Lectures on Field Theory and Topology

Daniel S. Freed

To {Alex, David, Sonia},
with love and gratitude

Contents

Preface

In early August of 2017 David Ayala and Ryan Grady organized a CBMS conference on *Topological and Geometric Methods in Quantum Field Theory* at Montana State University in Bozeman. I gave a series of ten lectures which form the basis for this volume. There were supplementary lectures by several other mathematicians and physicists. Some of those are collected in a companion volume [**AFG**]. Many students and postdocs attended, as did several more senior mathematicians. The beautiful natural setting and relaxed atmosphere were perfect catalysts for interactions among the participants.

I thank David and Ryan for their outstanding work bringing the conference together. Agnes Beaudry, Robert Bryant, Tudor Dimofte, Sam Gunningham, Max Metlitski, Dave Morrison, and Andy Neitzke gave terrific lectures at the conference. Beth Burroughs, Jane Crawford, and Katie Sutich in the Department of Mathematical Sciences at MSU provided massive assistance during the planning and execution of the conference. The College of Letters and Science and the VP of Research at Montana State University provided support for the CBMS conference as well. The Aspen Center for Physics provided me hospitality during the period when I wrote the first draft of the lectures. I thank them all.

Dan Freed
December, 2018

Introduction

The motivating problem for these lectures comes from condensed matter physics:

$$(\star) \qquad \text{Classify invertible gapped phases of matter}$$

Phases of matter are familiar in everyday life. Ice, water, and vapor are different forms of H_2O, more different from each other than, say, water at 1°C and water at 99°C. If we consider "forms of H_2O" as a function of temperature, then there are two special temperatures—0°C and 100°C—at which there is a transition, from solid to liquid and from liquid to vapor. The real line of temperatures minus those two transition points has three path components: the three phases of solid, liquid, and vapor. In other words, forms of H_2O connected continuously by a path are considered to be in the same phase. So far we have only introduced one parameter—temperature—with pressure fixed to be the one you are experiencing right now. If we also allow pressure to vary, then there are only two phases of H_2O.

Our task is to (1) build a mathematical model of the physics problem, (2) solve the mathematics questions which arise in the model, and (3) apply the solution back to physics. This is the classic Three-Step Procedure of external applications of mathematics. From the example of H_2O we see the outlines of a mathematical framework: there is a space $\mathcal{M}$ of "systems" with a "singular" locus $\Delta \subset \mathcal{M}$, and we are interested in $\pi_0(\mathcal{M}\backslash\Delta)$. But what are the "systems"? How do we construct the space $\mathcal{M}$ which parametrizes them? And which systems are "singular"? What is clear is that problem $(\star)$ is topological in nature. For example, the answer only depends on the homotopy type of $\mathcal{M}\backslash\Delta$, so our solution need only construct a homotopy type, not a more precise algebraic or smooth space.

There are many physical models of a given "system" in nature. They roughly fall into two boxes: discrete and continuous. Discrete models include various types of lattice models (discrete space, continuous time) and statistical mechanics models (discrete space, discrete time); they are prevalent in condensed matter physics. There is an extensive mathematical literature on these systems, but not as far as we know general definitions which apply directly to the problem at hand. (Nonetheless, Alexei Kitaev has made great strides in this direction.) Field theories, such as those of Maxwell and Einstein, are continuous models of nature. There are general physical principles which guide passage between discrete and continuous models. Our mathematical model of 'phases of matter' is grounded in field theory, and more specifically in an Axiom System introduced by Graeme Segal. This structural vision of field theory is the starting point for constructing a space $\mathcal{M}$ which parametrizes mathematical objects that in turn model some physical reality. The model retains minimal information—the long range physics—which on the one hand is robust enough to determine the phase and on the other hand is flabby enough to be amenable to topological techniques.

These lectures are based on a joint paper [**FH1**] with Mike Hopkins. In these lectures we offer complementary and supplementary background, motivation, and results; we leave out several detailed proofs which are in [**FH1**]. In this extended introduction we outline the lectures which make up this volume. The reader may wish to flip back and forth between this broad idea sketch and the more detailed lectures which follow.

Moduli spaces and deformation classes

Problems analogous to ($\star$) are familiar in mathematics. As a simple example, fix a positive integer n and let $\mathcal{M}_n$ be the space of configurations of n points on the real line $\mathbb{R}$. The position of the i^{th} point is a function $x^i\colon \mathcal{M}_n \to \mathbb{R}$. Together the positions define an isomorphism $(x^1,\dots,x^n)\colon \mathcal{M}_n \xrightarrow{\cong} \mathbb{R}^n$. We say $\mathcal{M}_n$ is the *moduli space* for this problem. There are natural "questions" which take the form of functions on $\mathcal{M}_n$. 'What is the distance between the 1^{st} and 3^{rd} points?' is the function $|x^1 - x^3|$. So far there is no interesting topology: $\mathcal{M}_n$ is contractible. Let $\Delta \subset \mathcal{M}_n$ be the locus of n-tuples $x = (x^1,\dots,x^n)$ in which not all x^i are distinct—the union of all diagonals. Configurations in $\mathcal{M}_n\backslash\Delta$ satisfy a "gap condition", and now there is nontrivial topology: $\mathcal{M}_n\backslash\Delta$ has $n!$ contractible components. A gapped configuration $x \in \mathcal{M}_n\backslash\Delta$ determines a permutation $\sigma(x) \in \mathrm{Sym}_n$, and the permutation is a complete invariant of the path component, or *deformation class*. In other words, $\sigma\colon \pi_0(\mathcal{M}_n\backslash\Delta) \to \mathrm{Sym}_n$ is an isomorphism.

It is worth pausing to contemplate the sophisticated mathematical theory underlying this example. Nowadays we confidently write '$\mathbb{R}$' because we have in hand a rich theory of real numbers. Historically it was not always so. Only after many hard-fought struggles and contradictions could victory be declared in the form of the three-word characterization of the real numbers: *complete ordered field*. That profound hard-won phrase is the starting point of every undergraduate real analysis class. The order on the real numbers underlies the isomorphism σ. But do you see any infinite decimals, Cauchy sequences, or Dedekind cuts? The topological problem 'Compute $\pi_0(\mathcal{M}_n\backslash\Delta)$' does not require such precision, whereas geometric questions about distance between points do. The characterization of the nature of real numbers is more useful for qualitative questions than any detailed construction would be.

This toy problem exhibits several features common to moduli problems. First, there are discrete parameters, here the positive integer n. Second, there is a singular locus Δ; off of Δ the parametrized objects satisfy a nonsingularity condition. Third, there are interesting functions on the moduli space which encode geometric information about the parametrized objects. Finally, there is a complete invariant of the deformation class, which is an isomorphism to a known or computable set. In this toy problem, as well as in problem ($\star$), there is a natural group structure: π_0 is a group, not merely a set. (It should be said the group structure is not obvious if we view points of $\mathcal{M}_n\backslash\Delta$ as configurations.) A known complete invariant of π_0 is not present in all situations.

As another example we might ask to parametrize "1-dimensional metric shapes". This is ill-defined as stated. Better said, it is *not* defined as stated: to make a mathematical theory we must provide a definition. Whereas the real numbers are characterized uniquely (complete ordered field), here there is no uniqueness. Still, a definition should capture general features, even though it does not characterize. We might decide that, intuitively, we do not want to allow the 1-dimensional shape to cross itself. That is a "gap condition". Once more there is a hard-won mathematical definition at hand: 1-dimensional smooth Riemannian manifold. Furthermore, we know what a family of such objects is: a smooth fiber bundle $\pi\colon X \to S$ with a Riemannian structure. (We leave the reader to ponder what the Riemannian structure is; the definition should lead to a unique Levi-Civita connection on the relative tangent bundle.) But now there is a second kind of "singularity" which is still in the game. Namely, there exists a smooth fiber bundle over $S = \mathbb{R}$ such that the fiber at $s < 0$ is diffeomorphic to a circle and the fiber at $s > 0$ is diffeomorphic to the disjoint union of two circles. The "singularity" at $s = 0$ is a noncompact fiber: two lines. To rule out the transition from one circle to two circles, restrict to proper fiber bundles. Demanding compactness is again a kind of gap condition—think spectrum of the Laplace operator. The moduli space $\mathcal{M}$ of closed Riemannian 1-manifolds has all the features enumerated in the previous paragraph. The discrete parameter is the dimension, the singular locus was already eliminated, functions such as total length are interesting geometric invariants, and the number of components is a complete invariant. There is a natural commutative monoid structure on $\pi_0\mathcal{M}$ given by disjoint union, but there are no inverses and so $\pi_0\mathcal{M}$ is not a group.

Problem ($\star$) is about families of quantum mechanical systems. There is a basic dichotomy determined by the energy spectrum: a system is *gapped* if there is a gap in the spectrum of the Hamiltonian above the minimal energy and otherwise is *gapless*. This problem only considers gapped systems. In this context the singular locus parametrizes *phase transitions*, which are bifurcated into first-order and higher-order. A phase transition occurs along a path of gapped quantum systems when the energy gap is closed—the energy spectrum comes down to the minimum. If discrete spectrum goes down the transition is first-order; if continuous spectrum, then it is higher-order. In any case we throw them all out and define two systems to be in the same phase if they can be joined by a continuous path of gapped systems with no phase transition. Problem ($\star$) includes another adjective—invertible—which we discuss below; the classification question makes sense in the absence of invertibility. Of course, there are interesting analytic questions which are expressed in terms of correlation functions, which are functions on $\mathcal{M}$, but problem ($\star$) is topological so we do not need anything so precise. The broad rough outline, then, is: construct a moduli space $\mathcal{M}$ for invertible gapped systems; throw out a locus Δ of phase transitions; compute $\pi_0(\mathcal{M}\backslash\Delta)$. In fact, our transformation to a problem in field theory obviates the need to consider Δ.

Axiom System for field theory

As stated above, we attack ($\star$) by shifting to a problem in field theory. Previously we used 'characterization' and 'definition' as monikers for the mathematical starting point, but for field theory we use 'axiom system'. Why? Certainly there is not a unique field theory, even with discrete parameters fixed, so 'characterization' is inappropriate. We shy away from 'definition', which to our ears suggests *stare decisis*—settled law—and the situation for quantum field theory is hardly that! Also, the Axiom System for field theory is perhaps more analogous to the Eilenberg-Steenrod axioms for (generalized) homology theories than, say, to the definition of a smooth Riemannian manifold. The Eilenberg-Steenrod axioms, which *do* play the formal role of a 'definition', tell what a homology theory is without constructing one. Their power lies in their simplicity. One can check whether a construction satisfies the axioms, and in this way know that highly disparate constructions yield the same mathematical object, something that was not at all apparent before the Eilenberg-Steenrod 1945 paper.

The analogy between pre-1945 algebraic topology and present-day quantum field theory is not perfect, but consider the strong commonalities: multiple starting points, constructions, and approaches. The Axiom System for field theory, introduced by Segal in the 1980s for conformal theories in two spacetime dimensions and later adapted by Atiyah for topological theories in all dimensions, is flexible. It applies to both classical and quantum theories in all dimensions. In the Axiom System a field theory is a map, and as such has a domain and codomain; part of the flexibility is the freedom to vary them. The domain and codomain are each a symmetric monoidal category, and the map is a symmetric monoidal functor. The domain is a bordism category of smooth manifolds equipped with fields. 'Field' has a precise definition[1] and includes traditional scalar fields of physics as well as topological structures (orientation, spin structure) and more exotic possibilities. The codomain for a physically relevant field theory is an appropriate category of complex topological vector spaces. This choice goes back to the early days of quantum mechanics: linearity encodes superposition and complex numbers encode interference. Ergo the Axiom System in a nutshell: *a field theory is a linear representation of a geometric bordism category.* Other choices for the codomain, such as the category of abelian groups, have proved useful in mathematical contexts. Indeed, the viewpoint of the Axiom System has proved its value many times over in *mathematics*: in low dimensional topology, symplectic geometry, geometric representation theory, category theory, etc. The story of these lectures is one small application to *physics*, and there are many more indications there of its pertinence and utility. However, it has not unified the disparate points of view on quantum field theory, and there are few rigorous examples; in non-topological contexts the Axiom System is not as established as, say, the Eilenberg-Steenrod axioms.

We explain two routes to the Axiom System in these lectures. The first route, explained in Lecture 1, is through classical bordism theory: a topological field theory is a categorification of a classical bordism invariant. The second is through quantum field theory. An essential point is that what is being axiomatized is *Wick-rotated* quantum theory—physics with purely imaginary time. In Lecture 2 we introduce the basic characters in quantum theory in the context of quantum mechanics: states, observables, and correlation functions. In fact, we give a unified picture of mechanics—classical and quantum—which goes back to early mathematical work on quantum theory. Wick rotation is straightforward in quantum mechanics, and one can already see the Axiom System emerging in this 1-dimensional case of field theory. Lecture 3 takes up relativistic quantum field theory, but only from a very structural perspective in order to explain where the Axiom Systems sits. Our starting point is definitely non-topological, so it is not surprising that the Axiom System applies to non-topological field theories. Although we invoke non-topological field

[1] which we do not include in these lectures; see [**FT1**, Appendix].

theories at many points in the subsequent lectures, the technical work is for topological field theory. The reader might mistakenly infer from our words that the Wick-rotated non-topological theories we refer to, such as Yang-Mills + Chern-Simons in three dimensions, are completely well-defined objects which have been mathematically worked out. Not so! Even the topological parts of field theory are under rapid development.

The Axiom System is, in a sense, a Wick-rotated version of the Schrödinger approach to quantum mechanics and to quantum field theory (Wightman *et al.*). That is, it emphasizes states and time-evolution of states, albeit imaginary time-evolution. By contrast, the Heisenberg approach to quantum mechanics emphasizes algebras of observables, as does that approach to quantum field theory (Haag *et al.*). There are also modern mathematical axiom systems based on the Heisenberg approach, most prominently in work of Costello and collaborators. It should also be said that the Axiom System does not distinguish classical and quantum; a classical field theory fits the formal axioms as a very special case—classical field theories are invertible. Many quantum theories have semiclassical limits in which they are described via quantization, say in terms of path integrals. Detailed descriptions in terms of fluctuating fields furnish important information about a quantum field theory, but they do not enter our approach to the classification problem ($\star$). Nevertheless, one of the original goals of the Axiom System was precisely to capture the formal properties of canonical quantization and the path integral.

Symmetries

The sample moduli problems introduced above all have discrete parameters—number of points on a line, dimension of a Riemannian manifold—and a moduli space is constructed for fixed values of these parameters. The discrete parameters in the phases of matter problem ($\star$) are the dimension of *space* and the symmetry group. Analogous parameters are present in effective long-range field theories, so it is important that we understand how these parameters manifest in the Axiom System. Dimension is a fundamental parameter evident in the domain bordism category. The dimension n, which is the dimension of *spacetime*, strongly affects not only topological theories, but also analytic aspects of usual quantum field theories. Symmetry is more complicated, and we devote much effort in these lectures and in [**FH1**] to this topic.

The initial arena for a relativistic quantum field theory is Minkowski spacetime $\mathbb{M}^n$, an n-dimensional affine space equipped with a translation-invariant Lorentz metric and a time-orientation. Its automorphism group $\mathcal{I}_n$ is the subgroup of the affine group which preserves the metric and time-orientation. A quantum field theory is a structure over $\mathbb{M}^n$, so its automorphism group $\mathcal{G}_n$ comes equipped with a homomorphism $\rho\colon \mathcal{G}_n \to \mathcal{I}_n$. In other words, a symmetry of a relativistic quantum field theory induces a symmetry of the underlying spacetime. Relativistic invariance is the requirement that the image of ρ contain the identity component of the Lie group $\mathcal{I}_n$. This geometrically natural setup places $\mathcal{I}_n$ or its identity component as a *quotient* of $\mathcal{G}_n$, whereas in traditional approaches to quantum field theory one assumes that the Poincaré group is a *subgroup* of $\mathcal{G}_n$. (The Poincaré group is a double cover of the identity component of $\mathcal{I}_n$.) In Lecture 3 we track symmetry through Wick rotation to Euclidean space and then to curved Riemannian manifolds. As we know from differential geometry, at this last stage it is natural to divide by translations and use the quotient as the structure group of a smooth manifold. Doing so we obtain from $\mathcal{G}_n$ a *compact* Lie group H_n and a homomorphism $\rho\colon H_n \to \mathrm{O}_n$ whose image is either SO_n or O_n. We call the pair (H_n, ρ_n) the *symmetry type* of the theory. It is an important discrete invariant in any field theory; we advocate articulating it explicitly in every example.

