrak{g})$ are simultaneously diagonaliseable.

Remark 2.7. (skip) Note $N_{\mathfrak{g}}(\mathfrak{h}) = \{X \in \mathfrak{g} : [\mathfrak{h}, X] \subset \mathfrak{h}\}$. If $\mathfrak{h}$ is a subalgebra then $\mathfrak{h} \subset N_{\mathfrak{g}}(\mathfrak{h}) \subset \mathfrak{g}$, where $\mathfrak{h}$ is an ideal if the second equality holds, and self-normalising if the first does.

Remark 2.8. The first dot point is mostly to emphasise that it is important for us to be in the semisimple world at this point.

Theorem 2.9. *[KK96, Theorem 2.9 and 2.15] Any complex Lie algebra has a Cartan subalgebra, the notion is well defined when $\mathfrak{g}$ is semisimple, and they are all conjugate via $Ad(g)$ for some $g \in G$.*

Theorem 2.10. *$\mathfrak{g}$ semisimple complex Lie algebra and $\mathfrak{h}$ a Cartan subalgebra, then:*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta \subset \mathfrak{h}^*} \mathfrak{g}_\alpha$$

Where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \, \forall H \in \mathfrak{h}\}$

Definition 2.11. $\alpha \in \mathfrak{h}^*$ such that $\mathfrak{g}_\alpha \neq 0$ are roots, and $\mathfrak{g}_\alpha$ are called the root spaces.

Remark 2.12. Δ is finite since $\mathfrak{g}$ has finite dimension.

2.3 Example of Root Space decomposition

Example 2.13. $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}) = \{X \in M_n(\mathbb{C}) : \mathrm{Tr}X = 0\}$. In this case our Cartan subalgebra is:

$$\mathfrak{h} = \{\mathrm{diag}(h_1, ..., h_n) \in \mathfrak{sl}_n\}$$

That is, diagonal matrices, whose entries sum to one.

Sanity check: This is abelian, feels maximal, and we will show simultaneous diagonaliseability of $\mathrm{ad}(\mathfrak{h})$.

$$e_i : \mathfrak{h} \to \mathbb{C},\ e_i(\mathrm{diag}(h_1, ..., h_n)) = h_i$$

Consider the vectors E_{ij} with a 1 in the ith row and jth column. Then let $H = \mathrm{diag}(h_1, ..., h_n) \in \mathfrak{h}$

$$[H, E_{ij}] = HE_{ij} - E_{ij}H = h_iE_{ij} - h_jE_{ij} = \underbrace{(e_i - e_j)}_{root}(H)E_{ij}$$

Therefore:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathbb{C}E_{ij} = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathfrak{g}_{e_i - e_j}$$

So $\Delta = \{e_i - e_j : i \neq j\} \subset \mathfrak{h}^*$.

DRAW THE HEXAGON FOR n=3

Example 2.14. $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C}) = \{X \in M_n(\mathbb{C}) : X^T + X = 0\}$. Here our Cartan subalgebra $\mathfrak{h}$ is given by matrices of the form:

$$H = \begin{pmatrix} \begin{pmatrix} 0 & ih_1 \\ -ih_1 & 0 \end{pmatrix} & & & \\ & \ddots & & \\ & & \begin{pmatrix} 0 & ih_n \\ -ih_n & 0 \end{pmatrix} & \\ & & & 0 \end{pmatrix}$$

Sanity check: $\mathfrak{h}$ is an abelian subalgebra, we will check diagonalisability of $\mathrm{ad}(\mathfrak{h})$.

