What is an anomaly?

Dan Freed
University of Texas at Austin

February 7, 2023
On the Use of Subtraction Fields and the Lifetimes of Some Types of Meson Decay

J. Steinberger
The Institute for Advanced Study, Princeton, New Jersey
(Received June 11, 1949)

The method of subtraction fields in current meson perturbation theory is described, and it is shown that it leads to finite results in all processes. The method is, however, not without ambiguities, and three are stated. It is then applied to the following problems in meson decay: Decay of a neutral meson into two and

Axial-Vector Vertex in Spinor Electrodynamics

J. S. Bell
Institute for Advanced Study, Princeton, New Jersey 08540
(Rceived 24 September 1968)

Working within the framework of perturbation theory, we show that the axial-vector vertex in spinor electrodynamics has an anomalous property which differs from those found by the formal manipulations of field equations. Specifically, because of the presence of closed-loop "triangle diagrams," the divergence of the axial-vector current does not satisfy the usual Ward identity. One consequence is that, even after the external-line
Anomalies and the Atiyah-Singer index theorem

Axial anomaly and Atiyah-Singer theorem

N. K. Nielsen, B. Schuster

Path integral for gauge theories with fermions

Kunio Fujikawa

Institute for Nuclear Study, University of Tokyo, Tokai, Tokyo 270, Japan

Received 25 June 1980

The Atiyah-Singer index theorem indicates that a naive scaling transformation of basic vectors for fermions interacting with gauge fields is not allowed in general. On the basis of this observation, it was previously shown that the path-integrated measure of a gauge-invariant fermion theory is transformed nontrivially under the chiral transformation, and thus leads to a single derivation of "anomalous" chiral Ward-identity identities. We here clarify some of the technical aspects associated with the discussion. It is shown that the Jacobian factor in the path-integrated measure, which corresponds to the Adler-Bell-Jackiw anomaly, is independent of any smooth regularization procedure of large eigenvalues of $D$ in Euclidean theory; this property holds in any one-dimensional space-time and also for the gravitational anomaly. The


Mathematics

Dirac operators coupled to vector potentials

(eigenvalues/induced theory/characteristic classes/anomalies/gauge fields)

M. F. Atiyah† and I. M. Singer‡

†Mathematical Institute, University of Oxford, Oxford, England; and Department of Mathematics, University of California, Berkeley, CA 94720

Contributed by I. M. Singer, January 6, 1984

THEOREM 4. A gauge covariant $\mathcal{F}(A)$ smooth in $A$ exists if and only if the determinant line bundle of $\text{Ind } \mathcal{F}$ is trivial—i.e., $d_2 = 0 \in H^2(\Omega/\mathbb{Z}, Z)$ or $t_1 = 0 \in H^1(\mathbb{Z}, Z)$.

The characteristic forms $d_2 \in H^2(\Omega/\mathbb{Z}, Z)$ are obstructions to the existence of a covariant propagator for $\mathcal{F}_{\text{Ind}}$. We ask the question: Do the higher obstructions have physical significance?

Hamiltonian Interpretation of Anomalies

Phillip Nelson$^\dagger$ and Luis Alvarez-Gaumé$^\ddagger$

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$^\ddagger$Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA

Abstract. A family of quantum systems parametrized by the points of a compact space can realize its classical symmetries via a new kind of nontrivial ray representation. We show that this phenomenon in fact occurs for the quantum mechanics of fermions in the presence of background gauge fields, and is responsible for both the nonabelian anomaly and Witten's $\text{SU}(2)$ anomaly. This provides a hamiltonian interpretation of anomalies: in the affected theories, Gauss' law cannot be implemented. The analysis clearly shows why there are no further obstructions corresponding to higher spheres in configuration space, in agreement with a recent result of Atiyah and Singer.

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Anomalies in Nonlinear Sigma Models

Gregory Moore and Philip Nelson

Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138

Received 1 June 1984

Certain nonlinear sigma models with fermions suffer from an anomaly similar to the one in non-Abelian gauge theory. We exhibit this anomaly using both perturbative and global methods. The effective theories are ill defined and hence unsuitable for describing low-energy dynamics. They include certain supersymmetric models in four-space dimensions.

ALGEBRAIC AND HAMILTONIAN METHODS IN THE THEORY OF NON-ABELIAN ANOMALIES

L. D. Faddeev and S. L. Shatashvili

The non-Abelian anomalies and the Wess-Zumino action are given a new interpretation in terms of infinitesimal and global cocycles of the representation of the gauge group acting on functionals of Yang-Mills fields. On the basis of this interpretation, two simple methods of nonperturbative calculation of the anomalies and the Wess-Zumino action are proposed.

Faddeev's anomaly in Gauss's law

Gromov Segal

51. General remarks

Faddeev [1] has pointed out that when a gauge theory is quantized the gauge operators act with anomalous commutation relations —so called “towards terms” — on the Hilbert space $\mathcal{H}$ of states. In mathematical language this means that the Lie algebra $\mathcal{L}$ of the gauge group does not act on $\mathcal{H}$, but an extension of $\mathcal{L}$ by the vector space $\mathcal{G}$ of vector-valued functions on the space of gauge fields does act. (Here $\mathcal{G}$ is regarded as an abelian Lie algebra.) The extension is described by a cocycle

$$c: \mathcal{G} \times \mathcal{H} \rightarrow \mathcal{H},$$

In this note I shall explain how the cocycle $c$ arises from simple topological considerations of a general kind. I am very grateful
Global Gravitational Anomalies

Edward Witten
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544, USA

Abstract. A general formula for global gauge and gravitational anomalies is derived. It is used to show that the anomaly free supergravity and superstring theories in ten dimensions are all free of global anomalies that might have ruined their consistency. However, it is shown that global anomalies lead to some restrictions on allowed compactifications of these theories. For example,

not obvious. Usually, the only simple way to study a diffeomorphism \( \pi \) is to investigate the associated manifold \((M \times S^1)_\pi\) discussed in Sect. II. The simplest properties of \((M \times S^1)_\pi\) are invariants of a manifold \(B\) which has it for boundary. The only evident connection between \((M \times S^1)_\pi\) and \(B\) in which spinors play a role is the Atiyah-Patodi-Singer theorem concerning the \(\eta\)-invariant [29]. The \(\eta\) invariant can be defined as

\[
\eta = \lim_{\epsilon \to 0} \sum_{\varepsilon, \phi \neq 0} (\text{sign} E_\phi) \exp -\varepsilon |E_\phi|,
\]

where \(E_\phi\) are the eigenvalues of the Dirac operator on \((M \times S^1)_\pi\). The Atiyah-Patodi-Singer theorem asserts (for the spin 1/2 case) that

\[
\frac{\eta}{2} = \text{index}_K(iD) - \int_B \hat{A}(R),
\]

WORLD-SHEET CORRECTIONS
VIA D-INSTANTONS

Edward Witten
School of Natural Sciences, Institute for Advanced Study
Olden Lane, Princeton, NJ 08540, USA

1.) Such a relation means that there is a three-manifold \(U \subset Y\) whose boundary is the union of the \(C_i\) (or more generally a three-manifold \(U\) with a map \(\phi : U \to Y\) such that the boundary of \(U\) is mapped diffeomorphically to the union of the \(C_i\)). In this situation, we can give a relation, which depends only on the gauge-invariant \(H\)-field and not on the mysterious B-field, for the product \(\prod_{i=1}^n F(C_i)\).

First of all, though the factors \(\exp \left( i \int_{C_i} \hat{B} \right)\) are mysterious individually, for their product we can write an obvious classical formula that depends only on \(H\) and \(U\):

\[
\prod_{i=1}^n \exp \left( i \int_{C_i} \hat{B} \right) = \exp \left( i \int_{U} H \right).
\]

This expression depends on \(U\), though this is not shown in the notation on the left-hand side.