For the theories in these lectures $\mathcal{G}_n$ is a Lie group, but in general $\mathcal{G}_n$ may be a *super* Lie group and may include "higher" symmetries. In the Axiom System the symmetry type is manifest in the domain bordism category, which consists of Riemannian manifolds equipped with an H_n-structure in the sense of Cartan. In physics the H_n-structure is a "background field". For example, if $H_n = \mathrm{O}_n \times K$ for a compact Lie group K, then the background fields are a Riemannian metric and a principal K-bundle with connection. More general background fields can also be considered part of the "symmetry type" of a theory. But, as already stated, in these lectures we stick to compact Lie groups H_n and their associated background gauge fields and Riemannian metrics.

The rigidity of compact Lie groups leads to general structure theorems (Proposition 3.16, Theorem 3.24) and to classification theorems (Example 3.22, Theorem 10.2). The discrete parameters of dimension and symmetry type appear in our main theorems about invertible reflection positive field theories (§8.6).

Extended Locality

In Minkowski spacetime $\mathbb{M}^n$ one expression of locality is that observables supported in spacelike separated regions commute. After Wick rotating to Euclidean space $\mathbb{E}^n$, every pair of regions is spacelike separated and so operators with disjoint supports commute. Another expression of locality in $\mathbb{M}^n$ is cluster decomposition, a factorization property of correlation functions. The Axiom System for an n-dimensional Wick-rotated theory on Riemannian manifolds captures locality in codimension one. That is, if we cut a closed n-manifold X along a codimension one separating submanifold $Y \subset X$, so that we obtain two manifolds X_1, X_2 with common boundary Y, then a correlation function on X factors as a product of a correlation function on X_1 and a correlation function on X_2, assuming the supports of all observables are disjoint from Y. However, in all but exceptionally simple cases the correlation functions on X_1, X_2 are not complex numbers but rather lie in a complex vector space, the state space obtained by quantization on Y. Composition in a bordism category encapsulates this factorization of manifolds. Since a field theory is a functor out of a bordism category, the factorization of correlation functions in codimension one follows.

The strong locality of quantum field theory is perhaps more evident in the Haag approach, in which one essentially considers the theory built up from information on arbitrary open subsets of $\mathbb{M}^n$. This suggests that in Wick-rotated field theory one should be able to reconstruct everything from invariants attached to small balls. Starting with an n-manifold X, we must make cuts in n "directions" to express X as a union of balls. An algebraic structure which encapsulates these cuts has n composition laws. In this way n-categories enter the picture: the domain of an *extended* field theory is a bordism n-category. What is not known is a natural choice of codomain n-category, so that choice is left flexible, though in physical examples it is still constrained by superposition and interference. An extended field theory is then a symmetric monoidal functor between symmetric monoidal n-categories.

Lecture 5 is an exposition of extended locality and the extended Axiom System. The ideas are most developed in topological field theory. A basic theorem, the *cobordism hypothesis*, is a precise version of the statement that a field theory can be reconstructed from its restriction to a small ball. (In a topological theory one usually shrinks the ball to a point.)

Extended locality in this form was introduced in the mathematical literature in the early 1990s in connection with 3-dimensional quantum Chern-Simons theory. It appeared earlier in the physics literature in the form of extended observables, such as line operators.

Invertibility and homotopy theory

There is a composition law for quantum systems: conjunction without interaction. The state space of the composite is the tensor product of constituent state spaces and the Hamiltonian is the sum of constituent Hamiltonians. In the (extended) Axiom System the symmetric monoidal structures on the domain and codomain categories as well as on the functor between them combine to give the Wick-rotated version of this composition law. The trivial theory is a unit for the composition law; on each closed space there is a single quantum state and zero Hamiltonian, and in terms of the Axiom System it is the theory whose values are tensor units. Thus invertibility is defined. It is immediate that a theory is invertible if and only if it factors through the maximal subgroupoid of the codomain: the state space of every closed $(n-1)$-manifold is 1-dimensional, for example. But then one can "localize" and factor through the groupoid quotient of the domain bordism category. In this way an invertible field theory becomes a symmetric monoidal functor between (higher) Picard groupoids. ('Picard' is short for 'symmetric monoidal with invertible objects'.)

Enter stable homotopy theory. One passes freely between higher groupoids and topological spaces, or rather homotopy types, via the homotopy hypothesis. A Picard structure on a higher groupoid goes over to an infinite loop structure on a topological space. After that transmogrification, an invertible field theory is an infinite loop map of infinite loop spaces. This is a far cry from our starting point in physics(!), yet is the result of a step-by-step progression. So as not to muddy the mathematical waters, in Lecture 6 we take this homotopical incarnation of an invertible field theory as an *ansatz*. But in reality it is a theorem derived from the Axiom System, at least for topological field theories; we give appropriate references in that lecture.

We begin in Lecture 6 with non-extended topological field theories. In Theorem 6.27 we use an elementary Morse theory argument to prove that the partition function of an invertible field theory is a bordism invariant. However, it is not in general a *Thom* bordism invariant, but rather a *Reinhart* bordism invariant. That same conclusion follows from a deeper theorem which identifies the result of inverting all morphisms in a bordism n-category. It is the infinite loop space associated to a *Madsen-Tillmann spectrum*, which is then the domain spectrum of a field theory (Ansatz 6.89). Since we do not have in hand a canonical codomain n-category for a field theory, we cannot take its maximal subgroupoid to determine a canonical codomain spectrum for an invertible field theory. However, there *is* a natural choice for the codomain spectrum: the *character dual* to the sphere spectrum. It is characterized in field-theoretic terms by the property that the partition function determines the theory, something not true in general but a desirable property. Magically, the boson/fermion dichotomy of states falls out automatically (Remark 6.91).

At this point we have a homotopy type for a space of invertible topological field theories, but for three reasons it is not the correct homotopy type to apply to problem ($\star$). First, the topology is wrong: π_0 is the abelian group of *isomorphism* classes of topological theories, not the group of *deformation* classes. It is as if we ask about deformation classes of nonzero complex numbers but use the discrete topology rather than the usual continuous topology. We introduce the Anderson dual to the sphere spectrum as a substitute for the continuous topology; see §6.8. This leads us to introduce *continuous* invertible topological field theories (§6.10), which we argue capture the deformation class of an invertible theory. The second consideration which tells we have the wrong homotopy type is that the low-energy description of an invertible gapped quantum system is not necessarily a discrete *topological* theory, as we discuss below. And, importantly, the Axiom System does not incorporate *unitarity*, an important property of quantum systems to which we now turn.

Extended unitarity

The two pillars of quantum field theory are locality and unitarity. We explained above that a strong form of locality is implemented in Wick-rotated field theory by an extended Axiom System. Our mission now is to implement unitarity in the Axiom System, both in non-extended and extended forms, a topic we take up in Lecture 7 and Lecture 8, respectively. We only succeed in defining extended unitarity for *invertible topological* field theories; it is an interesting open question to define extended unitarity in general.

It is well-known that *reflection positivity* in Euclidean field theory is the Wick-rotated manifestation of unitarity in relativistic field theory. In fact, reflection and positivity are separate concepts. In the non-extended version discussed in Lecture 7, reflection is a structure and positivity is a condition. Reflection is implemented in the Axiom System as an involution on both the domain bordism category and the codomain category of complex vector spaces. If the domain consists of oriented manifolds, then the reflection involution is orientation-reversal. We define an analog for any symmetry type (H_n, ρ_n) in terms of a co-extension of the Lie group H_n; see Theorem 7.13. On the codomain the reflection involution is complex conjugation. A reflection structure is equivariance data for the functor which defines the field theory. This realizes the slogan "orientation-reversal maps to complex conjugation". A *reflection structure* induces a nondegenerate hermitian inner product on the state space attached to every closed $(n-1)$-manifold. In a non-extended theory positivity is the requirement that all of these hermitian inner products be positive definite. The reflection structure/positivity condition in the Axiom System on curved manifolds is a direct generalization of standard reflection positivity in Euclidean field theory.

In Lecture 8 we narrow the focus to invertible topological field theories. Recall that we model an extended invertible theory in stable homotopy theory as a map between appropriate spectra. An extended reflection structure in this invertible case is a lift of this map to an equivariant map between spectra with involutions. We give arguments to justify specific involutions in the domain and codomain, as appear in (8.45), (8.46). Not surprisingly, extended positivity is no longer just a condition in codimension one; it is also data in lower codimensions. To formulate it in our stable homotopy world, we first recast naive positivity of hermitian inner products in categorical terms (§7.1). This motivates the involution which models complex conjugation (§8.3), Definition 8.33 of a spectrum of positive definite Hermitian lines, and finally Definitions 8.53 and 8.55 of a space, or homotopy type, of reflection positive invertible topological field theories of fixed dimension and symmetry type. It turns out that for an extended invertible n-dimensional field theory with reflection structure, an extended positivity structure is a trivialization of an associated "real" $(n-1)$-dimensional field theory (Definition 8.62).

Non-topological invertible theories

Throughout the lectures we use non-topological field theories to guide our modeling, although the mathematical theorems pertain only to topological theories. Non-topological *invertible* field theories are relevant for the solution to $(\star)$, as we explain in Lecture 9. A standard hypothesis is that some long range behavior of a physical system, including its phase, is captured by a scale-invariant field theory. If the physical system is gapped, then this effective field theory is almost topological, but it may be off by a non-topological invertible theory; we coin the term "topological*" for this class of theories. If the entire effective field theory is invertible, then there is no reason for it to be topological. So to solve $(\star)$ we should produce a homotopy type of not-necessarily-topological invertible reflection positive theories. What we argue in Lecture 9 is that this is the homotopy type of *continuous* invertible reflection positive theories. Our proposal (Conjecture 9.34) has a more specific incarnation: invertible not-necessarily-topological field theories correspond to appropriate cocycles for *generalized differential cohomology*. The partition function of a theory is a secondary geometric invariant, the "integral" of a generalized differential cohomology class, and the theory itself provides a fully local description of the secondary invariant. A typical example is the exponentiated η-invariant of Atiyah-Patodi-Singer, which from this point of view is fully local.

Theorems

At this point we have carried out the First Step in the Three-Step Procedure of applications of mathematics: we have built a mathematical model of the physics problem $(\star)$. That model consists of a well-defined homotopy type $\mathcal{M}$ of appropriate field theories. The Second Step is to prove mathematical theorems which determine $\pi_0\mathcal{M}$. In §8.6 we state these results but do not include the proofs, which may be found in [**FH1**, §8]. We not only determine π_0 but identify the entire homotopy type in familiar terms, namely as maps from a Thom spectrum to the Anderson dual to the sphere spectrum. (This is the answer for continuous theories; for discrete theories there is a slightly more complicated answer.) Therefore, the entire effect of imposing unitarity in its Wick-rotated manifestation is to replace a Madsen-Tillmann spectrum by a Thom spectrum. We remark that in our general study of symmetry we prove (Theorem 3.24) that a symmetry type (H_n, ρ_n) has a stabilization (H, ρ) as $n \to \infty$. This produces a sequence (8.7) of Madsen-Tillmann spectra which limit to a Thom spectrum, and leads to the notion of a *stable* invertible topological field theory, i.e., one which factors through the Thom spectrum. Thus a reflection positive continuous invertible theory is stable.

Part B of the Second Step in the Three-Step Procedure is to compute the abelian group $\pi_0\mathcal{M}$ of deformation classes for physically relevant values of the discrete parameters (dimension and symmetry type). Fortunately, the mathematical techniques to make the computations are already well-developed. A major tool is the Adams spectral sequence, but it is not the only one. In Lecture 10 we present some special computations; many more computations are contained in the references, which also include pedagogical introductions to the Adams spectral sequence.

While these are the mathematical theorems and techniques directly applicable to ($\star$), we also take the opportunity to present other classification theorems in Lecture 4. The main techniques there are Morse and Cerf theory, not homotopy theory. The first theorem of this type is the classification of oriented 2-dimensional topological theories in terms of Frobenius algebras. We quote without proof an analogous result in a non-topological case, but where the theory only depends on an area form, not on a full Riemannian metric. We also sketch the proof of a 1-dimensional case of the cobordism hypothesis.

Free spinor fields

We are ready for Step Three of the Three-Step Procedure: application of the mathematical theorems to the physics problem ($\star$). We test our theorems and homotopy theoretical computations against known results in the physics literature, and we also derive new results. What is known and what is new is entirely a function of the discrete parameters: dimension and symmetry type. To apply our field theoretic theorems to discrete systems in condensed matter physics, we must understand how these parameters match up in the two descriptions of the same physical system; this is one of the topics treated in §9.1. The condensed matter literature primarily deals with spatial dimensions $d = 0, 1, 2, 3$, which of course corresponds to spacetime dimensions $n = 1, 2, 3, 4$. There are many computations in the physics literature for various symmetry types in varying dimension. By now many of the corresponding homotopy theoretic computations have been made, and there is complete agreement; see the references to Lecture 10. New results have also been obtained, and essentially any case in low dimensions can be computed on demand by an appropriately skilled young homotopy theorist.

In Lecture 10 we focus on symmetry types for which there is a notion of a "free fermion" system. There is a map from free systems to interacting systems, and thus three pieces of data for each symmetry type: the map from free to interacting together with its domain and codomain. Hence for these special symmetry types the test against physics literature is richer; for other symmetry types there is only one piece of data to check. Our first task is to classify appropriate symmetry types. It turns out there are ten of them (Theorem 10.2), another instance of the famous 10-fold way, which goes back to Dyson. We remark that the two flavors of pin group occur among the ten fermionic symmetry types, and our ideas about free fermions provide insight into the theory of Dirac operators on pin manifolds. A classical free fermion field theory, massless or massive, is specified by an appropriate Clifford module, as we recount in §10.2. The crucial Lemma 10.21 tells how a nondegenerate mass is equivalent to an extra Clifford generator. To pass from the classical free fermion to its free field quantization requires a choice beyond the Clifford data, as we explain shortly in the discussion about anomalies. In the massive case, which is the effective field theory of a gapped free fermion lattice system, Conjecture 10.25 tells the deformation class of the invertible low energy approximation in terms of the Clifford data. The conjectural formula employs (1) the connection between Clifford modules and KO-theory, and (2) the map from spin bordism to KO-theory, both of which were elucidated by Atiyah-Bott-Shapiro. In §10.3 we present the results of computations for two of the ten symmetry types.

Anomalies

Bonus Lecture 11 is a piece on the general topic of anomalies in quantum field theory; it was not part of the CBMS conference. There is a rich theory of anomalies which exhibits a multitude of approaches. We focus on geometric aspects, a point of view which fits in best with these lectures. Also, we use free spinor fields as motivation for the discussion of anomalies in general. The modern view on anomalies is: to an n-dimensional field theory F is canonically attached a truncated invertible $(n+1)$-dimensional field theory α_F, its *anomaly*. (The truncation means we only evaluate on manifolds of dimension $\leqslant n$; in many cases α_F extends to a full non-truncated theory.) This is half of Thesis 11.29. The other half asserts that to produce a well-defined quantum field theory one must specify a trivialization τ of the anomaly α_F. The ratio of two trivializations is an invertible n-dimensional field theory δ. Often what appears to be natural to define is the *relative* theory F with anomaly α_F; an absolute theory $\tau \circ F$ is only determined up to tensoring by an invertible theory δ. Quantum mechanics illustrates this indeterminacy (Example 11.34); shifting by δ tensors the state space by a line and shifts the energy by a constant. This indeterminacy also helps explain why a gapped physical system may have an effective long range field theory which is topological only up to tensoring by a possibly-non-topological invertible theory.

We return to free massive spinor fields in Example 11.36. We explain how to an n-dimensional massive free spinor field theory is canonically associated an $(n-1)$-dimensional massless free spinor field theory. The n-dimensional anomaly theory of the latter is a canonical choice for the long range effective field theory of the former. It is this choice which is used in Conjecture 10.25. This maneuver implicitly defines a canonical trivialization τ of the anomaly of a massive free spinor field theory, but in families of theories—for example with variable mass—one does not necessarily use this canonical choice.

Final remarks

An appendix to the lectures summarizes some relevant facts about 1-categories. In these lectures we treat higher categories heuristically and leave the full theory, including construction of models, to the literature.

Problem $(\star)$ concerns physical systems which exist on any manifold which represents space. One can also ask for the classification of phases on a particular space:

$(\star\star)$ Classify invertible gapped phases of matter on a space Y, possibly with group action

The theory developed in [**FH1**], recounted in these lectures, leads to a solution to problem ($\star\star$), again based on field theoretic ideas. The answer is a Borel-Moore (Borel equivariant) homology group of Y, as we explain in [**FH3**].