$\Delta = \{\pm e_i \pm e_j : i \neq j \in [n]\} \cup \{\pm e_k : k \in [n]\} \subset \mathfrak{h}$

DRAW BOX DIAGRAM FOR WHEN n=2 (so(3))

Some Properties:

Theorem 2.15. *[KK96, Chapter 2, section 4] If Δ is the collection of roots attached to a semisimple complex Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$.*

- $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$
- $\alpha, \beta \in \Delta \cup \{0\}$ *and* $\alpha + \beta \neq 0$, *then* $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ *(root decomp is almost orthogonal)*
- $\alpha \in \Delta$ *then B is non-degenerate on* $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$
- $\alpha \in \Delta \implies -\alpha \in \Delta$.
- $B|_{\mathfrak{h}\times\mathfrak{h}}$ *is non-degenerate.*
- Δ *spans* $\mathfrak{h}^*$
- *————elementary above this line, deep below————*
- $\alpha \in \Delta \implies \dim \mathfrak{g}_\alpha = 1$.
- $\alpha \in \Delta$, *then* $n\alpha \in \Delta \implies n \in \{-1, 1\}$.
- $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$
- $\mathfrak{h}_0 \subset \mathfrak{h}$ *is the real subspace on which all roots are real ($\mathbb{R}$ span of Δ in our examples). Then $B|_{\mathfrak{h}_0\times\mathfrak{h}_0}$ is a non-degenerate inner-product.*

2.4 Root Systems

One reason to restrict to $\mathfrak{h}_0$ is that angles are sensible for real inner product spaces.

Let $B|_{\mathfrak{h}_0\times\mathfrak{h}_0}$ non-degenerate induces an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{h}_0^*$. The pictures are faithful.

We have for each $\alpha \in \Delta \subset \mathfrak{h}_0^*$ a reflection in α:

$$s_\alpha(\beta) = \beta - 2\frac{\langle \beta, \alpha \rangle}{\|\alpha\|^2}\alpha = \beta - 2\langle \beta, \hat{\alpha} \rangle \hat{\alpha}$$

DRAW s_α on the pictures

The rigidity of these conditions tells us that Δ forms a root system.

Definition 2.16. A reduced root system is a finite set of points Δ in a real inner product space V such that:

- Δ spans V
- The s_α preserve Δ
- $\frac{2\langle \beta, \alpha \rangle}{\|\alpha\|^2} \in \mathbb{Z}$ (reflections are adding integer multiples of α).
- $\alpha \in \Delta \implies 2\alpha \notin \Delta$.

Definition 2.17. Δ is reducible if $\Delta = \Delta' \cup \Delta''$ with $\Delta' \perp \Delta''$. The root system of a Lie algebra is irreducible iff the Lie algebra is simple.

Definition 2.18. Positive systems Δ^+ are roots landing in a hyperplane. Simple systems Π can get all the positive roots with positive integer combinations.

Remark 2.19. These notions are "equivalent"

Theorem 2.20. *[KK96, Theorem 2.108] Let $(\mathfrak{g}, \mathfrak{h}), (\mathfrak{g}', \mathfrak{h}')$ be complex semisimple Lie algebras with Cartan subalgebras whose associated root systems are Δ and Δ'. Suppose $\phi : \mathfrak{h} \to \mathfrak{h}'$ is such that ϕ induces a bijection $\Delta \cong \Delta'$. Then there exists a unique map of Lie algebras $\tilde{\phi} : \mathfrak{g} \to \mathfrak{g}'$ "up to scaling" such that $\tilde{\phi}|_{\mathfrak{h}} = \phi$. (in fact, up to a choice of non-zero root vectors E_α for a positive system $\alpha \in \Pi \subset \Delta$).*

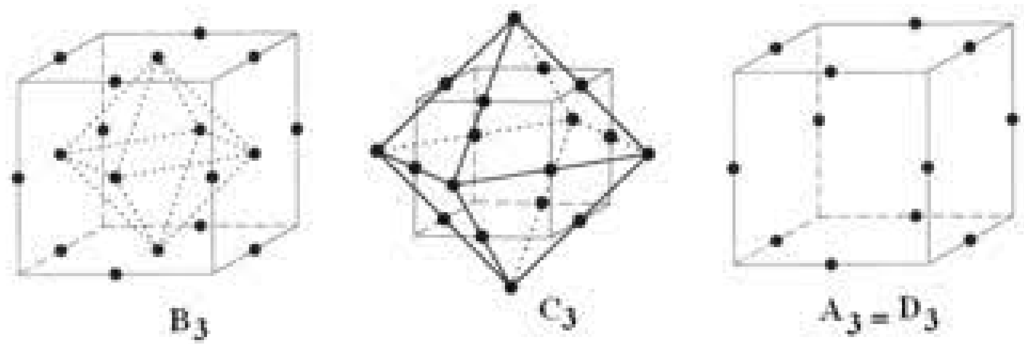
$$\mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \xrightarrow{\tilde{\phi}} \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