More subtle is the product of the Pfaffians. We recall that each fermion path integral \(\text{Pf}((D_\xi \hat{C}_i))\) takes values in a complex line \(\mathbb{C}\). However, according to a theorem of Dai and Freed [1], for every choice of a three-manifold \(U\) whose boundary is the union of the \(C_i\) (together with an extension of all of the bundles over \(U\)), there is a canonical trivialization of the product \(\otimes_i \hat{C}_i\). This trivialization is obtained by suitably interpreting the quantity \(\exp(\pi i \eta(U)/24)\), where \(\eta(U)\) is an eta-invariant of a Dirac operator on \(U\) defined using global (Atiyah-Patodi-Singer) boundary conditions on the \(C_i\). We write the trivialization
Two myths

Just in case...

**Myth 1:** Anomalies are only caused by fermionic fields

**Myth 2:** Anomalies are only associated to symmetries
Two myths

Just in case...

**Myth 1:** Anomalies are only caused by fermionic fields

**Mythbuster 1:** The flavor symmetry of QCD is anomalous—indeed, that anomaly involves fermions—but the anomaly persists in the effective theory of pions, which is a bosonic theory

**Myth 2:** Anomalies are only associated to symmetries
Two myths

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**Myth 2:** Anomalies are only associated to symmetries

**Mythbuster 2:** The theory of a free spinor field has an anomaly
Main thesis

Quantum theory is projective. Quantization is linear.
Main thesis

Quantum theory is projective. Quantization is linear.

The anomaly of a quantum theory expresses its projectivity
**Main thesis**

Quantum theory is projective. Quantization is linear.

The anomaly of a quantum theory expresses its projectivity.

The anomaly is a feature, not a bug (’t Hooft).
Main thesis

**Quantum theory is projective. Quantization is linear.**

The anomaly of a quantum theory expresses its projectivity.

The anomaly is a feature, not a bug ('t Hooft).

The anomaly is an obstruction only when quantizing.
Outline

- Projective spaces, linearization, and symmetry
- Quantum mechanics as a projective system
- Quantum field theory as a projective system
- Invertible field theories
- Anomalies as an obstruction to quantization
- Anomaly of a spinor field
Projectivization of a linear space

$W$  (complex) vector space
$\mathbb{P}(W)$  projective space of lines $L \subset W$
$\text{End}(W)$  algebra of linear maps $T : W \rightarrow W$

```
If $K$ is any line (1-dimensional vector space), then there are
canonical isomorphisms

$\mathbb{P}(W) \xrightarrow{\cong} \mathbb{P}(K) \xrightarrow{T \cong T_K}$

A linear symmetry of $W$ induces a projective symmetry of $\mathbb{P}(W)$
A projective symmetry of $\mathbb{P}(W)$ has a $\mathbb{C}^\ast$-torsor of lifts to a linear symmetry of $W$
```
Projectivization of a linear space

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If $K$ is any line (1-dimensional vector space), then there are *canonical* isomorphisms

\[
\begin{align*}
\mathbb{P}(W) &\rightarrow \mathbb{P}(W \otimes K) \\
L &\rightarrow L \otimes K
\end{align*}
\]

\[
\begin{align*}
\text{End}(W) &\rightarrow \text{End}(W \otimes K) \\
T &\rightarrow T \otimes \text{id}_K
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Projectivization of a linear space

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$$
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\mathbb{P}(W) & \rightarrow \mathbb{P}(W \otimes K) & \text{End}(W) & \rightarrow \text{End}(W \otimes K) \\
L & \mapsto L \otimes K & T & \mapsto T \otimes \text{id}_K 
\end{align*}
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Projectivization of a linear space

$W$ (complex) vector space
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L & \mapsto L \otimes K \\
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A linear symmetry of $W$ induces a projective symmetry of $\mathbb{P}(W)$

A projective symmetry of $\mathbb{P}(W)$ has a $\mathbb{C}^\times$-torsor of lifts to a linear symmetry of $W$
Projective symmetries

\[ \mathbb{C}^\times \rightarrow \text{GL} \rightarrow \text{PGL} \]

Short exact sequence of Lie groups
Projective symmetries

\[ \mathbb{C}^\times \rightarrow \text{GL} \rightarrow \text{PGL} \]

Short exact sequence of Lie groups

Lie group \( G \) of projective symmetries
Projective symmetries

\[ \mathbb{C}^\times \longrightarrow \text{GL} \longrightarrow \text{PGL} \]

\[ \mathbb{C}^\times \longrightarrow \tilde{G} \longrightarrow G \]

Short exact sequence of Lie groups

Lie group \( G \) of projective symmetries

Pullback group extension; linear action of \( \tilde{G} \)
Projective symmetries

\[ \mathbb{C}^\times \to GL \to PGL \]

\[ \mathbb{C}^\times \to \tilde{G} \to G \]

Short exact sequence of Lie groups

Lie group \( G \) of projective symmetries

Pullback group extension; linear action of \( \tilde{G} \)

Lift to linear symmetries \( \leftrightarrow \) splitting of group extension
Projective symmetries

Short exact sequence of Lie groups

Lie group $G$ of projective symmetries

Pullback group extension; linear action of $\tilde{G}$

Lift to linear symmetries $\leftrightarrow$ splitting of group extension

Obstruction to lifting
Projective symmetries

\[ \mathbb{C}^\times \to \tilde{G} \to G \]

\[ \alpha \]

\[ G \to BC^\times \quad \leftrightarrow \quad \text{group extension} \]
Projective symmetries

\[ \mathbb{C}^\times \rightarrow \text{GL} \rightarrow \text{PGL} \rightarrow BC^\times \]

\[ \mathbb{C}^\times \rightarrow \tilde{G} \rightarrow G \]

\[ G \rightarrow BC^\times \quad \text{group extension} \]

Projective action of \( G \) with projectivity \( \alpha \) \( \longleftrightarrow \) linear action of \( \tilde{G} \) s.t. \( \mathbb{C}^\times \) acts by scalar mult
Projective symmetries

$G \rightarrow BC^\times \quad \longleftrightarrow \quad \text{group extension}$

Projective action of $G$ with projectivity $\alpha \leftrightarrow$ linear action of $\tilde{G}$ s.t. $C^\times$ acts by scalar mult

In QM one has analogs of the projective action

In QFT one has analogs of the anomaly $\alpha$ and the linear action
Projective symmetries

Projective action of $G$ with projectivity $\alpha \longleftrightarrow$ linear action of $\tilde{G}$ s.t. $\mathbb{C}^\times$ acts by scalar mult

In QM one has analogs of the projective action

In QFT one has analogs of the anomaly $\alpha$ and the linear action

The analog of the splitting is a linearization or trivialization of the anomaly $\alpha$
The projectivity has an equivalence class in $H^2(G; \mathbb{C}^\times)$ for some cohomology theory.
Cohomological interpretation; splittings

The projectivity has an equivalence class in $H^2(G; \mathbb{C}^\times)$ for some cohomology theory.

The extension is a “cocycle” for this cohomology class.
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Splittings of the extension—trivializations of $\alpha$—form a torsor over characters of $G$. 
Cohomological interpretation; splittings

The projectivity has an equivalence class in $H^2(G; \mathbb{C}^\times)$ for some cohomology theory. The extension is a “cocycle” for this cohomology class. Splittings of the extension—trivializations of $\alpha$—form a torsor over characters of $G$. Characters—invertible linear representations—are elements of $H^1(G; \mathbb{C}^\times)$.
Cohomological interpretation; splittings

The projectivity has an equivalence class in $H^2(G; \mathbb{C}^\times)$ for some cohomology theory

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Splittings of the extension—trivializations of $\alpha$—form a torsor over characters of $G$

Characters—invertible linear representations—are elements of $H^1(G; \mathbb{C}^\times)$

Summary: Projectivity is a “suspended” invertible linear representation
What is a projective space?