As should be crystal clear by now, these lectures are based on joint work with Mike Hopkins, whom I thank for many enjoyable collaborations. He and several other confederates and correspondents—including the Aspen 7 Gang—Mike Freedman, Anton Kapustin, Alexei Kitaev, Greg Moore, and Constantin Teleman, in addition to Mike Hopkins and me; David Ben-Zvi; Jonathan Campbell; Jacques Distler; Davide Gaiotto; Zheng-Cheng Gu; Matt Hastings; Jacob Lurie; Max Metlitski; Andy Neitzke; Graeme Segal; Nati Seiberg; Stephan Stolz; Peter Teichner; Ulrike Tillmann; Senthil Todadri; Kevin Walker; Xiao-Gang Wen; and Ed Witten—have influenced the ideas and work recounted in these lectures, though none of them should be held responsible for any errors of fact or judgment. The reader will join me in thanking Arun Debray for his expert proofreading, which greatly improved the manuscript. I am grateful to all.

LECTURE 1

Bordism and Topological Field Theories

Bordism is a notion which can be traced back to Henri Poincaré at the end of the 19$^{\text{th}}$ century, but it comes into its own mid-20$^{\text{th}}$ century in the hands of Lev Pontrjagin [**P**] and René Thom [**T**]. Poincaré originally tried to develop homology theory using smooth manifolds, but eventually simplices were used instead. Recall that a *singular q-chain* in a topological space S is a formal sum of continuous maps $\Delta^q \to S$ from the standard q-simplex. There is a boundary operation ∂ on q-chains, constructed by restriction to the boundary of Δ^q, and a q-chain c is a *cycle* if $\partial c = 0$; a cycle c is a *boundary* if there exists a $(q+1)$-chain b with $\partial b = c$. If S is a point, then every cycle of positive dimension is a boundary. In other words, abstract chains carry no information. In bordism theory one replaces cycles by *closed smooth manifolds* mapping continuously into S. A chain is replaced by a compact smooth manifold X and a continuous map $X \to S$; the boundary of this chain is the restriction $\partial X \to S$ to the boundary. Now there is information even if $S = \mathrm{pt}$. For not every closed smooth manifold is the boundary of a compact smooth manifold. For example, $Y = \mathbb{RP}^2$ is *not* the boundary of a compact 3-manifold.

We begin in §1.1 by using bordism to construct an equivalence relation on closed manifolds, and so obtain *sets* of bordism classes of manifolds. There is an algebraic structure: the set of bordism classes is an *abelian group* with group law derived from disjoint union. These ideas date from the 1950s. A modern take on bordism extracts more intricate algebraic gadgets from smooth manifolds: *categories* and their more complicated cousins. (See Appendix A for a review of the relevant algebra.) In this context the analog of a classical bordism invariant, such as the signature of a manifold, is a *topological field theory*. The bordism category—but not topological field theories—already appeared in the classic book [**Mi3**, §1]. In §1.2 we use bordism as a quick route to the Axiom System for field theories. In Lecture 2 and Lecture 3 we will see how these same axioms arise from quantum mechanics and quantum field theory. In this lecture we give several examples in the topological framework, including the basic Example 1.23. In §1.3 we introduce "tangential" structures into bordism and topological field theory, though in the generality we consider they need hardly relate to intrinsic geometry. In §1.5 we leave field theory to recall the Pontrjagin-Thom theorem, the basic relationship between bordism and stable homotopy theory. It will be used in our study of invertible field theories, beginning in Lecture 6.

The course notes [**F1**] contain more details on much of this material.

1.1. Classical bordism

All manifolds in these lectures are smooth, perhaps with boundary or corners, and maps between manifolds are also smooth. A manifold is *closed* if it is compact without boundary.

Fix an integer $d \geqslant 0$.

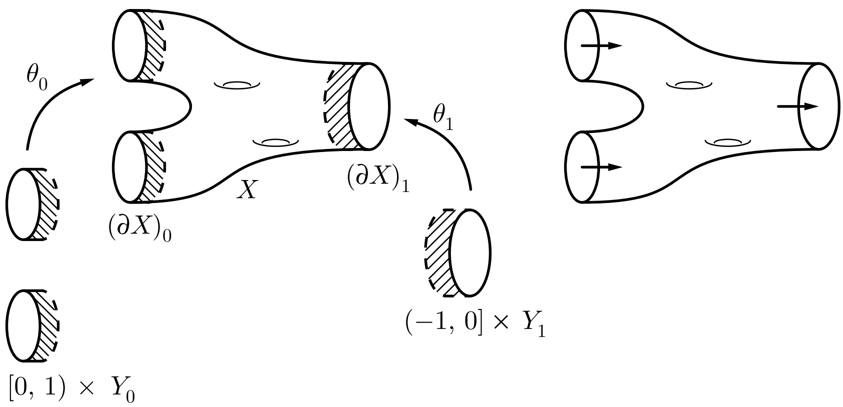

FIGURE 1.1. X is a bordism from Y_0 to Y_1

DEFINITION 1.1. Let Y_0, Y_1 be closed d-manifolds. A *bordism* $\big(X\,,\,p\,,\,\theta_0\,,\,\theta_1\big)$ *from* Y_0 *to* Y_1 consists of a compact $(d+1)$-manifold X with boundary, a partition $p\colon \partial X \to \{0,1\}$ of ∂X, and embeddings

$$\theta_0\colon [0,+1) \times Y_0 \longrightarrow X \tag{1.2}$$

$$\theta_1\colon (-1,0] \times Y_1 \longrightarrow X \tag{1.3}$$

with disjoint images, such that $\theta_i(0, Y_i) = (\partial X)_i$, $i = 0, 1$, where $(\partial X)_i = p^{-1}(i)$.

Each of $(\partial X)_0, (\partial X)_1$ is a union of components of ∂X; note that there is a finite number of components since X, and so too ∂X, is compact. Terminology: $(\partial X)_0$ is the *incoming boundary* and $(\partial X)_1$ the *outgoing boundary.* The map θ_i is a diffeomorphism onto its image, which is a collar neighborhood of $(\partial X)_i$; we require that the closures of the images of θ_1 and θ_2 are disjoint. The collar neighborhoods are included to make it easy to glue bordisms. The first drawing in Figure 1.1 illustrates the definition, but we picture a bordism more simply as the second in which we drop the collars. The arrows on the boundary encode the value of p. If the context is clear, we notate a bordism $\big(X\,,\,p\,,\,\theta_0\,,\,\theta_1\big)$ as 'X'. The *dual bordism* $X^\vee = \big(X^\vee\,,\,p^\vee\,,\,\theta_0^\vee\,,\,\theta_1^\vee\big)$ from Y_1 to Y_0 is the same underlying manifold $X^\vee = X$, but with $p^\vee = 1 - p$: reverse the arrow at each boundary component.

DEFINITION 1.4. Closed d-manifolds Y_0, Y_1 are *bordant* if there exists a bordism X from Y_0 to Y_1.

EXERCISE 1.5. Prove that bordism is an equivalence relation. Write carefully the details of the transitivity property, for which Figure 1.2 will be helpful.

EXERCISE 1.6. Show that diffeomorphic d-manifolds are bordant.

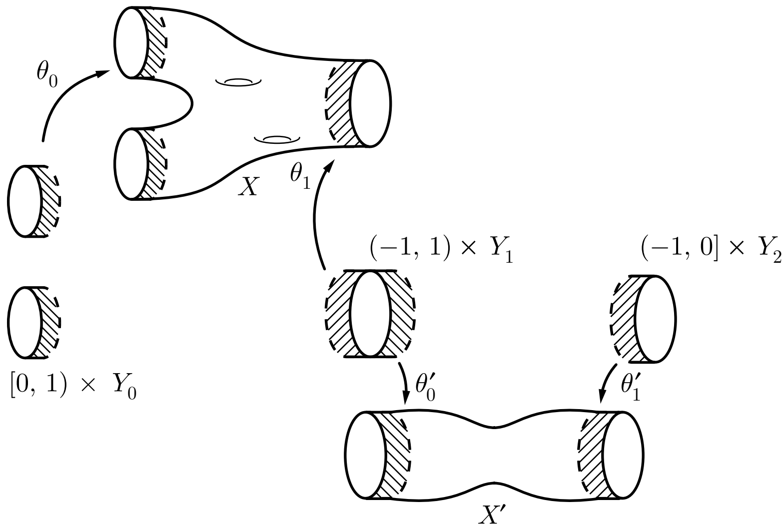

FIGURE 1.2. Gluing bordisms

Let Ω_d denote the set of equivalence classes of closed d-manifolds under the equivalence relation of bordism. An element of Ω_d is called a *bordism class*. The empty manifold $\varnothing^d$ is a special element of Ω_d, so we may consider Ω_d as a *pointed set*. (We let '$\varnothing^d$' denote the empty set with the structure of a d-dimensional smooth manifold.) Disjoint union is an operation on manifolds which passes to bordism classes: if Y_0 is bordant to Y_0' and Y_1 is bordant to Y_1', then $Y_0 \amalg Y_1$ is bordant to $Y_0' \amalg Y_1'$. So $(\Omega_d, \amalg)$ is a commutative monoid.[2] The identity element is represented by $\varnothing^d$. A *null bordant* manifold is one which is bordant to $\varnothing^d$. In fact, $(\Omega_d, \amalg)$ is an abelian group: every element is invertible—in fact of order at most two—since for any closed d-manifold Y, the disjoint union $Y \amalg Y$ is null-bordant via the null bordism $[0,1] \times Y$. The abelian group $(\Omega_d, \amalg)$ is finitely generated, though we do not prove that here. Denote this abelian group as 'Ω_d'.

In §1.3 we discuss bordism of manifolds with general tangential structure. An important special case is an orientation. Thus in Definition 1.1 we can take Y_0, Y_1, X oriented and ask that the maps θ_0, θ_1 be orientation-preserving, where in the domains we use the Cartesian product orientations. The resulting bordism groups are denoted[3] '$\Omega_d(\mathrm{SO})$'.

DEFINITION 1.7. A *classical bordism invariant* of oriented manifolds is a homomorphism

$$f\colon \Omega_d(\mathrm{SO}) \longrightarrow \mathbb{Z}. \tag{1.8}$$

For example, the signature $\mathrm{Sign}\colon \Omega_{4k}(\mathrm{SO}) \to \mathbb{Z}$ plays an important role in classical manifold theory. There are many other important bordism invariants in geometry, including the $\hat{A}$-genus $\hat{A}\colon \Omega_{4k}(\mathrm{Spin}) \to \mathbb{Z}$ in spin geometry and index theory, as

[2] A *commutative monoid* is a set with a commutative, associative composition law and identity element.

[3] For consistency, then, the unoriented bordism groups may be denoted '$\Omega_d(\mathrm{O})$'.

well as the Todd genus Todd: $\Omega_{2k}(\mathrm{U}) \to \mathbb{Z}$ in complex geometry. The latter and its cousins encode enumerative invariants of complex manifolds.

REMARK 1.9. One needn't choose the codomain to be $\mathbb{Z}$; there are also important $\mathbb{Z}/2\mathbb{Z}$-valued bordism invariants, as in Exercise 1.10 and Example 1.59.

EXERCISE 1.10. Prove that the mod 2 Euler number defines a bordism invariant $\Omega_d(\mathrm{O}) \to \mathbb{Z}/2\mathbb{Z}$. On the other hand, show that the integer-valued Euler number is not generally a bordism invariant.

REMARK 1.11. Bordism invariants with fixed domain (tangential structure) and codomain form an abelian group, e.g., $\mathrm{Hom}\big(\Omega_d(\mathrm{SO}), \mathbb{Z}\big)$.

1.2. Topological field theories

In this section we promote bordism from a condition to data. This process is often referred to as *categorification*:[4] rather than a set of equivalence classes of closed d-manifolds, we construct a bordism category[5] whose objects are closed d-manifolds and whose morphisms keep track of bordisms between them. The categorification of a classical bordism invariant (Definition 1.7) is a *topological field theory*.

Set $n = d + 1$ so that bordisms between d-manifolds are n-dimensional.[6] As our emphasis moves from the d-manifolds to the bordisms between them, we phase out 'd' and use 'n' instead.

Suppose $X = (X, p, \theta_0, \theta_1)$ and $X' = (X', p', \theta'_0, \theta'_1)$ are bordisms from Y_0 to Y_1. A *diffeomorphism* $F\colon X \to X'$ is a diffeomorphism of manifolds with boundary which commutes with the structure maps.

DEFINITION 1.12. Fix $n \in \mathbb{Z}^{\geqslant 0}$. The *bordism category* $\mathrm{Bord}_{\langle n-1,n\rangle}$ is the symmetric monoidal category defined as follows.

(1) Objects are closed $(n-1)$-manifolds.
(2) Morphisms in $\mathrm{Bord}_{\langle n-1,n\rangle}(Y_0, Y_1)$ are diffeomorphism classes of bordisms $X\colon Y_0 \to Y_1$.
(3) Composition of morphisms is by gluing (Figure 1.2).
(4) For each Y the bordism $[0,1] \times Y$ is the identity morphism $\mathrm{id}_Y\colon Y \to Y$.
(5) The monoidal product is disjoint union.
(6) The empty manifold $\varnothing^{n-1}$ is the tensor unit.

Even though technically we take an object to be an $(n-1)$-manifold Y, conceptually we think of Y as coming with an embedding into an n-manifold, or better a germ of an embedding. Germs of collar neighborhoods allow us to glue along open sets, as in Figure 1.2. When we add n-dimensional structures to the manifolds in §1.3, then we will necessarily have to thicken up the $(n-1)$-dimensional objects. In this topological setting it is enough to "thicken up" the tangent bundle $TY \to Y$,

[4]Historically, one of the first examples of categorification is the passage from Betti numbers to (Noether) homology groups. Atiyah's question in the early 1960s—Why is the $\hat{A}$-genus of a spin manifold an integer?—was one impetus for another important example of categorification: the construction/rediscovery by Atiyah-Singer of the Dirac operator on a Riemannian spin manifold and the identification of the $\hat{A}$-genus as the dimension of a super vector space.

[5]See Appendix A for some basics about categories.

[6]In the physics setting, d is the dimension of *space* and n the dimension of *spacetime*.

replacing it by the rank n bundle $\underline{\mathbb{R}} \oplus TY \to Y$, but in the geometric setting we encounter in later lectures we use germs of n-manifolds [**Se3**, **ST**].

EXERCISE 1.13. For each object Y in $\mathrm{Bord}_{\langle n-1,n\rangle}$ let $\mathrm{Diff}\, Y$ denote the group of diffeomorphisms of Y. Construct a homomorphism

$$\pi_0(\mathrm{Diff}\, Y) \to \mathrm{Bord}_{\langle n-1,n\rangle}(Y,Y) \tag{1.14}$$

by constructing a bordism $X_f = (X_f, p, \theta_0, \theta_1)$ associated to a diffeomorphism $f\colon Y \to Y$. Is (1.14) necessarily injective?

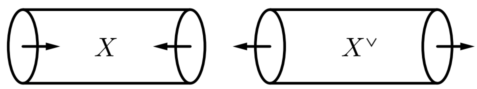

FIGURE 1.3. Some bordisms in $\mathrm{Bord}_{\langle 1,2\rangle}$

EXERCISE 1.15 ($\mathrm{Bord}_{\langle 1,2\rangle}$). Objects are closed 1-manifolds, so finite unions of circles. As depicted in Figure 1.3 the cylinder can be interpreted as a bordism $X\colon (S^1)^{\amalg 2} \to \varnothing^1$; the dual bordism $X^\vee$ is a map $X^\vee\colon \varnothing^1 \to (S^1)^{\amalg 2}$. Let $\rho\colon S^1 \to S^1$ be reflection; set $f = 1 \amalg \rho$, a diffeomorphism of $(S^1)^{\amalg 2}$; and let X_f be the associated bordism (Exercise 1.13). Then verify

$$\begin{aligned} X \circ X_{\mathrm{id}} \circ X^\vee &\cong \text{torus} \\ X \circ X_f \circ X^\vee &\cong \text{Klein bottle} \end{aligned} \tag{1.16}$$

These diffeomorphisms become equations in the monoid $\mathrm{Bord}_{\langle 1,2\rangle}(\varnothing^1, \varnothing^1)$ of diffeomorphism classes of closed 2-manifolds.

Let $\mathrm{Vect}_{\mathbb{C}}$ be the symmetric monoidal category whose objects are complex vector spaces and whose morphisms are linear maps. The monoidal product is the tensor product of vector spaces. We may consider[7] $\mathrm{Vect}_{\mathbb{C}}$ to be a categorification of the abelian group $\mathbb{Z}$. The following definition[8] categorifies Definition 1.7 of a classical bordism invariant.

AXIOM SYSTEM 1.17 (Topological field theory). Fix $n \in \mathbb{Z}^{\geqslant 0}$. An *$n$-dimensional topological field theory* is a symmetric monoidal functor

$$F\colon \mathrm{Bord}_{\langle n-1,n\rangle} \longrightarrow \mathrm{Vect}_{\mathbb{C}}\,. \tag{1.18}$$

Such a functor attaches to each closed $(n-1)$-manifold Y a vector space $F(Y)$ and to each bordism $X\colon Y_0 \to Y_1$ a linear map $F(X)\colon F(Y_0) \to F(Y_1)$. Gluings of bordisms map to compositions of linear maps, disjoint unions map to tensor products, the tensor unit $\varnothing^{n-1}$ maps to the tensor unit $\mathbb{C}$, and so a closed n-manifold X maps to a complex number $F(X)$. Anticipating the physics terminology, the vector space $F(Y)$ is the *state space* and the number $F(X)$ the *partition function*.