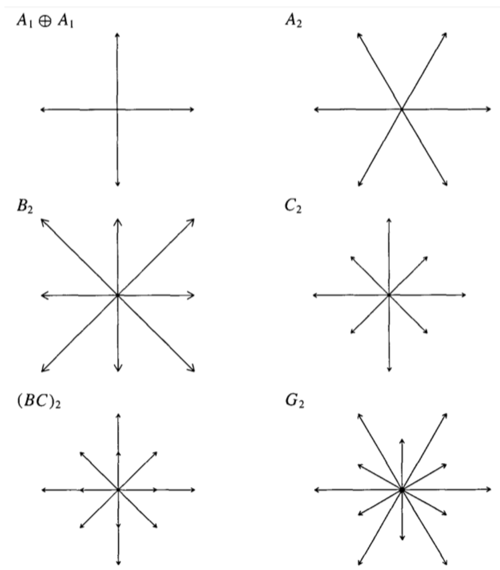
Theorem 2.21. *All irreducible root systems are of the form*

- $A_n : \{e_i - e_j : i \neq j \in [n]\} \leftrightarrow \mathfrak{sl}_n$.
- $B_n : \{\pm e_i \pm e_j : i \neq j \in [n]\} \cup \{\pm e_k\} \leftrightarrow \mathfrak{so}_{2n+1}$
- $C_n : \{\pm e_i \pm e_j : i \neq j \in [n]\} \cup \{\pm 2e_k\} \leftrightarrow \mathfrak{sp}_{2n}$
- $D_n : \{\pm e_i \pm e_j : i \neq j \in [n]\} \leftrightarrow \mathfrak{so}_{2n}$
- $E_6, E_7, E_8 \leftrightarrow \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$
- $F_4 \leftrightarrow \mathfrak{f}_4$
- G_2 : *see hand out* $\leftrightarrow \mathfrak{g}_2$

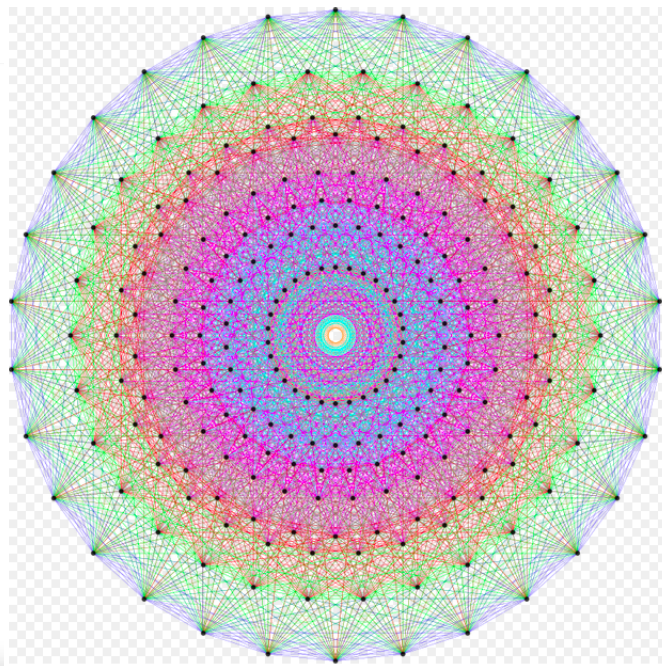
Remark 2.22. The third dot point above gives a restriction on the possible angles of simple roots: $\{\underset{1}{90}, \underset{1}{120}, \underset{\sqrt{2}}{135}, \underset{\sqrt{3}}{150}\}$.

Dynkin diagram: vertices indexed proportional to size squared.





ch FIGURE 2.2. Abstract root systems with $V = \mathbb{R}^2$



Bibliography

[KK96] Anthony W Knapp and Anthony William Knapp. *Lie groups beyond an introduction*, volume 140. Springer, 1996.