Goal: Define a projective space $\mathbb{P}$ without committing to a linearization $\mathbb{P} \xrightarrow{\cong} \mathbb{P}(W)$
What is a projective space?

**Goal**: Define a projective space \( \mathbb{P} \) without committing to a linearization \( \mathbb{P} \to \mathbb{P}(W) \).

Geometric structure à la Klein-Cartan specified by a model geometry \( H \subset X \).
What is a projective space?

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Geometric structure à la [Klein-Cartan](https://en.wikipedia.org/wiki/Kleinian_geometry) specified by a model geometry $H \subset X$

An instance of that geometry is associated to a right $H$-torsor $T$ by mixing: $X_T := T \times_H X$
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**Parametrized family:** principal $H$-bundle $P \to S$  **symmetry:** a groupoid/stack $S = *//G$
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Model geometries for complex projective space:

- $\text{PGL}_{n+1} \subset \mathbb{CP}^n$ (complex manifold)
- $\text{PU}_{n+1} \subset \mathbb{CP}^n$ (Kähler manifold)
- $\widehat{\text{PGL}}_{n+1} \subset \mathbb{CP}^n$ (+ antiholomorphic)
- $\text{PQ}_{n+1} \subset \mathbb{CP}^n$ (+ antiunitary)

($= \text{Fubini-Study isoms}$)
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- $\text{PQ}_{n+1} \subset \mathbb{CP}^n$ (+ antiunitary)

($= \text{Fubini-Study isoms}$)

There are infinite dimensional analogs.
Outline

- Projective spaces, linearization, and symmetry
- **Quantum mechanics as a projective system**
- Quantum field theory as a projective system
- Invertible field theories
- Anomalies as an obstruction to quantization
- Anomaly of a spinor field
Quantum mechanics as a linear system

$\mathcal{H}$

complex separable Hilbert space

$\mathcal{P}_H$

space of pure states

$H \in \text{End}(\mathcal{H})$

Hamiltonian

$p : \mathcal{P}_H \times \mathcal{P}_H \rightarrow [0, 1]$

transition probability function ($\psi_i \in L_i$ unit norm)

$L_0, L_1 \mapsto |\langle \psi_0, \psi_1 \rangle|^2$
Quantum mechanics as a linear system

$\mathcal{H}$
complex separable Hilbert space

$\mathbb{P}\mathcal{H}$
space of pure states

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$p: \mathbb{P}\mathcal{H} \times \mathbb{P}\mathcal{H} \rightarrow [0, 1]$
transition probability function ($\psi_i \in L_i$ unit norm)

$L_0$, $L_1 \mapsto |\langle \psi_0, \psi_1 \rangle|^2$

Probability:

$p \left( L_f, e^{-i(t_f-t_n)H/\hbar} A_n \cdots e^{-i(t_2-t_1)H/\hbar} A_1 e^{-i(t_1-t_0)H/\hbar} L_0 \right) \in [0, 1]$

$t_0 < t_1 < \cdots < t_n < t_f$ real numbers, $A_1, \ldots, A_n \in \text{End} \mathcal{H}$, $L_0, L_f \in \mathbb{P}\mathcal{H}$
Quantum mechanics as a linear system

$\mathcal{H}$ complex separable Hilbert space

$\mathcal{PH}$ space of pure states

$H \in \text{End}(\mathcal{H})$ Hamiltonian

$p: \mathcal{PH} \times \mathcal{PH} \rightarrow [0,1]$ transition probability function ($\psi_i \in L_i$ unit norm)

$L_0, L_1 \rightarrow |\langle \psi_0, \psi_1 \rangle|^2$

Probability: $p \left( L_f, e^{-it_f} A_n \ldots e^{-it_1} A_1 e^{-it_0} L_0 \right) \in [0,1]$ $t_0 < t_1 < \cdots < t_n < t_f$ real numbers, $A_1, \ldots, A_n \in \text{End} \mathcal{H}$, $L_0, L_f \in \mathcal{PH}$

Amplitude: $\langle \psi_f, e^{-it_f} A_n \ldots e^{-it_1} A_1 e^{-it_0} \psi_0 \rangle_{\mathcal{H}} \in \mathbb{C}$ if we choose vectors $\psi_0 \in L_0, \psi_f \in L_f$; as a function of $L_0, L_f$ the amplitude lies in the hermitian line $(L_0 \otimes L_f)^*$; the probability is the norm square: $|\text{Amplitude}|^2 = \text{Probability}$
Quantum mechanics as a projective system

We only need a projective space, not a linear space:

\( \mathbb{P} \)  

projective space

\( \mathcal{A}_\mathbb{P} \) 

complex algebra

\( H \in \text{End}(\mathcal{A}_\mathbb{P}) \)  

Hamiltonian

\[
p: \mathbb{P} \times \mathbb{P} \rightarrow [0, 1]
\]

\( \sigma_0, \sigma_1 \mapsto |\langle \psi_0, \psi_1 \rangle|_\mathcal{H}^2 \)

for any linearization \( \mathbb{P} \rightrightarrows \mathbb{P}\mathcal{H} \)
Quantum mechanics as a projective system

We only need a projective space, not a linear space:

\[ \mathbb{P} \]  
\[ \mathcal{A}_\mathbb{P} \]  
\[ H \in \text{End}(\mathcal{A}_\mathbb{P}) \]  

\( p: \mathbb{P} \times \mathbb{P} \rightarrow [0, 1] \)
\[ \sigma_0, \sigma_1 \mapsto |\langle \psi_0, \psi_1 \rangle|_\mathcal{H}^2 \]

for any linearization \( \mathbb{P} \xrightarrow{\cong} \mathbb{P}\mathcal{H} \)

Probability: \( p(\sigma_f, e^{-i(t_f-t_n)H/h}A_n \ldots e^{-i(t_2-t_1)H/h}A_1e^{-i(t_1-t_0)H/h} \sigma_0) \in [0, 1] \)
Quantum mechanics as a projective system

We only need a projective space, not a linear space:

\[ \mathbb{P} \]  
\[ \mathcal{A}_\mathbb{P} \]  
\[ H \in \text{End}(\mathcal{A}_\mathbb{P}) \]

Projective space
Complex algebra
Hamiltonian

\[ p: \mathbb{P} \times \mathbb{P} \rightarrow [0, 1] \]
\[ \sigma_0, \sigma_1 \mapsto |\langle \psi_0, \psi_1 \rangle|^2_H \]

for any linearization \( \mathbb{P} \xrightarrow{\cong} \mathbb{P}H \)

Probability:  
\[ p \left( \sigma_f, e^{-i(t_f-t_n)H/h} A_n \ldots e^{-i(t_2-t_1)H/h} A_1 e^{-i(t_1-t_0)H/h} \sigma_0 \right) \in [0, 1] \]

Amplitude:  
\[ \langle -, e^{-i(t_f-t_n)H/h} A_n \ldots e^{-i(t_2-t_1)H/h} A_1 e^{-i(t_1-t_0)H/h} - \rangle \in \mathcal{L}_{\sigma_0,\sigma_f} \]
The symmetry/structure group of quantum mechanics