[7]But it does not stand up to scrutiny: (i) equivalence classes of objects under isomorphism form the commutative monoid $\mathbb{Z}^{\geqslant 0}$; and (ii) the monoidal structure we use, tensor product, categorifies multiplication in $\mathbb{Z}$, not addition. Nonetheless, it is a useful motivation for Axiom System 1.17.

[8]We use 'Axiom System' in place of 'Definition' with an eye to our later, non-topological, context in which the Axiom System (§3.1) is not meant to be the final word on a mathematical definition of quantum field theory. For *topological* field theories the mathematical underpinnings are much more developed, and we would use 'Definition' were it not for the broader context.

Remark 1.19. There are many variations of Axiom System 1.17 in these lectures. For example, we will consider theories of manifolds with tangential structure, such as orientation. Also, analogous to Remark 1.9, we can vary the codomain: for example, as in the passage from Betti numbers to homology groups we can replace $\mathrm{Vect}_{\mathbb{C}}$ with the symmetric monoidal category Ab of abelian groups under tensor product. Our choice $\mathrm{Vect}_{\mathbb{C}}$ comes from quantum mechanics: the linearity of vector spaces encodes *superposition* and the complex numbers encode *interference*. In Lecture 6 we will implement an extended notion of *locality* in which we move from categories to n-categories. Then in Lecture 8 we implement *unitarity*. Even more fundamentally, Axiom System 1.17 is stated for *topological* field theories. There are extensions for non-topological theories, which of course is the typical case of physical interest. The original formulation of the Axiom System is for 2-dimensional conformal field theories and is due to Segal [**Se1**]. The topological version is due to Atiyah [**A1**]. For this reason the term 'Atiyah-Segal Axioms' is sometimes used for Axiom System 1.17.

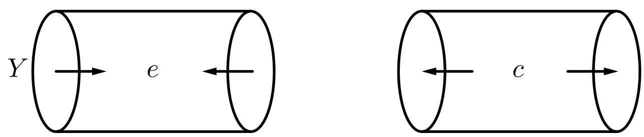

Figure 1.4. Evaluation e and coevaluation c morphisms for $Y = S^1$

Duality places stringent finiteness restrictions on topological field theories. (See §A.2 in the appendix for a review of duality in symmetric monoidal categories.) The main observation is quite simple.

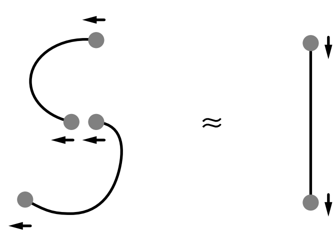

Figure 1.5. The S-diagram argument

Lemma 1.20. *Every object* Y *in the bordism category* $\mathrm{Bord}_{\langle n-1,n\rangle}$ *is dualizable.*

The dual object $Y^\vee = Y$; the evaluation and coevaluation bordisms are depicted in Figure 1.4. The compositions (A.29) are the cylinder as a bordism $Y \to Y$, which is the identity map id_Y in the bordism category (Definition 1.12). This is often referred to as the "S-diagram argument" in view of Figure 1.5. The image of a dualizable object under a symmetric monoidal functor is dualizable, from which we deduce the following.

Corollary 1.21. *Let* $F\colon \mathrm{Bord}_{\langle n-1,n\rangle} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$ *be a topological field theory. Then for all objects* $Y \in \mathrm{Bord}_{\langle n-1,n\rangle}$*, the vector space* $F(Y)$ *is finite dimensional.*

EXERCISE 1.22. Prove that if an object V in $\mathrm{Vect}_{\mathbb{C}}$ is dualizable, then V is finite dimensional, and then provide a complete proof of Corollary 1.21.

Next, we give an important example of a topological field theory. In the high energy physics literature it appears in a paper of Dijkgraaf-Witten [**DW**]; it appears in nascent form earlier in the condensed matter literature [**We**]. See [**FQ**] for careful proofs of the gluing laws which show that this finite gauge theory satisfies Axiom System 1.17. The lecture notes [**HL**] contain a more modern exposition and place the "finite path integral" (1.27) below in a proper categorical setting.

EXAMPLE 1.23 (Finite gauge theory). Fix a finite group G. For any topological space S there is an associated groupoid $\mathrm{Bun}_G(S)$ whose objects are principal G-bundles over S and whose morphisms are isomorphisms of G-bundles covering id_S. A continuous map $f\colon S' \to S$ induces $f^*\colon \mathrm{Bun}_G(S) \to \mathrm{Bun}_G(S')$ by pullback. In particular, a bordism $X\colon Y_0 \to Y_1$ in $\mathrm{Bord}_{\langle n-1,n\rangle}$ induces a correspondence

$$\begin{tikzcd} & \mathrm{Bun}_G(X) \arrow[dl, "s"'] \arrow[dr, "t"] & \\ \mathrm{Bun}_G(Y_0) & & \mathrm{Bun}_G(Y_1) \end{tikzcd} \tag{1.24}$$

in which the source and target maps are restriction to the incoming and outgoing boundary components. The finite gauge theory

$$\mathscr{G}_G\colon \mathrm{Bord}_{\langle n-1,n\rangle} \longrightarrow \mathrm{Vect}_{\mathbb{C}} \tag{1.25}$$

linearizes this correspondence diagram. Namely, let

$$\mathscr{G}_G(Y) = \mathrm{Map}\big(\mathrm{Bun}_G(Y), \mathbb{C}\big), \qquad Y \text{ an object of } \mathrm{Bord}_{\langle n-1,n\rangle}, \tag{1.26}$$

be the vector space of functions[9] of G-bundles and let

$$\mathscr{G}_G(X) = t_* \circ s^* \tag{1.27}$$

be the composition of pullback and pushforward of functions. The fibers of t are finite, so the pushforward of a function is computed by a finite sum, but the fiber is a groupoid and so that sum must be normalized: if $\mathcal{G}$ is a groupoid whose set $\pi_0\mathcal{G}$ of equivalence classes is finite and if the automorphism group $\mathrm{Aut}(x)$ of any object x is finite, then the sum/integral/pushforward of a function $f\colon \mathcal{G} \to \mathbb{C}$ is

$$\sum_{[x]\in\pi_0\mathcal{G}} \frac{f(x)}{\#\,\mathrm{Aut}(x)}. \tag{1.28}$$

The sum is over representatives of equivalence classes of objects. In particular, if X is a closed n-manifold, then the partition function $\mathscr{G}_G(X) \in \mathbb{C}$ counts the G-bundles on X weighted by the inverse number of automorphisms:

$$\mathscr{G}_G(X) = \sum_{[P]\in\pi_0\,\mathrm{Bun}_G(X)} \frac{1}{\#\,\mathrm{Aut}\,P}. \tag{1.29}$$

Notice that a variant of $\mathscr{G}_G$ exists with codomain the symmetric monoidal category $\mathrm{Vect}_{\mathbb{Q}}$ of rational vector spaces.

[9] A function is a functor from $\mathrm{Bun}_G(Y)$ to the category with set of objects $\mathbb{C}$ and only identity morphisms, which amounts to a function on equivalence classes of G-bundles.

Remark 1.30. The pushforward t_* in (1.27) is a finite version of the *Feynman path*[10] *integral*. The G-bundle is a *fluctuating*, or *quantum* field, and it is summed over so does not appear in the domain of (1.25).

Exercise 1.31. Prove that (1.26) and (1.27) define a symmetric monoidal functor. To check compositions you'll need to prove that for a composition of bordisms the correspondences (1.24) compose by fiber product; see [**FQ**].

Exercise 1.32. Compute the partition function of a closed connected n-manifold X. Hint: Choose a basepoint and construct a presentation of $\mathrm{Bun}_G(X)$ using the theory of covering spaces and the fundamental group.

As in Remark 1.11 we can consider the collection of field theories with fixed domain and codomain. This collection

$$\mathrm{TFT}_{\langle n-1,n\rangle} = \mathrm{Hom}\bigl(\mathrm{Bord}_{\langle n-1,n\rangle}, \mathrm{Vect}_{\mathbb{C}}\bigr) \tag{1.33}$$

of symmetric monoidal functors forms a symmetric monoidal category: given theories F_1, F_2 there is a tensor product theory $F_1 \otimes F_2$ defined object-wise. The tensor unit theory $\mathbf{1}$ assigns the vector space $\mathbb{C}$ to every closed $(n-1)$-manifold and $\mathrm{id}_{\mathbb{C}}$ to every n-dimensional bordism. The set of morphisms from F_1 to F_2 is the set of natural transformations between the functors. The state space attached to a closed $(n-1)$-manifold Y is the tensor product $F_1(Y) \otimes F_2(Y)$ and the linear map attached to a bordism X is the tensor product $F_1(X) \otimes F_2(X)$ of linear maps. This monoidal structure has a manifestation in physical systems; see Remark 2.10 and Remark 2.14. In this topological case we have the following strong consequence of duality.

Proposition 1.34. *A morphism* $(\eta\colon F \to G) \in \mathrm{TFT}_{\langle n-1,n\rangle}$ *is invertible:* $\mathrm{TFT}_{\langle n-1,n\rangle}$ *is a groupoid.*

This follows from Proposition A.44(2), since every object in the bordism category is dualizable (Lemma 1.20).

1.3. Structures on manifolds; further examples

Tangential structures on smooth manifolds are incorporated into bordism theory from the beginning. We have already mentioned orientations. Other common examples are spin structures, framings, and almost complex structures. Given a topological space S we can have a structure on an n-manifold X which is a continuous map $X \to S$. In the physics context these structures are *background fields*, and we will later use rigid geometric models; see §3.5. In our current topological context we take a flabbier approach.

A smooth n-manifold X has a rank n real tangent bundle $TX \to X$. Choose a universal rank n real vector bundle

$$W_n \longrightarrow B\mathrm{GL}_n\,\mathbb{R}. \tag{1.35}$$

Recall that a classifying bundle $EH \to BH$ for a Lie group H is a topological principal H-bundle with the property that EH is contractible (and a sufficiently nice[11] topological space). The homotopy type of BH is independent of the realization.

[10] It is better called the Feynman *functional* integral. In quantum mechanics, as first considered by Feynman, the integral *is* over paths.

[11] metrizable infinite dimensional topological manifold, for example

The base of (1.35) is a classifying space of $GL_n\mathbb{R}$, and (1.35) is the vector bundle associated to the standard linear action of $GL_n\mathbb{R}$ on $\mathbb{R}^n$. A *classifying map* for the tangent bundle is a pullback diagram

$$\begin{tikzcd} TX \arrow[r] \arrow[d] & W_n \arrow[d] \\ X \arrow[r] & B\mathrm{GL}_n\,\mathbb{R} \end{tikzcd} \tag{1.36}$$

of vector bundles. Classifying maps form a contractible space, and so without changing the equivalence class of the bordism category we can assume that every manifold comes equipped with a classifying map of its tangent bundle.

Suppose we have a homomorphism

$$\rho_n \colon H_n \to \mathrm{GL}_n\,\mathbb{R} \tag{1.37}$$

of Lie groups. Then a *flabby H_n-structure* on X is a lift of its classifying map (1.36) in the diagram

$$\begin{tikzcd} & & (B\rho_n)^*W_n \arrow[dl] \arrow[dd] \\ TX \arrow[r] \arrow[urr, dashed] \arrow[dd] & W_n \arrow[dd] & \\ & & BH_n \arrow[dl, "B\rho_n"] \\ X \arrow[r] \arrow[urr, dashed] & B\mathrm{GL}_n\,\mathbb{R} & \end{tikzcd} \tag{1.38}$$

Exercise 1.39. Express orientations and spin structures in this language. Reconcile this flabby definition with any other that you know.

In the end, then, we only need the space $\mathcal{X}_n = BH_n$ and a rank n real vector bundle over it, or equivalently a continuous map $\pi_n \colon \mathcal{X}_n \to B\mathrm{GL}_n\,\mathbb{R}$. Let $B\mathrm{GL} = \operatorname*{colim}_{n\to\infty} B\,\mathrm{GL}_n\,\mathbb{R}$.

Definition 1.40. An *n-dimensional flabby tangential structure* is a topological space $\mathcal{X}_n$ and a fibration $\pi_n \colon \mathcal{X}_n \to B\,\mathrm{GL}_n\,\mathbb{R}$. A *stable flabby tangential structure* is a topological space $\mathcal{X}$ and a fibration $\pi\colon \mathcal{X} \to B\mathrm{GL}$. It gives rise to an n-dimensional flabby tangential structure for each $n \in \mathbb{Z}^{\geqslant 0}$ by defining $\pi_n \colon \mathcal{X}_n \to B\mathrm{GL}_n\,\mathbb{R}$ as the fiber product

$$\begin{tikzcd} \mathcal{X}_n \arrow[r, dashed] \arrow[d, dashed, "\pi_n"'] & \mathcal{X} \arrow[d, "\pi"] \\ B\mathrm{GL}_n\,\mathbb{R} \arrow[r] & BGL \end{tikzcd} \tag{1.41}$$

If M is a k-dimensional manifold, $k \leqslant n$, then a *flabby $\mathcal{X}_n$-structure* on M is a lift[12] $M \to \mathcal{X}_n$ of a classifying map $M \to B\mathrm{GL}_n\,\mathbb{R}$ of the rank n stabilized tangent bundle[13] $\underline{\mathbb{R}^{n-k}} \oplus TM \to M$. A *flabby $\mathcal{X}$-structure* on M is a family of coherent flabby $\mathcal{X}_n$-structures for n sufficiently large.

The terminology is inconsistent for $\mathcal{X}_n = BH_n$, but hopefully that does not cause confusion: we use 'H_n-structure' in place of 'BH_n-structure'.

[12]The lift includes an isomorphism of rank n bundles, as in (1.36) and (1.38).

[13]Let S, F be spaces. Then $\underline{F} \to S$ is the constant fiber bundle with fiber F over S.

EXAMPLE 1.42 (Framings). A *framing* of a vector bundle is an isomorphism with a trivial bundle, that is, a bundle with constant fibers. If the bundle has rank n, then we might call this an 'n-framing'. This corresponds to the n-dimensional flabby tangential structure $\pi_n\colon E\mathrm{GL}_n\,\mathbb{R} \to B\mathrm{GL}_n\,\mathbb{R}$ which is a universal principal $\mathrm{GL}_n\,\mathbb{R}$-bundle. Stable framings correspond to the stable universal bundle $\pi\colon E\mathrm{GL} \to B\mathrm{GL}$.

EXAMPLE 1.43. Let S be a topological space and set $\mathcal{X}_n = B\mathrm{GL}_n\,\mathbb{R} \times S$ with $\pi_n\colon \mathcal{X}_n \to B\mathrm{GL}_n\,\mathbb{R}$ projection. Then an $\mathcal{X}_n$-structure on M is a continuous map $M \to S$.

Fix an n-dimensional flabby tangential structure $\pi_n\colon \mathcal{X}_n \to B\mathrm{GL}_n\,\mathbb{R}$. There is a bordism category $\mathrm{Bord}_{\langle n-1,n\rangle}(\mathcal{X}_n)$ of $\mathcal{X}_n$-manifolds; in case $\mathcal{X}_n = BH_n$ we use the notation '$\mathrm{Bord}_{\langle n-1,n\rangle}(H_n)$'. We can replace the domain in (1.18) with $\mathrm{Bord}_{\langle n-1,n\rangle}(\mathcal{X}_n)$ to define an n-dimensional topological field theory of $\mathcal{X}_n$-manifolds. Here are a few examples.

EXAMPLE 1.44 (Dijkgraaf-Witten [**DW**]). This is a variation of Example 1.23 defined on oriented manifolds, so a functor $\mathrm{Bord}_{\langle n-1,n\rangle}(\mathrm{GL}_n^+\,\mathbb{R}) \to \mathrm{Vect}_{\mathbb{C}}$, where $\mathrm{GL}_n^+\,\mathbb{R} \subset \mathrm{GL}_n\,\mathbb{R}$ is the index two subgroup of orientation-preserving automorphisms of $\mathbb{R}^n$. Fix a space of homotopy type BG and a singular cocycle $\lambda \in C^n(BG;\mathbb{R}/\mathbb{Z})$. The isomorphism class of the theory we construct depends only on the cohomology class $\bar\lambda \in H^n(BG;\mathbb{R}/\mathbb{Z})$. If X is a closed oriented n-manifold and $P \to X$ a principal G-bundle, then the characteristic class $\bar\lambda$ determines a cohomology class $\bar\lambda(P) \in H^n(X;\mathbb{R}/\mathbb{Z})$ and so a nonzero complex number $\exp(2\pi i\,\langle\bar\lambda(P),[X]\rangle)$, where $[X] \in H_n(X)$ is the fundamental class of the orientation. A more elaborate construction [**FQ**, Appendix B] produces for each closed oriented $(n-1)$-manifold Y a complex line bundle $L_{G,\lambda}(Y) \to \mathrm{Bun}_G(Y)$ and for each bordism $X\colon Y_0 \to Y_1$ an isomorphism

$$\alpha_{G,\lambda}(X)\colon s^*L_{G,\lambda}(Y_0) \longrightarrow t^*L_{G,\lambda}(Y_1) \tag{1.45}$$

in the correspondence diagram (1.24). Define the twisted gauge theory

$$\mathscr{G}_{G,\lambda}\colon\ \mathrm{Bord}_{\langle n-1,n\rangle}(\mathrm{GL}_n^+\,\mathbb{R}) \longrightarrow \mathrm{Vect}_{\mathbb{C}} \tag{1.46}$$

by replacing (1.26) with the space of sections of $L_{G,\lambda}(Y) \to \mathrm{Bun}_G(Y)$ and (1.27) with the composition

$$\mathscr{G}_{G,\lambda}(X) = t_* \circ \alpha_{G,\lambda}(X) \circ s^*. \tag{1.47}$$

REMARK 1.48. Continuing Remark 1.30 we observe that if $X\colon \varnothing^{n-1} \to \varnothing^{n-1}$ is a closed oriented n-manifold, then

$$\alpha_{G,\lambda}(X) = e^{2\pi i\,\langle\bar\lambda(-),[X]\rangle} \tag{1.49}$$

is multiplication by the exponentiated characteristic number of a principal G-bundle $P \to X$. This is called the *exponentiated (classical) action* in a physical context. Then t_* is again the Feynman path integral, but now with the usual integrand of the exponentiated action (1.49).

EXAMPLE 1.50 (Classical Dijkgraaf-Witten). The Dijkgraaf-Witten theory in Example 1.44 uses the principal G-bundle as a fluctuating field. Before summing over it we obtain another topological field theory in which the principal G-bundle is a background field, so part of the (flabby) tangential structure. Set $\mathcal{X}_n = B\mathrm{GL}_n^+\,\mathbb{R}\times BG$ with the obvious map to $B\mathrm{GL}_n\,\mathbb{R}$. Then define[14]

$$\alpha_{G,\lambda}\colon\ \mathrm{Bord}_{\langle n-1,n\rangle}(\mathcal{X}_n)\longrightarrow \mathrm{Vect}_{\mathbb{C}} \tag{1.51}$$

on closed $(n-1)$-manifolds using $L_{G,\lambda}$ and on n-dimensional bordisms using (1.45); see [**FQ**] for details. In physics terminology $\alpha_{G,\lambda}$ is the *classical*[15] field theory whose quantization is the Dijkgraaf-Witten theory. It is an *invertible*[16] theory in the sense that it factors through the maximal Picard subgroupoid $\mathrm{Line}_{\mathbb{C}}\subset\mathrm{Vect}_{\mathbb{C}}$ of complex lines (1-dimensional vector spaces) and invertible linear maps. We will say much more about invertible theories in Lecture 6.

Example 1.52 (Characteristic numbers). Any characteristic number determines a topological field theory [**F2**]. These theories often require an orientation.[17] The characteristic number can be computed by cutting an n-manifold along codimension one submanifolds. (There is a stronger locality which allows cutting along higher codimension submanifolds; see Lecture 6.) The simplest example comes from the Euler number. Fix a dimension n and a nonzero complex number $\mu\in\mathbb{C}^\times$. Then define

$$\epsilon_\mu\colon\ \mathrm{Bord}_{\langle n-1,n\rangle}\longrightarrow\mathrm{Vect}_{\mathbb{C}} \tag{1.53}$$

by $\epsilon_\mu(Y)=\mathbb{C}$ for every object Y in $\mathrm{Bord}_{\langle n-1,n\rangle}$, and $\epsilon_\mu(X)=\mu^{\mathrm{Euler}(X)}$ for every morphism X, where $\mathrm{Euler}(X)\in\mathbb{Z}$ is the Euler number of the compact topological space underlying the bordism X.

1.4. Varying the codomain: super vector spaces

We introduce the symmetric monoidal category of *super vector spaces*. For more detail on superalgebra see [**DeM**]. The word 'super' is a synonym for '$\mathbb{Z}/2\mathbb{Z}$-graded'. A *super vector space* is a pair (V,ϵ) consisting of a vector space (over a field k of characteristic not equal[18] to 2) and an endomorphism $\epsilon\colon V\to V$ such that $\epsilon^2=\mathrm{id}_V$. The $\pm$-eigenspaces of ϵ provide a decomposition $V=V^0\oplus V^1$; elements of the subspace V^0 are called *even* and elements of the subspace V^1 are called *odd*. A morphism $(V,\epsilon)\to(V',\epsilon')$ is a linear map $T\colon V\to V'$ such that $T\circ\epsilon=\epsilon'\circ T$. It follows that T maps even elements to even elements and odd elements to odd elements. The monoidal structure is defined as

$$(V_1,\epsilon_1)\otimes(V_2,\epsilon_2)=(V_1\otimes V_2,\epsilon_1\otimes\epsilon_2) \tag{1.54}$$

If $v\in V$ is a homogeneous element, define its *parity* $|v|\in\{0,1\}$ so that $v\in V^{|v|}$. Then for homogeneous elements $v_i\in V_i$ the symmetry is

$$\begin{aligned}\sigma\colon(V_1,\epsilon_1)\otimes(V_2,\epsilon_2)&\longrightarrow(V_2,\epsilon_2)\otimes(V_1,\epsilon_1)\\ v_1\otimes v_2&\longmapsto(-1)^{|v_1|\,|v_2|}\,v_2\otimes v_1\end{aligned} \tag{1.55}$$

This is called the *Koszul sign rule*. Let $s\mathrm{Vect}_k$ denote the symmetric monoidal category of super vector spaces. The obvious forgetful functor $s\mathrm{Vect}_k\to\mathrm{Vect}_k$

[14] The domain of (1.51) can also be notated '$\mathrm{Bord}_{\langle n-1,n\rangle}(\mathrm{GL}_n^+\,\mathbb{R}\times G)$'.

[15] Axiom System 1.17 does not distinguish between classical and quantum.

[16] invertible in the symmetric monoidal category (1.33)

[17] For characteristic numbers in generalized cohomology theories, such as K-theory, an appropriate generalized orientation, such as a spin^c structure, is required.

[18] There is a different description in characteristic 2.

is *not* a symmetric monoidal functor, though it is a monoidal functor. Diligent application of (1.55) leads to a plethora of signs yet avoids sign problems.

REMARK 1.56. In quantum mechanics we use super vector spaces over $\mathbb{C}$ to model quantum systems with bosonic and fermionic states. In that context the parity of a homogeneous element encodes the *statistics* of the corresponding quantum state.

EXAMPLE 1.57. There is an invertible 1-dimensional theory α of unoriented manifolds which assigns the odd line Π to a point. (Π is the vector space $\mathbb{C}$ with odd grading.) You can check that $\alpha(S^1) = -1$. Write

$$\alpha\colon \mathrm{Bord}_{\langle 0,1\rangle} \longrightarrow s\mathrm{Line}_{\mathbb{C}}, \tag{1.58}$$

where $s\mathrm{Line}_{\mathbb{C}} \subset s\mathrm{Vect}_{\mathbb{C}}$ is the *underlying groupoid* whose objects are super lines and whose morphisms are even *iso*morphisms, i.e., degree-preserving isomorphisms.

EXAMPLE 1.59 (Arf theory [**MS**, **Gu**, **DeGu**]). This is an invertible 2-dimensional theory of spin manifolds:

$$\alpha\colon \mathrm{Bord}_{\langle 1,2\rangle}(\mathrm{Spin}_2) \longrightarrow s\mathrm{Line}_{\mathbb{C}}\,. \tag{1.60}$$

The partition function of a closed spin 2-manifold is

$$\alpha(X) = (-1)^{\mathrm{Arf}(X)}, \tag{1.61}$$

where $\mathrm{Arf}(X) \in \mathbb{Z}/2\mathbb{Z}$ is the Arf invariant of the quadratic form $q_X\colon H_1(X) \to \mathbb{Z}/2\mathbb{Z}$ determined by the spin structure [**KT1**]: if $S \subset X$ is an embedded circle representing a homology class, then the value of q_X on that class is 0 or 1 according as the induced spin structure on S bounds or not.

REMARK 1.62. There is an important qualitative distinction between the theories in Example 1.57 and Example 1.59: the latter is unitary whereas the former is not. We discuss unitarity for Axiom System 1.17 in Lecture 7.

EXERCISE 1.63. Complete the definition of the Arf theory α. You should find that for S a circle with spin structure, $\alpha(S)$ is one-dimensional and is even or odd according as S bounds or not.

1.5. Bordism and homotopy theory

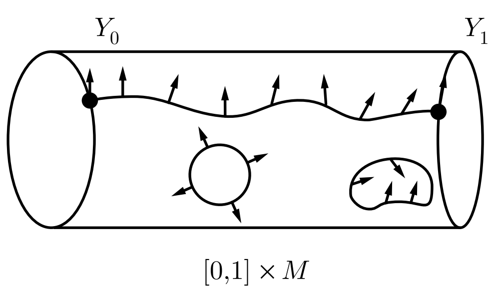

FIGURE 1.6. A framed bordism in M

The relationship between bordism and homotopy theory was originally developed by Pontrjagin [**P**] for *framed* bordism; subsequently, it was extended by

Thom [**T**]. Here we review the basic Pontrjagin-Thom construction. Fix a closed m-dimensional manifold M. Let $Y \subset M$ be a submanifold and $\nu \to Y$ the normal bundle, which is the quotient of the restriction of TM to Y by TY. A *framing* of the submanifold $Y \subset M$ is a trivialization of the normal bundle $\nu \to Y$. Fix a positive integer $q < m$. Framed submanifolds of M of codimension q arise as follows. Let $f\colon M \to S^q$ be a smooth map. Suppose $p \in S^q$ is a *regular value* of f. Then $Y := f^{-1}(p) \subset M$ is a submanifold and the differential f_* at y maps the subspace $T_yY \subset T_yM$ to zero, so f_* factors down to a map $\nu_y \to T_pS^q$. The fact that p is a regular value implies that the latter is an isomorphism, and this defines a framing of $\nu \to Y$. Of course, regular values are not unique. In fact, Sard's theorem asserts that they form an open dense subset of S^q. The inverse images $Y_0 := f^{-1}(p_0)$ and $Y_1 = f^{-1}(p_1)$ of two regular values $p_0, p_1 \in S^q$ are *framed bordant in* M. (See Figure 1.6.) Framed bordism in M is an equivalence relation; the set of equivalence classes is denoted '$\Omega^{\mathrm{fr}}_{m-q;M}$'. This *Pontrjagin construction* of a normally framed manifold as the inverse image of a regular value defines a map

$$[M, S^q] \longrightarrow \Omega^{\mathrm{fr}}_{m-q;M} \tag{1.64}$$

from homotopy classes of maps $M \to S^q$ to normally framed submanifolds of M.

Theorem 1.65 (Pontrjagin-Thom). *The map* (1.64) *is an isomorphism.*

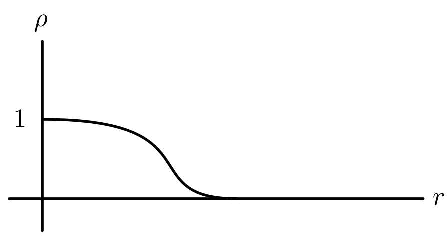

Figure 1.7. Cutoff function for collapse map

There is a geometrically defined inverse map to (1.64), known as the *Pontrjagin-Thom collapse.* Let $Y \subset M$ be a framed submanifold of codimension q, so equipped with an isomorphism $\nu \cong \underline{\mathbb{R}^q}$. Recall that any submanifold Y has a *tubular neighborhood*, which is an open neighborhood $U \subset M$ of Y, a submersion $U \to Y$, and an isomorphism $\varphi\colon \nu \to U$ which makes the diagram

$$\begin{tikzcd} \nu \arrow[rr, "\varphi"] \arrow[dr] & & U \arrow[dl] \\ & Y & \end{tikzcd} \tag{1.66}$$

commute. The framing of ν then leads to a map $h\colon U \to \mathbb{R}^q$. The collapse map $f_Y\colon M \to S^q$ is

$$f_Y(x) = \begin{cases} \dfrac{h(x)}{\rho\big(|h(x)|\big)}, & x \in U; \\ \infty, & x \in M\backslash U. \end{cases} \tag{1.67}$$

Here we write $S^q = \mathbb{R}^q \cup \{\infty\}$ and we fix a cutoff function ρ as depicted in Figure 1.7. We represent a collapse map in Figure 1.8.

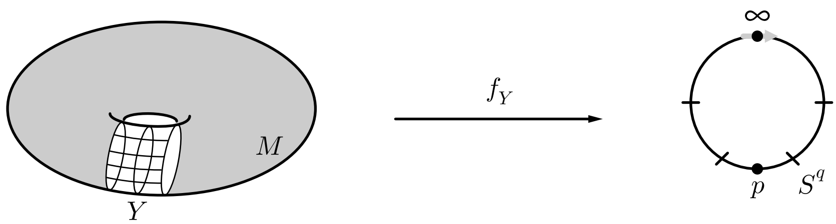

FIGURE 1.8. Pontrjagin-Thom collapse

There are choices (regular value, tubular neighborhood, cutoff function) in these constructions, but the map (1.64) and its inverse are independent of these choices. See [**Mi1**, **F1**] for a proof of Theorem 1.65.

Now specialize to the case $M = S^m$ is a sphere. Introduce basepoints and pass to pointed maps and pointed homotopy equivalences; the reader can check that the sets of homotopy classes of maps is unchanged in (1.68) when basepoints are accounted for. Suspension Σ of pointed topological spaces takes spheres to spheres—there is a homeomorphism $\Sigma S^m \cong S^{m+1}$. Hence suspension induces a sequence of maps

$$[S^m, S^q] \xrightarrow{\Sigma} [S^{m+1}, S^{q+1}] \xrightarrow{\Sigma} [S^{m+2}, S^{q+2}] \xrightarrow{\Sigma} \cdots \tag{1.68}$$

This is a sequence

$$\pi_m S^q \xrightarrow{\Sigma} \pi_{m+1} S^{q+1} \xrightarrow{\Sigma} \pi_{m+2} S^{q+2} \xrightarrow{\Sigma} \cdots \tag{1.69}$$

of homomorphisms of abelian groups, homotopy groups of the sphere. We state the following without proof, though we indicate one route to a proof below.

THEOREM 1.70 (Freudenthal [**Fr**]). *The sequence* (1.68) *stabilizes in the sense that all but finitely many maps are isomorphisms.*

Set $n = m - q$. The limiting group is the n^{th} *stable homotopy group of the sphere*, or n^{th} *stable stem*, and is denoted $\pi_n S^0$. The structure of the stable stem is central in stable homotopy theory. The first few groups are exhibited in the following table:

(1.71)

n	$\pi_n S^0$
4	0
3	$\mathbb{Z}/24\mathbb{Z}$
2	$\mathbb{Z}/2\mathbb{Z}$
1	$\mathbb{Z}/2\mathbb{Z}$
0	$\mathbb{Z}$

By the Pontrjagin-Thom Theorem 1.65 we rewrite (1.69) as a sequence of maps

$$\Omega^{\text{fr}}_{n;S^m} \xrightarrow{\sigma} \Omega^{\text{fr}}_{n;S^{m+1}} \xrightarrow{\sigma} \Omega^{\text{fr}}_{n;S^{m+2}} \xrightarrow{\sigma} \cdots \tag{1.72}$$

Representatives of these framed bordism groups are submanifolds of S^m. Write $S^m = \mathbb{A}^m \cup \{\infty\}$, and since each framed bordism class is represented by a framed submanifold $Y \subset \mathbb{A}^m$; we can arrange $\infty \notin Y$. This is the analog of passing from unpointed maps to pointed maps in (1.68). One can prove that (1.72) stabilizes using techniques used to prove the Whitney Embedding Theorem, namely affine projection onto lower dimensional affine spaces. The Freudenthal Suspension Theorem 1.70 follows, then, as a corollary of the Pontrjagin-Thom Theorem.

The stabilization maps σ in (1.72) have a geometric description. The stabilization sits $S^m \subset S^{m+1}$ as the equator and prepends the standard normal vector field $\partial/\partial x^1$ to the framing. The normal framing induces a *stable tangential framing* of Y, and the homotopy class of the stable tangential framing is unchanged under the stabilization map σ in the sequence (1.72). Conversely, if Y^n has a stable tangential framing, then by the Whitney embedding theorem we realize $Y \subset S^m$ as a submanifold for some m, and then we can pass from the stable tangential framing to a stable framing of the normal bundle, and so to an element of $\Omega^{\text{fr}}_{n;S^{m+k}}$ for some k. This sketch proves the following.