\( \mathbb{P} \)  

projective space

\( p : \mathbb{P} \times \mathbb{P} \rightarrow [0, 1] \)  

transition probability function

Fix a linearization \( \mathbb{P} \cong \mathbb{P} \mathcal{H} \); then the group \( \text{Aut}(\mathbb{P}, p) \) of maps \( \mathbb{P} \rightarrow \mathbb{P} \) preserving \( p \) is the isometry group of the Fubini-Study metric

\[
d : \mathbb{P} \mathcal{H} \times \mathbb{P} \mathcal{H} \rightarrow \mathbb{R}^{\geq 0} \quad \cos(d) = 2p - 1
\]
The symmetry/structure group of quantum mechanics

\( \mathbb{P} \)  
projective space

\( p: \mathbb{P} \times \mathbb{P} \to [0,1] \)  
transition probability function

Fix a linearization \( \mathbb{P} \to \mathbb{P}\mathcal{H} \); then the group \( \text{Aut}(\mathbb{P}, p) \) of maps \( \mathbb{P} \to \mathbb{P} \) preserving \( p \) is the isometry group of the Fubini-Study metric

\[ d: \mathbb{P}\mathcal{H} \times \mathbb{P}\mathcal{H} \to \mathbb{R}^{\geq 0} \quad \cos(d) = 2p - 1 \]

Example: \( \dim\mathcal{H} = 2, \mathbb{P} = \mathbb{C}\mathbb{P}^1 \approx S^2 \) (round metric), \( \text{Aut}(\mathbb{P}, p) = O_3 \)

\[
egin{align*}
T &\to U_2 \to SO_3 \\
T &\to Q_2 \to O_3 = PQ_2
\end{align*}
\]
The symmetry/structure group of quantum mechanics

\[ P \]  
projective space

\[ p: P \times P \rightarrow [0, 1] \]  
transition probability function

Fix a linearization \( P \overset{\cong}{\rightarrow} PH \); then the group \( \text{Aut}(P, p) \) of maps \( P \rightarrow P \) preserving \( p \) is the isometry group of the Fubini-Study metric \( d: PH \times PH \rightarrow \mathbb{R}_{\geq 0} \) \( \cos(d) = 2p - 1 \)

**Example:** \( \dim \mathcal{H} = 2, P = \mathbb{CP}^1 \approx S^2 \) (round metric), \( \text{Aut}(P, p) = O_3 \)

\[ T \rightarrow U_2 \rightarrow SO_3 \]

\[ T \rightarrow Q_2 \rightarrow O_3 = PQ_2 \]

**Theorem (von Neumann-Wigner):** The group \( PQ \) of projective QM symmetries fits into a group extension \( T \rightarrow Q \rightarrow PQ \), where \( Q = \text{group of unitaries and antiunitaries} \)
The symmetry/structure group of quantum mechanics

\[ \mathbb{P} \] projective space

\[ p: \mathbb{P} \times \mathbb{P} \to [0, 1] \] transition probability function

Fix a linearization \( \mathbb{P} \cong \mathbb{P}\mathcal{H} \); then the group \( \text{Aut}(\mathbb{P}, p) \) of maps \( \mathbb{P} \to \mathbb{P} \) preserving \( p \) is the isometry group of the Fubini-Study metric \( d: \mathbb{P}\mathcal{H} \times \mathbb{P}\mathcal{H} \to \mathbb{R}_{\geq 0} \) \( \cos(d) = 2p - 1 \)

**Example:** \( \dim \mathcal{H} = 2, \mathbb{P} = \mathbb{C}P^1 \cong S^2 \) (round metric), \( \text{Aut}(\mathbb{P}, p) = \text{O}_3 \)

\[
\begin{align*}
T & \to U_2 \to \text{SO}_3 \\
T & \to Q_2 \to \text{O}_3 = \text{PQ}_2
\end{align*}
\]

**Theorem (von Neumann-Wigner):** The group \( \text{PQ} \) of projective QM symmetries fits into a group extension \( T \to Q \to \text{PQ} \), where \( Q = \) group of unitaries and antiunitaries

Therefore, \( \text{PQ}_n \subset \mathbb{C}P^n \) or \( \text{PQ}_\infty \subset \mathbb{C}P^\infty \) is the model geometry for QM
The extension of QM symmetry groups is classified by a twisted cocycle $\alpha$. 

$\mathcal{T} \rightarrow \mathcal{Q} \rightarrow \mathcal{PQ} \xrightarrow{\alpha} \widetilde{\mathcal{B}_\mathcal{T}}$
Linearization and anomalies

\[ T \rightarrow Q \rightarrow PQ \overset{\alpha}{\rightarrow} \widetilde{B\Sigma} \]

The extension of QM symmetry groups is classified by a twisted cocycle \( \alpha \)

A family \( \mathcal{X} \rightarrow S \) of QM systems over \( S \) is specified by a principal \( PQ \)-bundle \( P \rightarrow S \)
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Associated “twisted gerbe” over \( S \) is the anomaly—obstruction to a linearization—which is a lift to a principal \( \mathbb{Q} \)-bundle over \( S \). Isomorphism class of projectivity lies in “\( H^2(S; \widetilde{\mathbb{T}}) \)”
Linearization and anomalies

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Linearizations, if they exist, are a “categorical torsor” (gerbe) over principal \( \tilde{\mathbb{T}} \)-bundles
The extension of QM symmetry groups is classified by a twisted cocycle $\alpha$.

A family $\mathcal{X} \to S$ of QM systems over $S$ is specified by a principal $PQ$-bundle $P \to S$.

Associated “twisted gerbe” over $S$ is the anomaly—obstruction to a linearization—which is a lift to a principal $Q$-bundle over $S$. Isomorphism class of projectivity lies in “$H^2(S; \tilde{T})$”.

Linearizations, if they exist, are a “categorical torsor” (gerbe) over principal $\tilde{T}$-bundles.

For $S = */G$ (single QM system with $G$-symmetry), reduce to group extension discussion.
Outline

• Projective spaces, linearization, and symmetry

• Quantum mechanics as a projective system

• Quantum field theory as a projective system

• Invertible field theories

• Anomalies as an obstruction to quantization

• Anomaly of a spinor field
Wick-rotated QFT as a linear representation

Graeme Segal (mid 1980’s): Wick-rotated QFT is a representation of a bordism category
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There are two “discrete parameters” that specify the species of bordism category: $n, \mathcal{F}$
Wick-rotated QFT as a linear representation

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$n$ is the dimension of “spacetime”
Wick-rotated QFT as a linear representation

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There are two “discrete parameters” that specify the species of bordism category: \( n, \mathcal{F} \)

\( n \) is the dimension of “spacetime”

\( \text{Man}_n \) category of smooth \( n \)-manifolds and local diffeomorphisms

\( \text{sSet} \) category of simplicial sets

**Definition:** A *Wick-rotated field* is a sheaf

\[
\mathcal{F}: \text{Man}_n^{\text{op}} \rightarrow \text{sSet}
\]

**Examples:** Riemannian metrics, \( G \)-connections, \( \mathbb{R} \)-valued functions, \( M \)-valued functions, orientations, spin structures, gerbes, …

\( \mathcal{F} \) can be a *collection* of fields; \( \mathcal{F}(M) \) is the simplicial set of fields on an \( n \)-manifold \( M \)
Axiom System: $\text{Bord}_n(\mathcal{F})$ bordism category

- $n$ dimension of spacetime
- $\mathcal{F}$ background fields (orientation, Riemannian metric, ...)

$X: \gamma_1 \cup \gamma_2 \cup \gamma_3 \rightarrow \varphi^{n-1}$
**Axiom System:** \( \text{Bord}_n(\mathcal{F}) \) bordism category

- \( n \) dimension of spacetime
- \( \mathcal{F} \) background fields (orientation, Riemannian metric, \ldots )

\( \text{Vect} \) linear category of topological vector spaces and linear maps
**Axiom System:** \( \text{Bord}_n(\mathcal{F}) \) bordism category

- \( n \) dimension of spacetime
- \( \mathcal{F} \) background fields (orientation, Riemannian metric, \ldots)

**Vect** linear category of topological vector spaces and linear maps

\( F : \text{Bord}_n(\mathcal{F}) \longrightarrow \text{Vect} \) linear representation of bordism category

\[ F(Y) = \left( F(Y_1) \otimes F(Y_2) \otimes F(Y_3) \rightarrow \mathbb{C} \right) \]
**Axiom System:** \( \text{Bord}_n(\mathcal{F}) \)  bordism category

\( n \) dimension of spacetime

\( \mathcal{F} \) background fields (orientation, Riemannian metric, . . .

\( \text{Vect} \) linear category of topological vector spaces and linear maps

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Fully local for *topological* theories; full locality in principle for general theories
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Fully local for *topological* theories; full locality in principle for general theories

Unitarity is encoded via an additional reflection positivity structure

**Kontsevich-Segal:** recent paper with these axioms for *nontopological* theories

geometric form of Wick rotation via admissible complex metrics

theorem constructing theory on globally hyperbolic Lorentz manifolds
Wick-rotated QFT as a projective representation; the anomaly

<table>
<thead>
<tr>
<th>Proj</th>
<th>category of “(holomorphic) projective spaces and holomorphic maps”</th>
</tr>
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Wick-rotated QFT as a projective representation; the anomaly

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\end{align*}

\[ \text{Line} \rightarrow \text{Vect} \rightarrow \text{Proj} \]
Wick-rotated QFT as a projective representation; the anomaly

**Proj** category of (holomorphic) projective spaces and holomorphic maps

**Vect** category of topological vector spaces and linear maps

**Line** category of complex lines and invertible linear maps

\[
\text{Line} \longrightarrow \text{Vect} \longrightarrow \text{Proj} \longrightarrow \Sigma(\text{Line})
\]
Wick-rotated QFT as a projective representation; the anomaly

**Proj** category of (holomorphic) projective spaces and holomorphic maps

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![Diagram](attachment:image.png)

Projective theory $\overline{F}$
Wick-rotated QFT as a projective representation; the anomaly

Proj category of (holomorphic) projective spaces and holomorphic maps
 Vect category of topological vector spaces and linear maps
 Line category of complex lines and invertible linear maps

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\text{Line} \rightarrow \text{Vect} \rightarrow \text{Proj} \rightarrow \Sigma(\text{Line})
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Projective theory $\overline{F}$
Its anomaly = projectivity $\alpha$
Wick-rotated QFT as a projective representation; the anomaly

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\[
\begin{align*}
\text{Line} & \longrightarrow \text{Vect} \longrightarrow \text{Proj} \longrightarrow \Sigma(\text{Line}) \\
\text{Line} & \longrightarrow \widetilde{\text{Bord}}_n(\mathcal{F}) \longrightarrow \text{Bord}_n(\mathcal{F}) \\
\end{align*}
\]

Projective theory $\overline{F}$

Its anomaly = projectivity $\alpha$ and resulting extension of the bordism category
Wick-rotated QFT as a projective representation; the anomaly

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\text{Proj} & \quad \text{category of (holomorphic) projective spaces and holomorphic maps} \\
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Projective theory $\overline{F}$

Its anomaly = projectivity $\alpha$ and resulting extension of the bordism category

Trivialization of $\alpha$ = linearization of $\overline{F}$ to $F$
Wick-rotated QFT as a projective representation; the anomaly

Projective theory $\overline{F}$

- Its anomaly = projectivity $\alpha$ and resulting extension of the bordism category
- Trivialization of $\alpha$ = linearization of $\overline{F}$ to $F$
- Ratio of trivializations: an invertible $n$-dimensional theory
Segal: 1980s paper on 2d conformal field theory

\[
\begin{align*}
\text{Line} & \longrightarrow \text{Vect} \longrightarrow \text{Proj} \\
\text{Line} & \longrightarrow \text{Bord}_n(\mathcal{F}) \longrightarrow \text{Bord}_n(\mathcal{F})
\end{align*}
\]

For any modular functor $E$ we have a map $E(X) @ E(Y) \rightarrow E(X \circ Y)$ when $X$ and $Y$ are composable morphisms in $\mathcal{C}$ with their boundaries compatibly labelled. So $E$ defines an extension $\mathcal{C}^E$ of the category $\mathcal{C}$. An object of $\mathcal{C}^E$ is a collection of circles each with a label from $\Phi$, and a morphism is a pair $(X, \epsilon)$, where $X$ is an morphism in $\mathcal{C}$ and $\epsilon \in E(X)$.

Definition (5.2). A weakly conformal field theory is a representation of $\mathcal{C}^E$ for some modular functor $E$, satisfying conditions as in (4.4).
Anomaly as an invertible field theory

$\Sigma$(Line) is a groupoid of gerbes, a categorification of Line
Anomaly as an invertible field theory

\[ \Sigma(\text{Line}) \]

\[ \alpha \]

\[ \text{Bord}_n(\mathcal{F}) \]

\( \Sigma(\text{Line}) \) is a groupoid of gerbes, a categorification of Line

The anomaly theory \( \alpha \) is a once-categorified \( n \)-dimensional invertible field theory
Anomaly as an invertible field theory

\[
\begin{array}{ccccc}
\text{Line} & \longrightarrow & \text{Vect} & \longrightarrow & \text{Proj} & \longrightarrow & \Sigma(\text{Line}) \\
\mid & & \downarrow \tilde{F} & & \downarrow F & & \alpha \\
\text{Line} & \longrightarrow & \text{Bord}_n(\mathcal{F}) & \longrightarrow & \text{Bord}_n(\mathcal{F}) \\
\end{array}
\]

\(\Sigma(\text{Line})\) is a groupoid of gerbes, a categorification of \(\text{Line}\).

The anomaly theory \(\alpha\) is a once-categorified \(n\)-dimensional invertible field theory.

An \(n\)-dimensional \textbf{theory} \(\tilde{F}\) \textit{relative to} \(\alpha\) assigns \(\tilde{F}(X^n): \mathbb{C} \longrightarrow \alpha(X^n)\) for \(X^n\) closed.

(Note: \textit{Relative} field theories are called \textit{twisted} theories by Stolz-Teichner.)