PROPOSITION 1.73. *The colimit of* (1.72) *is the bordism group* Ω^{fr}_n *of* n*-manifolds with a stable tangential framing.*

A bordism between two stably framed manifolds Y_0, Y_1 is, informally, a compact $(n+1)$-manifold X with boundary $Y_0 \amalg Y_1$ and a stable tangential framing of X which restricts on the boundary to the given stable tangential framings of Y_i. The formal definition follows Definition 1.1. Combining Theorem 1.65 and these stabilizations we deduce

COROLLARY 1.74 (Stable Pontrjagin-Thom). *There is an isomorphism*

$$\phi\colon \pi_n S^0 \longrightarrow \Omega^{\text{fr}}_n \tag{1.75}$$

for each $n \in \mathbb{Z}^{\geqslant 0}$.

The Pontrjagin-Thom theorem establishes a relationship between bordism groups and stable homotopy groups. It is a foundational result and we will make extensive use of several of its variations. In Lecture 6 we introduce a bordism theory of manifolds with a general stable flabby tangential structure. To close this lecture we sketch the basic idea behind Thom's extension of Theorem 1.65 in the simplest case of no tangential structure: unoriented manifolds.

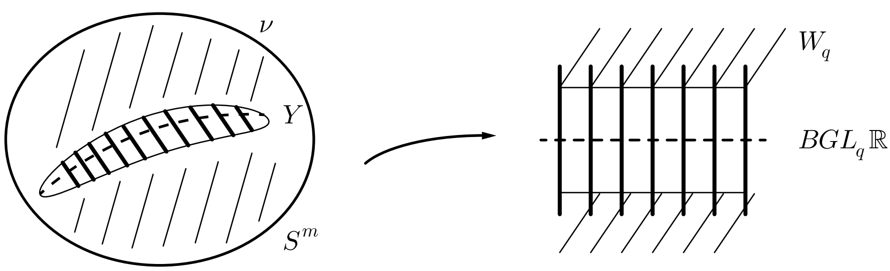

FIGURE 1.9. Pontrjagin-Thom collapse

Let $Y \subset S^m$ be a codimension q manifold. No framing of any sort is assumed, so we do not expect to construct Y as the inverse image of a point under a map to S^q. Rather, we classify its normal bundle $\nu \to Y$ by a map to the universal bundle (1.35):

$$
\begin{tikzcd}
\nu \arrow[r] \arrow[d] & W_q \arrow[d] \\
Y \arrow[r] & BGL_q\mathbb{R}
\end{tikzcd}
\tag{1.76}
$$

Now perform Pontrjagin-Thom collapse fiberwise in (1.76). The resulting pointed space is called the *Thom complex*, and composing with a further Pontrjagin-Thom collapse from S^m to the Pontrjagin-Thom collapse of a tubular neighborhood of Y, we obtain a map with domain S^m, as illustrated in Figure 1.9. In this construction Y appears as the transverse inverse image of the zero section of the universal bundle. The appropriate Pontrjagin-Thom theorem is an isomorphism of the bordism group Ω_n to the $(n+q)^{\text{th}}$ homotopy group of a Thom complex, at least in the stable limit $q \to \infty$.

LECTURE 2

Quantum Mechanics

In this lecture we turn from topology to physics. We begin with a description of an axiomatic framework for mechanics, both classical and quantum. We do not attempt a detailed development here, but include it to provide a mathematical lens for understanding examples. In the case of quantum mechanics these are often called the Dirac-von Neumann axioms [**D**, **vN1**]; the subsequent mathematical development was carried out by many mathematicians, including H. Weyl, I. Segal, and G. Mackey. We give a version of these axioms in §2.1. There is a substantial body of work on passing from classical systems to quantum—quantization—but we do not say anything about that here. (We encountered it in Lecture 1 in the discussion of finite gauge theory; see Remark 1.30.) A different set of axioms for quantum mechanics is discussed in [**Ka2**].

In §2.2 and §2.3 we give examples of quantum mechanical systems. The first is a continuous and the second discrete. The discrete system is called a *lattice system* and is the type of quantum mechanical system one often studies in condensed matter physics. In §2.4 we take up deformation classes of quantum systems, or *phases*. The problem which motivates these lectures is a class of classification problems for phases of quantum systems [**FH1**]. We return to it in Lecture 9. The important consequence of positivity of energy—*Wick rotation*—is the subject of §2.5. Finally, in §2.6 we indicate how Wick-rotated quantum mechanics fits with Axiom System 1.17 in the 1-dimensional case. The higher dimensional case, in the physical setting, is taken up in Lecture 3.

2.1. An axiomatic view of Hamiltonian mechanics

A *convex space* is a subset $\mathcal{S}$ of an affine space (over a topological vector space) with the property that if $\sigma_0, \dots, \sigma_k \in \mathcal{S}$ and $x^0, \dots, x^k \in \mathbb{R}^{>0}$ satisfy $x^0 + \cdots + x^k = 1$, then $x^0\sigma_0 + \cdots + x^k\sigma_k \in \mathcal{S}$. A point $\sigma \in \mathcal{S}$ is *extreme* if $\sigma = x\sigma' + (1-x)\sigma''$ for $0 \leqslant x \leqslant 1$ and $\sigma', \sigma'' \in \mathcal{S}$ implies $\sigma = \sigma'$ or $\sigma = \sigma''$. A *probability measure* on $\mathbb{R}$ is a positive measure of total measure 1 on the σ-algebra of Borel sets. The set of probability measures is denoted $\mathrm{Prob}(\mathbb{R})$. If $\mu \in \mathrm{Prob}(\mathbb{R})$, then its *expected value* is $\int_{\mathbb{R}} \lambda \, d\mu(\lambda)$. Let $\mathrm{Borel}(\mathbb{R})$ denote the space of real-valued Borel functions on $\mathbb{R}$.

Axiom System 2.1 (Mechanics). A *mechanical system* consists of the following data:

(1) A convex space $\mathcal{S}$ whose elements are called *states*. Extreme points are called *pure states* and comprise a subset denoted $\mathcal{S}_0$. Elements of $\mathcal{S} \backslash \mathcal{S}_0$ are often called *mixed states*.

(2) A complex topological vector space $\mathcal{O}$ with a real structure whose real elements $\mathcal{O}_{\mathbb{R}}$ are called *observables*. A dense subspace $\mathcal{O}^{\infty} \subset \mathcal{O}$ has the structure of a complex Lie algebra.[19]
(3) A choice of observable $H \in \mathcal{O}^{\infty}_{\mathbb{R}}$.
(4) A pairing $p\colon \mathcal{O}_{\mathbb{R}} \times \mathcal{S} \to \mathrm{Prob}(\mathbb{R})$; elements $A \in \mathcal{O}_{\mathbb{R}}$ and $\sigma \in \mathcal{S}$ pair to σ_A.
(5) A map $\mathrm{Borel}(\mathbb{R}) \times \mathcal{O}_{\mathbb{R}} \to \mathcal{O}_{\mathbb{R}}$; elements $f \in \mathrm{Borel}(\mathbb{R})$ and $A \in \mathcal{O}_{\mathbb{R}}$ pair to $f(A)$.
(6) A map from $\mathcal{O}^{\infty}_{\mathbb{R}}$ to one-parameter groups of automorphisms of $\mathcal{S}$ and of $\mathcal{O}$.

The data satisfy several axioms, some of which we list here. The pairing p separates states and observables: the induced maps $\mathcal{O} \to \mathrm{Map}\bigl(\mathcal{S}, \mathrm{Prob}(\mathbb{R})\bigr)$ and $\mathcal{S} \to \mathrm{Map}\bigl(\mathcal{O}, \mathrm{Prob}(\mathbb{R})\bigr)$ are injective. For $f \in \mathrm{Borel}(\mathbb{R})$, $A, A_1, A_2 \in \mathcal{O}$, $\sigma, \sigma_i \in \mathcal{S}$, and positive real numbers x^i summing to 1 we require

$$\begin{aligned} (x^i\sigma_i)_A &= x^i(\sigma_i)_A \\ \sigma_{A_1+A_2} &= \sigma_{A_1} * \sigma_{A_2} \\ \sigma_{f(A)} &= f_*\sigma_A, \end{aligned} \tag{2.2}$$

where '$*$' is the convolution product of measures and in the last equation we use the pushforward measure. The one-parameter groups $A^{(t)}$, $\sigma^{(t)}$ of automorphisms generated by $H \in \mathcal{O}^{\infty}_{\mathbb{R}}$ satisfy

$$\frac{dA^{(t)}}{dt} = -[H, A^{(t)}], \quad A \in \mathcal{O}^{\infty}, \quad t \in \mathbb{R}, \tag{2.3}$$

$${\sigma^{(t)}}_{A^{(t)}} = \sigma_A, \qquad \sigma \in \mathcal{S}, \quad A \in \mathcal{O}_{\mathbb{R}}, \quad t \in \mathbb{R}. \tag{2.4}$$

This is not a complete list of axioms, nor a definitive list, but it is representative of the structure. (For example, we have not commented on the topologies on states and observables.) A brief summary of the axioms appears at the beginning of [**Fa1**] and a more elaborate account is in [**Ma**, §2-2]. Other mathematical axiom systems for quantum mechanics may be found in [**Str**, §1.3] and [**Ta**, §2]. We make several remarks.

- There are dual points of view on the *time evolution* generated by an observable H. In the *Schrödinger picture* the states evolve $\sigma \mapsto \sigma^{(t)}$, and in the *Heisenberg picture* the observables evolve $A \mapsto A^{(t)}$. Equation (2.4) expresses the compatibility of these pictures.
- The probability measure σ_A models measurement of an observable A in a state σ. The *expectation value*

$$\langle A \rangle_\sigma = \int_{\mathbb{R}} \lambda \, d\sigma_A \tag{2.5}$$

  defines a real-valued pairing of observables and states, and it is also assumed to separate states and observables.
- The special observable $H \in \mathcal{O}^{\infty}_{\mathbb{R}}$ is called the *Hamiltonian*, or *energy operator*. Our sign convention [**DF2**] is that $-H$ generates time translation.
- No associative algebra structure is assumed on observables, only a Lie algebra structure.

[19]For $A_1, A_2 \in \mathcal{O}^{\infty}$ the real structure $A \mapsto A^*$ satisfies $[A_1, A_2]^* = [A_1^*, A_2^*]$.

- Often a state $\sigma \in \mathcal{S}$ is fixed as part of the data. It may be assumed invariant under the time-evolution $\sigma^{(t)} = \sigma$ in which case it is called a *vacuum state*.

Now we bring in time more explicitly and rework the axioms. Time is modeled as the standard Euclidean line $\mathbb{M}^1$; the notation anticipates Minkowski spacetime, which we introduce in Lecture 3. The inner product on the underlying group $\mathbb{R}$ of time translations models a clock: we can measure differences of time. No time orientation is assumed. The symmetry group of $\mathbb{M}^1$ is the Euclidean group of translations and reflections, the latter in this context called time-reversing symmetries. Not every mechanical system is time-reversal invariant.

Conceptually, it is useful to spread the observables over time as a complex vector bundle

$$\mathcal{O} \longrightarrow \mathbb{M}^1 \tag{2.6}$$

with real structure and a dense subbundle $\mathcal{O}^\infty \to \mathbb{M}^1$ with Lie algebra structure; this replaces Axiom (2). There is a trivialization so as to recover a single space of observables. Time evolution of observables in Axiom (6) is a lift of the translation action of $\mathbb{R}$ on $\mathbb{M}^1$ to (2.6). The pairing p in Axiom (4) is replaced by the *correlation functions*. Let $t_1 < \cdots < t_\ell$ be a finite sequence of times in $\mathbb{M}^1$ and $A_{t_i} \in \mathcal{O}_{t_i}$ observables at those times. Fix a state $\sigma \in \mathcal{S}$. Then the correlation function in the state σ is denoted

$$\langle A_{t_\ell} \cdots A_{t_1} \rangle_\sigma \in \mathbb{C}. \tag{2.7}$$

Physical quantities of interest can usually be extracted from correlation functions.

Remark 2.8. We do not expect (2.7) to be defined in the generality stated here: for example, in quantum mechanics we would have to multiply unbounded operators. For a single real observable we expect to recover the pairing p of Axiom (4); see [**Str**, §2.4] for the case of quantum mechanics using the C^*-algebra perspective.

In the next two (meta) examples we illustrate how both classical and quantum mechanics fit into Axiom System 2.1.

Example 2.9 (Classical mechanics). The data which defines a classical mechanical system is a symplectic manifold $(\mathcal{N}, \omega)$ called the *phase space*, and a smooth function $H\colon \mathcal{N} \to \mathbb{R}$, the Hamiltonian. For a particle moving on a smooth manifold M (see §2.2) the phase space is TM, the space of positions and velocities of the particle. The space of states $\mathcal{S} = \mathrm{Prob}(\mathcal{N})$ is then the set of probability measures on $\mathcal{N}$. The subspace of pure states are the point measures, which can be identified with $\mathcal{S}_0 = \mathcal{N}$. The mixed states—probability measures not supported on a point—are the states of classical statistical mechanics. The complex vector space of Borel functions $\mathcal{O} = \mathrm{Borel}_{\mathbb{C}}(\mathcal{N})$ on the phase space has real points the usual observables. The dense subspace $\mathcal{O}^\infty = C^\infty(\mathcal{N})$ of smooth functions carries the usual Poisson bracket of symplectic geometry, which makes $\mathcal{O}^\infty$ into a complex Lie algebra. If $A\colon \mathcal{N} \to \mathbb{R}$ is a real Borel function and $\sigma \in \mathrm{Prob}(\mathcal{N})$ a probability measure, then the pushforward measure $\sigma_A = A_*\sigma$ on $\mathbb{R}$ tells the probability distribution of measurements of A in the state σ. If $\sigma \in \mathcal{O}_0$ is a point measure supported at $m \in \mathcal{N}$—a non-probabilistic state of the system—then σ_A is a point measure supported at $A(m)$ and the observable A has a definite value. Functions of observables in Axiom (5) are formed by composition. The smooth function $H\colon \mathcal{N} \to \mathbb{R}$ has a symplectic gradient vector field ξ_H which generates[20] a flow $\varphi_t\colon \mathcal{N} \to \mathcal{N}$. This induces a 1-parameter group of automorphisms of probability measures ($\mathcal{S}$) and Borel functions ($\mathcal{O}$) by pushforward and pullback, respectively.

To bring time into the picture we define $\mathcal{N}'$ as the manifold of trajectories $\mathbb{M}^1 \to \mathcal{N}$ of ξ_H. At each time $t \in \mathbb{M}^1$ we obtain a diffeomorphism $\mathcal{N}' \to \mathcal{N}$ by evaluation of a trajectory at time t. Let the space of states be $\mathcal{S} = \mathrm{Prob}(\mathcal{N}')$. Define the vector space of observables $\mathcal{O}_t$ at t to consist of Borel functions on $\mathcal{N}'$ which depend only on the germ of a trajectory at t. Use the Hamiltonian equation of motion—that trajectories are integral curves of ξ_H—to see that we need only consider functions of the value (0-jet) of the trajectory at t. To define the correlation function (2.7), set $A = A_{t_\ell} \cdots A_{t_1} : \mathcal{N}' \to \mathbb{R}$ to be the product function, and compute the expectation value (2.5) of the pushforward measure $A_*\sigma \in \mathrm{Prob}(\mathbb{R})$. Note that the correlation function multiplies observables, and it is in this form that an algebra structure on observables appears in Axiom System 2.1. We say more about algebra structures in §5.5.

Remark 2.10. There is a composition law on classical mechanical systems which combines them without interaction. Let $(\mathcal{N}_i, H_i)$, $i = 1, 2$, be two symplectic manifolds with chosen Hamiltonian functions. The composite system has symplectic manifold $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$ and Hamiltonian function $\pi_1^* H_1 + \pi_2^* H_2$, where $\pi_i\colon \mathcal{N} \to \mathcal{N}_i$ is projection.