Anomaly as an invertible field theory

![Diagram of Line → Vect → Proj → Σ(Line)

\[\text{Line} \xrightarrow{\text{Line}} \text{Vect} \xrightarrow{\text{Proj}} \text{Proj} \xrightarrow{\Sigma(\text{Line})} \]

\[\tilde{F} \quad F \quad \alpha\]

\[\text{Line} \xrightarrow{\text{Bord}_n(\mathcal{F})} \text{Bord}_n(\mathcal{F})\]

\[\Sigma(\text{Line})\] is a groupoid of gerbes, a categorification of Line

The anomaly theory \(\alpha\) is a once-categorified \(n\)-dimensional invertible field theory

An \(n\)-dimensional theory \(\overline{F}\) relative to \(\alpha\) assigns \(\overline{F}(X^n): \mathbb{C} \rightarrow \alpha(X^n)\) for \(X^n\) closed

To \(Y^{n-1}\) closed, \(\overline{F}\) assigns a projective space with projectivity \(\alpha(Y^{n-1})\)
Anomaly as an invertible field theory

\[ \begin{array}{cccc}
\text{Line} & \rightarrow & \text{Vect} & \rightarrow & \text{Proj} & \rightarrow & \Sigma(\text{Line}) \\
\| & \downarrow & \| & \downarrow & \| & \downarrow & \\
\text{Line} & \rightarrow & \widetilde{\text{Bord}}_n(\mathcal{F}) & \rightarrow & \text{Bord}_n(\mathcal{F})
\end{array} \]

\( \Sigma(\text{Line}) \) is a groupoid of gerbes, a categorification of \( \text{Line} \)

The \textbf{anomaly theory} \( \alpha \) is a \textit{once-categorified} \( n \)-dimensional invertible field theory

An \( n \)-dimensional \[ \text{theory } \overline{F} \text{ relative to } \alpha \] assigns \( \overline{F}(X^n): \mathbb{C} \rightarrow \alpha(X^n) \) for \( X^n \) closed

To \( Y^{n-1} \) closed, \( \overline{F} \) assigns a projective space with projectivity \( \alpha(Y^{n-1}) \)

\textbf{Ratios} of trivializations of \( \alpha \): a standard type of \( n \)-dimensional invertible theory
Extension of anomaly theory; relative theory $\rightarrow$ boundary theory

\[
\Sigma(\text{Line}) \xrightarrow{\alpha} \lambda \xrightarrow{\tilde{\alpha}} \text{Bord}_n(\mathcal{F}) \xrightarrow{} \text{Bord}_{n+1}(\tilde{\mathcal{F}})
\]

In many cases the once-categorified $n$-dimensional anomaly theory $\alpha$ has an extension to an $(n + 1)$-dimensional theory $\tilde{\alpha}$.
Extension of anomaly theory; relative theory $\longrightarrow$ boundary theory

In many cases the once-categorified $n$-dimensional anomaly theory $\alpha$ has an extension to an $(n + 1)$-dimensional theory $\tilde{\alpha}$.

In that case a theory relative to $\alpha$ is promoted to a boundary theory for $\tilde{\alpha}$.
Extension of anomaly theory; relative theory \(\longrightarrow\) boundary theory

\[
\Sigma(\text{Line}) \xrightarrow{\alpha} \Bord_n(F) \xrightarrow{\tilde{\alpha}} \Bord_{n+1}(\tilde{F})
\]

In many cases the once-categorified \(n\)-dimensional anomaly theory \(\alpha\) has an extension to an \((n + 1)\)-dimensional theory \(\tilde{\alpha}\).

In that case a theory \emph{relative} to \(\alpha\) is promoted to a \emph{boundary} theory for \(\tilde{\alpha}\).

The extended anomaly theory \(\tilde{\alpha}\) assigns a nonzero number to a closed \((n + 1)\)-manifold which, though not part of an \(n\)-dimensional anomalous theory, is a useful quantity.
Extension of anomaly theory; relative theory $\longrightarrow$ boundary theory

\[
\begin{align*}
\Sigma(\text{Line}) & \\
\begin{array}{c}
\alpha \\
\downarrow \tilde{\alpha}
\end{array} & \\
\text{Bord}_n(\mathcal{F}) & \longrightarrow \text{Bord}_{n+1}(\tilde{\mathcal{F}})
\end{align*}
\]

In many cases the once-categorified $n$-dimensional anomaly theory $\alpha$ has an extension to an $(n+1)$-dimensional theory $\tilde{\alpha}$.

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The extended anomaly theory $\tilde{\alpha}$ assigns a nonzero number to a closed $(n+1)$-manifold which, though not part of an $n$-dimensional anomalous theory, is a useful quantity.

Anomaly theories $\alpha, \tilde{\alpha}$ are not in general topological; if so, topological tools are available.
Outline

- Projective spaces, linearization, and symmetry
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- Anomaly of a spinor field
Preliminary: differential cohomology

\( h^\bullet \) cohomology theory (on CW complexes)

\( \tilde{h}^\bullet \to h^\bullet \) differential refinement (on smooth manifolds)
Preliminary: differential cohomology

\[ h^\bullet \] cohomology theory (on CW complexes)

\[ \hat{h}^\bullet \longrightarrow h^\bullet \] differential refinement (on smooth manifolds)

\[ \overline{H\mathbb{Z}}^1(M) \longrightarrow H\mathbb{Z}^1(M) = H^1(M; \mathbb{Z}) \]

\[ \{\phi: M \to \mathbb{R}/\mathbb{Z}\} \quad \{\phi: M \to \mathbb{R}/\mathbb{Z}\} / \text{homotopy} \]
Preliminary: differential cohomology

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\[
\begin{align*}
\tilde{H}\mathbb{Z}^1(M) & \to H\mathbb{Z}^1(M) = H^1(M; \mathbb{Z}) \\
\{\phi: M \to \mathbb{R}/\mathbb{Z}\} & \sim \{\phi: M \to \mathbb{R}/\mathbb{Z}\} / \text{homotopy}
\end{align*}
\]

\[
\begin{align*}
\tilde{H}\mathbb{Z}^2(M) & \to H\mathbb{Z}^2(M) = H^2(M; \mathbb{Z}) \\
\{\mathbb{R}/\mathbb{Z}\text{-connections on } M\} & \sim \{\text{principal } \mathbb{R}/\mathbb{Z}\text{-bundles on } M\} / \sim
\end{align*}
\]
\[ \widetilde{H}_{\mathbb{Z}}^2(M) \xrightarrow{\text{curvature}} \Omega_{\text{closed}}^2(M) \]

Chern class

\[ H_{\mathbb{Z}}^2(M) \xrightarrow{} H_{\text{dR}}^2(M) \cong H_{\mathbb{R}}^2(M) \]

\[ \mathbb{R}/\mathbb{Z} \xrightarrow{} 0 \]

\[ M = S^1 : \]

\[ 0 \xrightarrow{} 0 \]
$\mathcal{H} \mathbb{Z}^2(M) \xrightarrow{\text{curvature}} \Omega^2_{\text{closed}}(M) \xrightarrow{\text{de Rham}} \mathbb{R}/\mathbb{Z} \rightarrow 0$

$\mathcal{H} \mathbb{Z}^2(M) \rightarrow H^2_{dR}(M) \cong H^2_{\mathbb{R}}(M)$

$M = S^1 :$

$0 \rightarrow 0$

$H^2_{dR}(M)$
Invertible field theories

Introduced by F-Moore, homotopical approach developed by F-Hopkins-Teleman
Invertible field theories

Introduced by F-Moore, homotopical approach developed by F-Hopkins-Teleman

Generalized differential cocycles on bordism, values in Anderson dual $IZ$ to sphere; based on ideas of Hopkins-Singer
Invertible field theories

Introduced by F-Moore, homotopical approach developed by F-Hopkins-Teleman

*Generalized* differential cocycles on bordism, values in Anderson dual $\mathbb{IZ}$ to sphere; based on ideas of Hopkins-Singer

Here is the diagram for an extended anomaly theory ($\mathcal{B}$ is a differential bordism spectrum)

$$
\begin{array}{ccc}
\mathbb{IZ}^{n+2}(\mathcal{B}) & \xrightarrow{\text{curvature}} & \Omega_{\text{closed}}^{n+2}(\mathcal{B}) \\
\Downarrow \text{deformation class} & & \Downarrow \text{“de Rham”} \\
\mathbb{IZ}^{n+2}(\mathcal{B}) & \xrightarrow{} & \mathbb{IR}^{n+2}(\mathcal{B})
\end{array}
$$

The curvature, or "anomaly polynomial", encodes the local anomaly. The deformation class is accessible via homotopical methods.
Invertible field theories

Introduced by F-Moore, homotopical approach developed by F-Hopkins-Teleman

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Here is the diagram for an extended anomaly theory ($\mathcal{B}$ is a differential bordism spectrum)

\[ \tilde{\mathcal{IZ}}^{n+2}(\mathcal{B}) \xrightarrow{\text{curvature}} \Omega_{\text{closed}}^{n+2}(\mathcal{B}) \]

\[ \text{deformation class} \quad \text{"de Rham"} \]

\[ \mathcal{IZ}^{n+2}(\mathcal{B}) \xrightarrow{\text{curvature}} \mathcal{IR}^{n+2}(\mathcal{B}) \]

The curvature, or "anomaly polynomial", encodes the *local anomaly*

The deformation class is accessible via homotopical methods
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Quantum theory is projective. Quantization is linear.

\[ \pi : \mathcal{F} \rightarrow \overline{\mathcal{F}} \]

- fiber bundle of collection of fields
- fluctuating fields

\[ \overline{\mathcal{F}} \]

- background fields

\[
\text{quantization: passage from a theory } \mathcal{F} \text{ on } \mathcal{F} \text{ to a theory } \mathcal{F} \text{ on } \mathcal{F} \text{ via integration over } \pi
\]
Quantum theory is projective. Quantization is linear.

\[ \pi : \mathcal{F} \longrightarrow \overline{\mathcal{F}} \quad \text{fiber bundle of collection of fields} \]

fibers of \( \pi \) fluctuating fields

\( \mathcal{F} \) background fields

Quantization: passage from a theory \( F \) on \( \mathcal{F} \) to a theory \( \overline{F} \) on \( \overline{\mathcal{F}} \) via integration over \( \pi \)
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- Fibers of \( \pi \): fluctuating fields
- \( \overline{\mathcal{F}} \): background fields

Quantization: passage from a theory \( F \) on \( \mathcal{F} \) to a theory \( \overline{F} \) on \( \overline{\mathcal{F}} \) via integration over \( \pi \)

Closed \( n \)-manifold \( X \): Feynman path integral
Quantum theory is projective. Quantization is linear.

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fiber bundle of collection of fields

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Closed \( n \)-manifold \( X \): Feynman path integral

Closed \((n - 1)\)-manifold \( Y \): canonical quantization

\[ \pi \left( \mathcal{F} \right) \]

\[ \overline{\mathcal{F}} \left( Y \right) \]
Quantum theory is projective. Quantization is linear.

\[ \pi : \mathcal{F} \longrightarrow \overline{\mathcal{F}} \quad \text{fiber bundle of collection of fields} \]

fibers of \( \pi \) fluctuating fields

\( \overline{\mathcal{F}} \) background fields

Quantization: passage from a theory \( F \) on \( \mathcal{F} \) to a theory \( \overline{F} \) on \( \overline{\mathcal{F}} \) via integration over \( \pi \)

Closed \( n \)-manifold \( X \): Feynman path integral

Closed \( (n - 1) \)-manifold \( Y \): canonical quantization

To carry out quantization we must \textit{descend} the projectivity/anomaly \( \alpha \):

\[ \text{Bord}_n(\mathcal{F}) \quad \xymatrix{ \alpha \ar[d] \ar[r]^-\alpha & \Sigma^{n+1} \mathbb{C}^\times \ar[d] \text{ anomaly is obstruction to existence} \ar[d] \ar[r]^-\alpha & \text{descents form a torsor over } n \text{-dimensional theories} \text{ Bord}_n(\overline{\mathcal{F}}) } \]
Anomalies: summary

- Quantum theory is projective—the ’t Hooft anomaly is the projectivity
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- Quantization is linear—the anomaly obstructs quantization
Anomalies: summary

- Quantum theory is projective—the ’t Hooft anomaly is the projectivity.
- Quantization is linear—the anomaly obstructs quantization.
- If the obstruction vanishes, one must specify descent data, which is a torsor over an abelian group of invertible field theories.
Anomalies: summary

- Quantum theory is projective—the ’t Hooft anomaly is the projectivity
- Quantization is linear—the anomaly obstructs quantization
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Anomalies: summary

- Quantum theory is projective—the ’t Hooft anomaly is the projectivity
- Quantization is linear—the anomaly obstructs quantization
- If the obstruction vanishes, one must specify descent data, which is a torsor over an abelian group of invertible field theories
- There is a well-developed theory of invertible field theories, so the projectivity of quantum field theory is accessible using geometric and topological tools
- The anomaly of a QFT is itself a field theory, so obeys locality and, typically, unitarity
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Free spinor field data on $\mathbb{M}^n$

$\mathbb{M}^n$

$C \subseteq \mathbb{R}^{1,n-1}$

$\text{Spin}_{1,n-1} \subseteq \text{Cliff}^0_{n-1,1}$

Minkowski spacetime (affine space, Lorentz metric) component of timelike vectors (time-orientation) Lorentz group
Free spinor field data on $\mathbb{M}^n$

$\mathbb{M}^n$  
$C \subset \mathbb{R}^{1,n-1}$  
$\text{Spin}_{1,n-1} \subset \text{Cliff}_{n-1,1}$  
$\mathcal{S}$  
$\Gamma: \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{R}^{1,n-1}$  
$m: \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{R}$

Minkowski spacetime (affine space, Lorentz metric)  
component of timelike vectors (time-orientation)  
Lorentz group  
real (ungraded) $\text{Cliff}_{n-1,1}$-module  
symmetric Spin$_{1,n-1}$-invariant form; $\Gamma(s, s) \in \mathcal{C}$ for all $s \in \mathcal{S}$  
skew-symmetric Spin$_{1,n-1}$-invariant (mass) form
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$C \subset \mathbb{R}^{1,n-1}$

$\text{Spin}_{1,n-1} \subset \text{Cliff}_{n-1,1}^0$

$\mathcal{S}$

$\Gamma : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{R}^{1,n-1}$

$m : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{R}$

- If $\mathcal{S}$ is irreducible, $\Gamma$ exists and is unique up to scale

Minkowski spacetime (affine space, Lorentz metric)

component of timelike vectors (time-orientation)

Lorentz group

real (ungraded) $\text{Cliff}_{n-1,1}^0$-module

symmetric $\text{Spin}_{1,n-1}$-invariant form; $\Gamma(s,s) \in \overline{C}$ for all $s \in \mathcal{S}$

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- If $\mathbb{S}$ is irreducible, $\Gamma$ exists and is unique up to scale
- Given a pairing $\Gamma$ there is a unique compatible $\text{Cliff}_{n-1,1}$-module structure on $\mathbb{S} \oplus \mathbb{S}^*$
Free spinor field data on $\mathbb{M}^n$

- $\mathbb{M}^n$ \hspace{1cm} Minkowski spacetime (affine space, Lorentz metric)
- $C \subset \mathbb{R}^{1,n-1}$ \hspace{1cm} component of timelike vectors (time-orientation)
- $\text{Spin}_{1,n-1} \subset \text{Cliff}_{n-1,1}$ \hspace{1cm} Lorentz group
- $\mathbb{S}$ \hspace{1cm} real (ungraded) $\text{Cliff}_{n-1,1}$-module
- $\Gamma : \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{R}^{1,n-1}$ \hspace{1cm} symmetric $\text{Spin}_{1,n-1}$-invariant form; $\Gamma(s,s) \in C$ for all $s \in \mathbb{S}$
- $m : \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{R}$ \hspace{1cm} skew-symmetric $\text{Spin}_{1,n-1}$-invariant (mass) form

- If $\mathbb{S}$ is irreducible, $\Gamma$ exists and is unique up to scale
- Given a pairing $\Gamma$ there is a unique compatible $\text{Cliff}_{n-1,1}$-module structure on $\mathbb{S} \oplus \mathbb{S}^*$
- Every finite dimensional $\text{Cliff}_{n-1,1}$-module is of this form
Free spinor field data on $\mathcal{M}^n$

$\mathcal{M}^n$

$C \subset \mathbb{R}^{1,n-1}$

$\text{Spin}_{1,n-1} \subset \text{Cliff}^0_{n-1,1}$

$\mathcal{S}$

$\Gamma : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{R}^{1,n-1}$

$m : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{R}$

Minkowski spacetime (affine space, Lorentz metric)

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Lemma (F–Hopkins): Nondegenerate mass terms for $\mathcal{S} \leftrightarrow \text{Cliff}_{n-1,2}$-module structures on $\mathcal{S} \oplus \mathcal{S}^*$ that extend the $\text{Cliff}_{n-1,1}$-module structure
Problem: For $(S, \Gamma)$ (with $m = 0$), deduce the $(n + 1)$-dimensional anomaly theory $\alpha_{(S, \Gamma)}$. 