Example 2.11 (Quantum mechanics). The data which defines a quantum mechanical system is a complex separable Hilbert space $\mathcal{H}$ and a self-adjoint operator H.[21] The Hilbert space $\mathcal{H}$ can be finite or countably infinite dimensional. The space of states $\mathcal{S}$ is the convex set of trace class nonnegative self-adjoint operators $\sigma\colon \mathcal{H} \to \mathcal{H}$ with $\mathrm{Tr}(\sigma) = 1$. A pure state is an orthogonal projection operator onto a 1-dimensional subspace $\ell \subset \mathcal{H}$; the space $\mathcal{S}_0 = \mathbb{P}\mathcal{H}$ of pure states may be identified with the projective space of $\mathcal{H}$. The observables in $\mathcal{O}$ are (densely defined) operators on $\mathcal{H}$, the real structure is $A \mapsto A^*$, and the real points are self-adjoint operators.[22] We allow unbounded operators and do not delve into the pursuant technicalities here. The spectral theorem associates a projection-valued measure π_A on $\mathbb{R}$ to a self-adjoint operator A on $\mathcal{H}$. Given a state σ the probability measure on $\mathbb{R}$ associated to A is the measure $E \mapsto \mathrm{Tr}\big(\pi_A(E) \circ \sigma\big)$, where $E \subset \mathbb{R}$ is a Borel subset. For a Borel function f the spectral theorem also gives a definition of $f(A)$. The self-adjoint operator H generates a 1-parameter group of unitary automorphisms

$$t \longmapsto U_t = \exp(-itH/\hbar) \tag{2.12}$$

of $\mathcal{H}$. (We include Planck's constant $\hbar$ so that the exponent is dimensionless, since H has units of energy.) The 1-parameter group (2.12) induces motions on states

[20]In general, there is only a local flow: we should restrict to functions which generate global flows.

[21]In fact, we should take $\mathcal{H}$ to be $\mathbb{Z}/2\mathbb{Z}$-graded (§1.4): lines in the even subspace are pure *bosonic* states and lines in the subspace of odd elements are pure *fermionic* states. For simplicity we omit the $\mathbb{Z}/2\mathbb{Z}$ grading here.

[22]It is mathematically more natural to use skew-adjoint operators [**DF2**], but for this exposition we don't bother.

and observables by conjugation. The Poisson bracket on (bracket-able) observables is the normalized operator commutator

$$[A_1, A_2] = \frac{-i}{\hbar}\,(A_1 \circ A_2 - A_2 \circ A_1). \tag{2.13}$$

REMARK 2.14. There is a composition law on quantum mechanical systems which combines them without interaction. Let $\big(\mathcal{H}_i, (U_i)_t\big)$, $i = 1, 2$, be two quantum mechanical systems. The composite system has Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and unitary time evolution $(U_1)_t \otimes (U_2)_t$.

REMARK 2.15. There is a basic dichotomy for quantum mechanical systems: *gapped* vs. *gapless*. In a gapped system the spectrum of the Hamiltonian is bounded below, the minimum is contained in the point spectrum, and there is a spectral gap above this lowest eigenvalue. If these conditions do not hold, the system is termed gapless. The low energy behavior of gapped systems is quite different from that of gapless ones.

REMARK 2.16. Let M, L, and T denote mass, length, and time. Then Planck's constant $\hbar$ and the speed of light c have dimensions

$$[\hbar] = \frac{ML^2}{T}, \qquad [c] = \frac{L}{T}. \tag{2.17}$$

In a quantum mechanical system $\hbar$ identifies energy (ML^2/T^2) with inverse time. So low energy is equivalent to long time. In a relativistic system c identifies length and time, so long time is long distance. We use the term 'long range' for either (and for both in a relativistic system). In a relativistic quantum system both $\hbar$ and c are in play, and we use $\hbar/c$ to identify mass with inverse length, so too inverse time, so too energy.

2.2. Example: particle on a manifold

Fix a Riemannian manifold (M, g) of dimension d. The mechanical system we consider models a particle moving on the manifold M, but variations of interpretation are possible. For example, if $M = (\mathbb{E}^3)^{\times N}$ is the Cartesian product of N copies of Euclidean space, then we use it to model N particles moving in space. There is a dichotomy in the behavior of the model depending if M is compact or not, but the formal setup is the same. Assume that the Riemannian manifold M is complete.

Axiom System 2.1 captures the *Hamiltonian* view of mechanics. For many physical systems there is a more powerful *Lagrangian* formulation from which the Hamiltonian data can be derived. We will not elaborate on that in these lectures, but remark that Example 1.23 and Example 1.44 are instances of the passage from Lagrangian to Hamiltonian in *quantum* theory using the Feynman path integral. For a detailed development of this passage in *classical* field theory, see [**DF1**]. The main object, the *Lagrangian density*, for a particle on M is $1/2\,|d\phi|^2\,|dt|$, where $\phi\colon \mathbb{R} \to M$ is a function which describes the trajectory of a particle as a function of (affine) time. The Euler-Lagrange equations are then used to derive the moduli space $\mathcal{N}'$ of classical trajectories as a symplectic manifold and also a Hamiltonian function $H\colon \mathcal{N}' \to \mathbb{R}$. Recall from Example 2.9 that from these two pieces of data we can define states and observables, their pairing to probability distributions, etc. For the particle on M (with zero potential energy function) we find $\mathcal{N}'$ is the space of geodesic motions on M. If we fix a time $t_0 \in \mathbb{R}$, then evaluation of the position and velocity of a geodesic $\phi\colon \mathbb{R} \to M$ at t_0 is a diffeomorphism $\mathcal{N}' \cong TM$. Under that diffeomorphism the symplectic structure becomes the pullback of the canonical symplectic form on T^*M under the isomorphism $TM \to T^*M$ determined by the Riemannian metric. The Hamiltonian function $H(\xi) = 1/2\,|\xi|^2$ generates the geodesic flow on TM.

The passage from the Lagrangian formulation to the quantum mechanical system (Example 2.11) is achieved via canonical, or geometric, quantization and the Feynman path integral. In this case we integrate over the infinite dimensional space of paths ϕ. Since it is integrated out the (fluctuating) field ϕ does not appear in the description. When we come to field theory this procedure does not have well-defined mathematical underpinnings in the generality needed, but for paths there is a rigorous theory of Wiener measure, or stochastic integration [**Ku**]. For the particle on M it is straightforward, at least heuristically, to derive the Hilbert space $\mathcal{H}$ and self-adjoint Hamiltonian operator H. Namely, $\mathcal{H} = L^2(M, \mu_g)$ is the space of complex L^2 functions with respect to the Riemannian measure, and $H = 1/2\,\Delta_g$ is the (unbounded) Hodge Laplace operator. The dichotomy we flagged in the first paragraph of this section can be made more concrete. If M is compact, then H has discrete spectrum. In particular, the minimum eigenvalue is 0 and there is a gap in the spectrum between 0 and the first nonzero eigenvalue. By contrast, if M is not compact then there can be continuous spectrum and perhaps no spectral gap. Again the spectrum is bounded below by 0, but if M has infinite volume then 0 is not an eigenvalue but rather is part of the continuous spectrum. For example, if $M = \mathbb{E}^d$ is Euclidean space then the spectrum of the (classical) Laplacian is continuous and is the subset $\mathbb{R}^{\geqslant 0}$ of $\mathbb{R}$.

Exercise 2.18. Let $V\colon M \to \mathbb{R}$ be a *potential energy* function. Work out the modification to both the classical and quantum models including the function V. How does the model behave in case $M = \mathbb{E}^1$ and the function V is quadratic? How does the sign of the quadratic term affect your answer?

Exercise 2.19. Let M be the circle of length $L > 0$, which we can write as $\mathbb{R}/L\mathbb{Z}$. Set $V = 0$.

(1) Work out the spectral decomposition of the Hamiltonian: eigenfunctions and eigenvalues.
(2) The orthogonal group O_2 acts by isometries on M so induces an action on the quantum mechanical system. How does that action appear relative to the spectral decomposition in (1)?

2.3. Example: a lattice system (toric code)

One often approximates a field theory by a lattice model. Conversely, lattice models are often the starting point in condensed matter physics, and there are field theory approximations. 'Lattice' refers to a combinatorial approximation to a manifold. Intuitively, the combinatorial structure replaces a Riemannian metric, which is a typical geometric structure on space in other quantum systems. Many models are *local*, so can be formulated on arbitrary d-manifolds for some positive integer d, the dimension of space. On a compact manifold the Hilbert space of states is usually finite dimensional. We do not attempt a mathematically precise formulation of lattice models in general, nor of their approximation at low energies by a field theory, but here give a geometric example in which the low energy field theory limit can easily be deduced. This particular model, called the *toric code*, is due to Kitaev [**Ki1**]. Lattice gauge theories were first introduced by Wegner [**We**]. For a twisted version of the toric code, see [**FHa**, **Deb**].

Let Y be a closed d-manifold with finite CW structure, for example from a Morse function. Let Y^i denote the i-skeleton and Δ^i the finite set of i-cells. Recall from Example 1.23 that $\mathrm{Bun}_G(Y)$ is the groupoid of principal G-bundles on Y. Let $\mathrm{Bun}_G(Y^1, Y^0)$ denote the groupoid of pairs (P, s) consisting of a principal G-bundle $P \to Y^1$ over the 1-skeleton together with a trivialization s of its restriction to the 0-skeleton. This groupoid is (equivalent to) a discrete groupoid; parallel transport along (oriented) 1-cells defines an equivalence

$$\mathrm{Bun}_G(Y^1, Y^0) \approx \mathrm{Map}(\Delta^1, G) \approx \bigtimes_{e \in \Delta^1} G \tag{2.20}$$

to the indicated set. Define the Hilbert space

$$\mathcal{H} = \mathrm{Map}\big(\mathrm{Bun}_G(Y^1, Y^0), \mathbb{C}\big) \cong \bigotimes_{e \in \Delta^1} \mathrm{Map}(G, \mathbb{C}) \tag{2.21}$$

of functions on this discrete groupoid. For the toric code we specialize to $G = \mathbb{Z}/2\mathbb{Z}$. The Hamiltonian is a sum of local commuting operators, one for each vertex and one for each 2-cell. For $v \in \Delta^0$ define a permutation φ_v of (2.20) by $(P, s) \mapsto (P, s_v)$, where

$$s_v(v') = \begin{cases} s(v'), & v' \neq v; \\ s(v) + 1, & v' = v, \end{cases} \tag{2.22}$$

and for $\psi \in \mathcal{H}$ set

$$H_v \psi = \frac{1}{2}(\psi - \varphi_v^* \psi). \tag{2.23}$$

The eigenvalues of H_v are 0 and 1; the eigenspaces the functions invariant/anti-invariant under changing the trivialization at v. For $f \in \Delta^2$ let

$$(H_f \psi)(P, s) = \mathrm{hol}_{\partial f}(P) \cdot \psi(P, s) \tag{2.24}$$

be multiplication by the holonomy $\mathrm{hol} \in \{0, 1\}$ around the boundary of f. Again the eigenvalues are 0 and 1; the kernel consists of bundles with trivial holonomy around ∂f. The operators H_v and H_f commute. Define the Hamiltonian

$$H = \sum_{v \in \Delta^0} H_v + \sum_{f \in \Delta^2} H_f. \tag{2.25}$$

Then $H \geqslant 0$ and the kernel of H is isomorphic to

$$\mathcal{H}_0 = \mathrm{Map}\big(\mathrm{Bun}_{\mathbb{Z}/2\mathbb{Z}}(Y), \mathbb{C}\big) \tag{2.26}$$

the space of functions on the stack of $\mathbb{Z}/2\mathbb{Z}$-bundles on Y. Note that this is precisely the state space $\mathcal{G}_G(Y)$ of the $(d+1)$-dimensional $\mathbb{Z}/2\mathbb{Z}$-gauge theory; see (1.26).

REMARK 2.27. This is an example in which we see very explicitly that the low energy sector is modeled as a topological field theory, at least as far as state spaces go. This model is *gapped* in the sense that there is a spectral gap above the minimum eigenvalue which is *uniform* over all CW structures. Here the spectrum is contained in $\mathbb{Z}^{\geqslant 0}$, so the gap is manifest. For other examples worked out in detail, see [**DeGu**, **Deb**].

2.4. Families of quantum systems

To define a good *moduli space*[23] $\mathcal{M}$ in geometry we usually need to (i) fix discrete data, and (ii) remove a locus Δ of "singular" objects. The set $\pi_0\mathcal{M}$ of path components of the moduli space is, by definition, the set of *deformation classes* of geometric objects under study. A deformation class consists of qualitatively similar objects. A general problem is to determine invariants which separate points of $\pi_0\mathcal{M}$.

EXAMPLE 2.28 (Symmetric bilinear forms). Fix a vector space V of dimension 2. Then $\operatorname{Sym}^2 V^*$ is the 3-dimensional vector space of symmetric bilinear forms on V. Let Δ denote the locus of degenerate forms, a quadratic cone in $\operatorname{Sym}^2 V^*$, and let $\mathcal{M} = \operatorname{Sym}^2 V^* \backslash \Delta$ be the space of nondegenerate forms. There are 3 components and the signature

$$\operatorname{Sign}\colon \pi_0\mathcal{M} \longrightarrow \{-2, 0, 2\} \tag{2.29}$$

is a complete invariant.

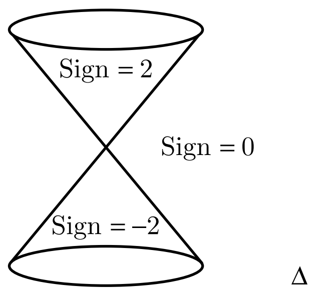

FIGURE 2.1. Nondegenerate symmetric bilinear forms in dimension 2

EXERCISE 2.30. The proper moduli space here is the set of equivalence classes of pairs (V, B) of a 2-dimensional vector space and a symmetric bilinear form, with degenerate forms removed. What is the picture?

EXAMPLE 2.31 (Complex surfaces). Moduli spaces are prevalent in algebraic geometry: moduli spaces of curves, moduli spaces of abelian varieties, moduli spaces of vector bundles, etc. As a concrete example, consider complex surfaces of general type. There are two numerical invariants to fix: the Euler number and the signature. Then there is a moduli space of surfaces of general type with fixed numerical invariants [**Ca**]. It has finitely many components, so there are finitely many deformation types. Deformation equivalent surfaces are diffeomorphic, but the converse is false.

We consider continuous families of quantum mechanical systems parametrized by a space. Care is needed with the topologies to arrive at the correct notion; see [**FM1**, Appendix D]. We can formulate a moduli problem by fixing discrete parameters, and indeed one of the main problems we address in these lectures is of that type.[24] For now the following picture is a cartoon, not rigorous mathematics.

[23] More precisely, moduli stack

[24] The problem lies in field theory, and we use a rigorous framework in which to address it. The application to lattice systems proceeds via heuristic leaps of faith.

Let $\mathcal{X}$ be the moduli space (really stack) of quantum theories with fixed discrete parameters, and $\Delta \subset \mathcal{X}$ the locus of *phase transitions*. A phase transition has an *order*, a positive integer, and they divide according as the order $= 1$ or the order > 1. At an order 1 phase transition the quantum system is still gapped, but some part of the discrete spectrum of nearby systems comes down to the minimum, so there is a discontinuity in the dimension of the vacuum space (eigenspace for the minimal eigenvalue). By contrast, at a phase transition of order > 1 a part of the continuous spectrum has come down to zero. The complement of Δ may contain gapless systems, which for our purposes we throw out. Then path components of $(\mathcal{X}\backslash\Delta)_{\text{gapped}}$ are called (gapped or topological) *phases*.

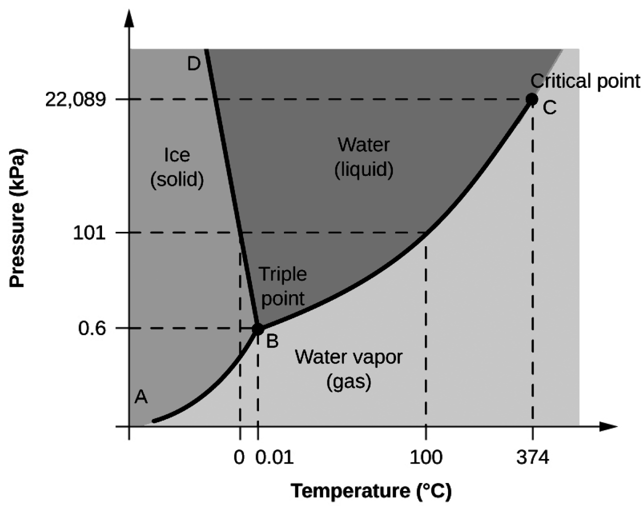

FIGURE 2.2. Phase diagram of water

EXAMPLE 2.32. The typical example is the phase diagram of H_2O. There the parameters are Temperature and Pressure; see Figure 2.2. The solid black line is the locus Δ of phase transitions. They are all first order except for the critical point C. At room pressure, labeled '101' in the figure, we see the usual 3 phases of solid, liquid, and gas. But in the two dimensional family there are only two phases: there is a path connecting liquid and gas at high temperature.