The "curvature" of the theory (local anomaly) is a degree $p_n$ differential form on $\text{Riem}$, a component of the Chern-Weil form for $\hat{A}$; it vanishes if $n \equiv 2 \pmod{4}$, in which case $p_S, q$ is a topological theory; it factors through $F = \text{Spin}$. Let $M_{p_Sq}$ denote the vector space of mass pairings. (It may be the zero vector space.) We can take $F = \text{Riem} \hat{\text{Spin}} \hat{M}_{p_Sq}$ and deduce the anomaly; see arXiv:1905.09315 with C'ordova-Lam-Seiberg. 

There is a related formula for the low energy effective theory in case $M_{p_Sq}$ is nondegenerate; see arXiv:1604.06527 with Hopkins.
Problem: For $(S, \Gamma)$ (with $m = 0$), deduce the $(n + 1)$-dimensional anomaly theory $\alpha_{(S, \Gamma)}$

- $\alpha_{(S, \Gamma)}$ is an invertible field theory with $\mathcal{F} = \text{Riem} \times \text{Spin}$
- We implicitly take a universal target for invertible field theories
**Problem:** For $(S, \Gamma)$ (with $m = 0$), deduce the $(n + 1)$-dimensional anomaly theory $\alpha_{(S, \Gamma)}$

- $\alpha_{(S, \Gamma)}$ is an invertible field theory with $\mathcal{F} = \text{Riem} \times \text{Spin}$
- We implicitly take a universal target for invertible field theories
- The “curvature” of the theory (local anomaly) is a degree $(n + 2)$ differential form on $\text{Riem}$, a component of the Chern-Weil form for $\hat{A}$; it vanishes if $n \not\equiv 2 \pmod{4}$, in which case $\alpha_{(S, \Gamma)}$ is a *topological* theory; it factors through $\mathcal{F} = \text{Spin}$
**Problem:** For \((\mathcal{S}, \Gamma)\) (with \(m = 0\)), deduce the \((n + 1)\)-dimensional anomaly theory \(\alpha(\mathcal{S}, \Gamma)\)

- \(\alpha(\mathcal{S}, \Gamma)\) is an invertible field theory with \(\mathcal{F} = \text{Riem} \times \text{Spin}\)
- We implicitly take a universal target for invertible field theories
- The “curvature” of the theory (local anomaly) is a degree \((n + 2)\) differential form on \(\text{Riem}\), a component of the Chern-Weil form for \(\mathring{A}\); it vanishes if \(n \not\equiv 2 \pmod{4}\), in which case \(\alpha(\mathcal{S}, \Gamma)\) is a *topological* theory; it factors through \(\mathcal{F} = \text{Spin}\)
- Let \(\mathcal{M}(\mathcal{S})\) denote the vector space of mass pairings. (It may be the zero vector space.) We can take \(\mathcal{F} = \text{Riem} \times \text{Spin} \times \mathcal{M}(\mathcal{S})\) and deduce the anomaly; see arXiv:1905.09315 with Córdova-Lam-Seiberg
Free fermion anomaly theory (F–Hopkins)

\[ S \]

real (ungraded) \( \text{Cliff}^0_{n-1,1} \)-module

\[ \Gamma : S \times S \rightarrow \mathbb{R}^{1,n-1} \]
symmetric \( \text{Spin}_{1,n-1} \)-invariant form; \( \Gamma(s, s) \in \overline{C} \) for all \( s \in S \)

**Lemma:** Nondegenerate mass terms for \( S \leftrightarrow \text{Cliff}_{n-1,2} \)-module structures on \( S \oplus S^* \) that extend the \( \text{Cliff}_{n-1,1} \)-module structure
Free fermion anomaly theory (F–Hopkins)

$S$  
real (ungraded) $\text{Cliff}_{n-1,1}^0$-module

$\Gamma : S \times S \to \mathbb{R}^{1, n-1}$  
symmetric $\text{Spin}_{1,n-1}$-invariant form; $\Gamma(s, s) \in \overline{C}$ for all $s \in S$

**Lemma:** Nondegenerate mass terms for $S \leftrightarrow \text{Cliff}_{n-1,2}$-module structures on $S \oplus S^*$ that extend the $\text{Cliff}_{n-1,1}$-module structure

$[S] \in \pi_{2-n}KO \cong [S^0, \Sigma^{n-2}KO]$  
(Atiyah–Bott–Shapiro)

$\Rightarrow$
Free fermion anomaly theory (F–Hopkins)

$\mathbb{S}$, real (ungraded) $\text{Cliff}^0_{n-1,1}$-module

$\Gamma: \mathbb{S} \times \mathbb{S} \to \mathbb{R}^{1,n-1}$, symmetric $\text{Spin}_{1,n-1}$-invariant form; $\Gamma(s,s) \in \overline{C}$ for all $s \in \mathbb{S}$

**Lemma:** Nondegenerate mass terms for $\mathbb{S} \leftrightarrow \text{Cliff}_{n-1,2}$-module structures on $\mathbb{S} \oplus \mathbb{S}^*$ that extend the $\text{Cliff}_{n-1,1}$-module structure

$[\mathbb{S}] \in \pi_{2-n}KO \cong [S^0, \Sigma^{n-2}KO]$ (Atiyah–Bott–Shapiro)

**Claim:** The isomorphism class of $\alpha(\mathbb{S}, \Gamma)$ is the differential lift of the composition

$$MS\text{Spin} \xrightarrow{\phi \wedge [\mathbb{S}]} KO \wedge \Sigma^{n-2}KO \xrightarrow{\mu} \Sigma^{n-2}KO \xrightarrow{\text{Pfaff}} \Sigma^{n+2}IZ$$
Free fermion anomaly theory (F–Hopkins)

\[ S \quad \text{real (ungraded) } \text{Cliff}_{n-1,1} \text{-module} \]

\[ \Gamma : S \times S \longrightarrow \mathbb{R}^{1,n-1} \quad \text{symmetric } \text{Spin}_{1,n-1} \text{-invariant form; } \Gamma(s,s) \in \overline{C} \text{ for all } s \in S \]

**Lemma:** Nondegenerate mass terms for \( S \leftrightarrow \text{Cliff}_{n-1,2} \)-module structures on \( S \oplus S^* \) that extend the \( \text{Cliff}_{n-1,1} \)-module structure

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**Claim:** The isomorphism class of \( \alpha(S,\Gamma) \) is the differential lift of the composition

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Partition function on a Riemannian spin \((n + 1)\)-manifold is an exponentiated \( \eta \)-invariant