The classification of *topological phases of matter* is a problem of current interest [**Nob**, **Cas**]. The discrete parameters to be fixed are the dimension d of space and the (internal) symmetry group I. Then we can imagine, but not define, a moduli space $\mathcal{M} = \mathcal{M}(d, I)$ of gapped systems after removing the locus of phase transitions. It is a very interesting question to rigorously define lattice systems and moduli spaces of lattice systems. For now we simply imagine $\mathcal{M}$ exists. A point $m \in \mathcal{M}$ represents an isomorphism class of lattice systems, and the *phase of* m is its component in $\pi_0\mathcal{M}$.

PROBLEM 2.33. Compute $\pi_0\mathcal{M}(d, I)$.

We emphasize that this is not a well-defined mathematical problem, so we invoke two physical principles to transform Problem 2.33 into a well-defined mathematical problem:

(1) the deformation class of a quantum system is determined by its low energy behavior;
(2) the low energy physics of a *gapped* system is well-approximated by a *topological*[25] field theory.

See [**G**] for some discussion of these physical principles. In Lecture 8 we prove classification theorems for certain field theories and apply them to Problem 2.33 using these principles. In other words, we model the question of classification of phases as a classification problem in field theory. Our attack on that problem is enabled by the existence of an Axiom System for field theory, which we discuss in Lecture 3.

REMARK 2.34. Let $\mathcal{M}'(d,I)$ be the moduli space of effective field theories with the given discrete parameters. The physical principles suggest that once $\mathcal{M}(d,I)$ is defined there is a map

$$\pi_0\mathcal{M}(d,I) \longrightarrow \pi_0\mathcal{M}'(d,I). \tag{2.35}$$

We do not expect a map $\mathcal{M}(d,I) \to \mathcal{M}'(d,I)$; there may be choices in writing a low energy approximation. But we can imagine that those choices[26] do not affect the deformation class of the effective field theory, so expect that the homotopy class of (2.35) is well-defined. Then (2) implies that the map (2.35) is injective. We will see in Lecture 10 that for our particular problem the computations we make of $\pi_0\mathcal{M}'$ agree with the deductions of $\pi_0\mathcal{M}$ in the physics literature, which provides evidence that, at least for this problem, the map (2.35) is bijective. We revisit this discussion in §9.1.

REMARK 2.36. In Riemannian geometry there is an illuminating analog of (1). Fix a dimension $d \in \mathbb{Z}^{\geqslant 0}$ and imagine a moduli space $\mathcal{M}_d$ of compact Riemannian manifolds (M,g) of dimension d. Let $\Delta_g^{(p)}$ denote the Laplace operator on p-forms. For each p this is analogous to a family of quantum mechanical systems—a family of self-adjoint operators over $\mathcal{M}_d$. Further, these systems are gapped since the Laplace operator has discrete spectrum. Now the deformation classes $\pi_0\mathcal{M}_d$ are diffeomorphism types of compact d-manifolds. The zero eigenspace consists of harmonic forms, and by the Hodge and de Rham theorems it has topological significance: its dimension is constant on components of $\mathcal{M}_d$ and equals the p^{th} Betti number of the underlying smooth manifold. These Betti numbers are not complete invariants of the diffeomorphism type, of course, but it is the only topological information in the spectrum.

REMARK 2.37. Consider the special case $d = 1$. Observe that if we allow noncompact manifolds, then there is a smooth path of complete Riemannian 1-manifolds which connects a single circle to two circles: elongate a circle to an ellipse to two lines and then each line to a circle. Similarly, if we restrict to compact spaces but allow simple singularities as in a figure eight, then we can connect a single circle to two circles (by passing through the figure eight). If we consider the spectrum of the Laplace operator along this path, then the first transition from one to two circles is analogous to a higher order phase transition whereas the second, through compact spaces, is analogous to a first order phase transition.

[25] In fact, the low energy effective field theory may be mildly non-topological, as we explain in Lecture 9.

[26] for example, of local terms in an effective lagrangian

2.5. Wick rotation in quantum mechanics

Let $(\mathcal{H}, U_t)$ be a quantum mechanical system, where the unitary evolution

$$U_t = e^{-itH/\hbar} \tag{2.38}$$

is the one-parameter group generated by a self-adjoint operator H on $\mathcal{H}$. If we assume that the energy operator H is nonnegative, then there is an "analytic continuation" of the system in the sense that the real time physics appears as the boundary value of a holomorphic theory on the lower half plane of $t \in \mathbb{C}$ with negative imaginary part. This applies to both the group of time evolution as well as the correlation functions, which are functions of *affine* time. To extend (2.38) to complex values of t, observe that if $\lambda \in \mathbb{R}^{\geqslant 0}$ then the function $t \mapsto e^{-i\lambda t}$, $t \in \mathbb{R}$, is the boundary value of a *bounded* holomorphic function on

$$\mathbb{C}_- = \bigl(t \in \mathbb{C} : \operatorname{Im} t < 0\bigr); \tag{2.39}$$

if $\lambda > 0$ then this function takes values in the open unit disk in $\mathbb{C}$. Hence if H is a nonnegative self-adjoint operator, the spectral theorem defines a *holomorphic* semigroup

$$t \longmapsto e^{-itH/\hbar}, \qquad t \in \mathbb{C}_-, \tag{2.40}$$

of bounded operators. The unitary evolution (2.38) appears on the boundary $\mathbb{R}$ of $\mathbb{C}_-$. Wick rotation is the restriction of this holomorphic semigroup to $t = -i\tau$ for positive real τ:

$$\tau \longmapsto e^{-\tau H/\hbar}, \qquad \tau \in \mathbb{R}^{>0}. \tag{2.41}$$

This is a *real* semigroup of bounded self-adjoint operators. The unitary group (2.38) can be reconstructed from the real semigroup (2.41).

2.6. The Axiom System in quantum mechanics

Wick-rotated quantum mechanics can be formulated as an $n = 1$ version of Axiom System 1.17, but with two important modifications:

- the 1-dimensional bordisms have a Riemannian metric;
- the codomain is an appropriate category of *topological* vector spaces [**Se2**, §1.4], [**C1**, Appendix 2], [**HST**, §6.1].

There is another technical modification: the objects are oriented 0-manifolds embedded in a germ of an oriented 1-dimensional Riemannian manifold. The Wick-rotated quantum mechanical system associated to $(\mathcal{H}, H)$ assigns $\mathcal{H}$ to a positively oriented point embedded in a symmetric germ, say the germ of $(-\epsilon, \epsilon) \subset \mathbb{E}^1$ for $\epsilon > 0$. (If the germ is not symmetric we expect a more complicated topological vector space [**Se3**, Lecture 2].) To the oriented closed interval X_τ of length $\tau > 0$ with one incoming and one outgoing boundary we assign the operator $e^{-\tau H/\hbar}$. This incorporates the basic structure. Other bordisms must be included to obtain a fully defined functor.

Remark 2.42. The circle S^1 of length $\tau > 0$ should map to $\operatorname{Tr}(e^{-\tau H/\hbar})$, but that operator need not be trace class in general. For a particle moving on a manifold M (§2.2), it is trace class if M is compact, but it need not be if M is not compact, e.g., $M = \mathbb{E}^d$.

FIGURE 2.3. Two geometric 1-dimensional bordisms

REMARK 2.43. There are more exotic bordisms, such as the oriented intervals depicted in Figure 2.3. The first, with both boundary components incoming, evaluates to the bilinear map

$$\begin{aligned} \mathcal{H} \times \mathcal{H} &\longrightarrow \mathbb{C} \\ \psi_1, \psi_2 &\longmapsto \langle \bar{\psi}_1,\, e^{-\tau H/\hbar}\psi_2 \rangle_{\mathcal{H}}, \end{aligned} \tag{2.44}$$

while the second need not be defined in the general (noncompact) case.

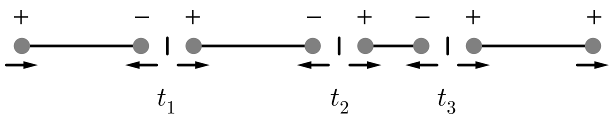

FIGURE 2.4. Wick-rotated correlation function

There is a notion of a "noncompact field theory", defined on bordisms with nonempty outgoing boundary on each component, to accommodate the issues raised in Remark 2.42 and Remark 2.43. This has been defined (in arbitrary dimensions—quantum mechanics is 1-dimensional in this sense) by Graeme Segal [**Se4**] and also by Kevin Costello in [**C2**].

REMARK 2.45. The vector space of observables at a point $x \in X$ in an oriented Riemannian 1-manifold X is the inverse limit as $\epsilon \to 0$ of the vector spaces attached to $\partial B_\epsilon(x)$, the boundary of the interval of radius $\epsilon > 0$ about x (with its induced germ); see [**Se3**, Lecture 2]. The Wick rotation of the correlation function (2.7) is then computed using Figure 2.4.

EXAMPLE 2.46 (Parallel transport as a quantum mechanical system). We conclude with an example of an *invertible*, but not topological, 1-dimensional field theory. It can be viewed as the Wick rotation of a quantum mechanical system with a single state. Let M be an auxiliary manifold and $L \to M$ a complex line bundle with covariant derivative. The oriented field theory α_k, $k \in \mathbb{Z}$, is defined on the bordism category of oriented 0- and 1-manifolds equipped with a smooth map to M. The value on a 0-manifold does not depend on a germ. There are no Riemannian metrics. Set

$$\begin{aligned} \alpha_k(Y, \phi\colon Y \to M) &= \bigotimes_{y \in Y} L_{\phi(y)}^{\otimes k}, && Y \text{ a 0-manifold}, \\ \alpha_k(X, \phi\colon X \to M) &= \big(\rho^{\otimes k}\colon L_{\phi(0)}^{\otimes k} \longrightarrow L_{\phi(1)}^{\otimes k}\big), && X = [0,1], \end{aligned} \tag{2.47}$$

where the bordism $X\colon \{0\} \to \{1\}$ has $\{0\}$ incoming and $\{1\}$ outgoing; ρ is the parallel transport on $\gamma^* L \to [0,1]$. The value of α_k on other bordisms is easily deduced. Of course, there is a variation which uses principal $\mathbb{T}$-bundles in place of complex line bundles. Here $\mathbb{T} \subset \mathbb{C}$ is the group of unit norm complex numbers.

EXERCISE 2.48. Find computations in the theory α_k which recover the integer k. (Hint: You may want to evaluate the theory on *families* of manifolds.[27].) You may regard k as a topological invariant of this non-topological theory; more precisely, it is a deformation invariant.

[27]It is important that the Axiom System include evaluations in families, thought we do not emphasize that in these lectures; see [**ST**, §2]

LECTURE 3

Wick-Rotated Quantum Field Theory and Symmetry

In Lecture 1 we saw how the categorification of bordism invariants, such as the signature of a closed oriented manifold, leads to the notion of a topological field theory (Axiom System 1.17). In Lecture 2 we saw how the state space and correlation functions of a quantum mechanical system, after Wick rotation, form a geometric analog of a 1-dimensional topological field theory (§2.6). It is a simple matter to combine these in the n-dimensional case ($n \in \mathbb{Z}^{\geqslant 1}$) to deduce the outlines of an Axiom System for Wick-rotated quantum field theory. This type of Axiom System was introduced in the 1980s by Graeme Segal [**Se1**] in the case of 2-dimensional conformal field theory, and since then he has advocated its more general use in quantum field theory. We do not attempt a formal definition, but rather tell the structure informally and point out some pitfalls. We present this as motivation for this lecture series as we will not delve deeply into non-topological field theories which are not invertible. We refer the reader to [**Se3**, **ST**] for further material. This approach is the quantum field theory version of the Schrödinger picture of quantum mechanics; the earlier axioms of Wightman [**SW**, **K**] depict quantum field theory on flat (non-Wick-rotated) spacetime. There is a Heisenberg picture analog as well. In flat spacetime it goes by the name 'algebraic quantum field theory' [**Ha**], and there is a modern geometric version—factorization algebras [**CG**]—which also has its roots in 2-dimensional conformal field theory [**BeDr**]. (For topological versions of factorization algebras, see [**L1**, §4.1], [**AFR**].)

We begin in §3.1 with an informal discussion of the Axiom System obtained by combining topological field theory and Wick-rotated quantum mechanics. Then in §3.2 and 3.3 we tell more systematically how to arrive at Axiom System 3.1 from the traditional starting point of relativistic quantum field theory. One of the main tasks in this lecture is to describe symmetry in quantum field theory, culminating in Definition 3.27 of the *symmetry type* of a theory. Wick rotation brings us to a *compact* Lie group[28] of symmetries, and the rigidity of compact Lie groups can be applied to deduce structure theorems. In particular, the Stabilization Theorem 3.24 plays an important role in these lectures when we come to stable homotopy theory in Lecture 6. The passage from Euclidean field theory, which is our landing point after Wick rotation, to curved manifolds and the Axiom System 3.1, is very much the passage in geometry from Klein to Cartan, as we explain in §3.5.

[28]There are theories whose symmetry group is a noncompact Lie group (the easiest example is a free massless real scalar field), a supergroup (any supersymmetric theory), a higher group [**GKSW**], or a mix of these. In these lectures we restrict to the simpler case of compact Lie groups of symmetries.

Two final points: (1) It is advantageous when contemplating any quantum field theory to identify its symmetry type (H,ρ). (2) The most radical step in Axiom System 3.1 is the restriction to *compact* manifolds, especially since we want to capture the *long range* behavior of a theory. It is perhaps not clear at first that compact manifolds retain that information; the explicit computations in Lecture 10 are evidence that they do.

3.1. Axiom System for quantum field theory

As in Axiom System 1.17 a (Wick-rotated, non-topological) field theory is a symmetric monoidal functor from a bordism category to a complex linear category. The domain bordism category consists of smooth manifolds with a specified n-dimensional geometric structure. The objects are closed $(n-1)$-manifolds embedded in a germ of an n-manifold with that structure; the morphisms are n-dimensional bordisms. The geometric structure can include, among many other possibilities: (i) topological structures as in §1.3; (ii) a Riemannian metric, a conformal structure, a connection on a principal bundle; (iii) a map to a fixed manifold M, a section of a fiber bundle; (iv) etc. The mathematical theory of geometric bordism categories is developed in Ayala's thesis [**Ay**]. Physicists call these geometric structures *background fields*, and a very general notion of field is used. There is a mathematical formulation of a field [**FT1**, Appendix A] which encodes the local nature of a field in terms of sheaves. In §3.6 we discuss a restricted class of geometric structures specified by a Lie group, the 'H-structures'[29] of Cartan; we gave a flabby topological analog in Definition 1.40. As for the codomain category, it is as in quantum mechanics (§2.6): a suitable symmetric monoidal category $t\mathrm{Vect}_{\mathbb{C}}$ of topological vector spaces with tensor product. To the extent that we have not formalized these definitions here, the following is informal.

Axiom System 3.1 (Segal). A *field theory* is a symmetric monoidal functor

$$F\colon\ \mathrm{Bord}_{\langle n-1,n\rangle}(\mathfrak{X}_n^{\nabla}) \longrightarrow t\mathrm{Vect}_{\mathbb{C}}\,. \tag{3.2}$$

The schematic notation $\mathfrak{X}_n^{\nabla}$ evokes a geometric structure of the type described above. We have already given several topological examples of this structure; some of the simplest non-topological examples are described in 4.3; Example 2.46 is even simpler. Notice that we do not use the adjectives 'quantum' and 'classical'. One possibility is to say that a field theory is classical if it is invertible, and it is quantum otherwise, but this is not terminology the author is ready to adopt since invertible field theories appear in several contexts outside of (semi)classical field theory.

Axiom System 3.1 encodes states, observables, and correlation functions:

- As already stated the geometric structure consists of *background* fields in the physics terminology. Quantum field theories are often described in terms of both background and *fluctuating* fields; the fluctuating fields are integrated out using the Feynman path integral. (See Example 1.23, Example 1.44 for topological cases.) So we can view Axiom System 3.1 as encoding the structure one obtains from performing the Feynman path integral, which is not a rigorous mathematical procedure in many cases of interest. Importantly, this formalization of Wick-rotated field theory then

[29] slightly generalized

applies to theories which do not have a known description in terms of fluctuating fields. (An example of much current interest is the 6-dimensional superconformal $(2,0)$-theory.)

- The topological vector space $F(Y)$ associated to a closed $(n-1)$-manifold is the *state space* of the system on the space Y. It can be thought of as the quantization of the system on the Lorentz manifold $\mathbb{R} \times Y$. (More precisely, Y is embedded in a germ of a Riemannian n-manifold and one can anti-Wick rotate the metric to a Lorentz metric.)
- If X is a closed n-manifold, then $F(X) \in \mathbb{C}$ is called the *partition function*.
- Let X be a closed n-manifold, $x_1, \dots, x_k \in X$, and $\widehat{X}_\epsilon = X \setminus \bigcup_{n=1}^k B_\epsilon(x_i)$ the bordism obtained by removing open balls of radius ϵ about the points x_i; each boundary component is taken to be incoming. The theory F applied to $\widehat{X}_\epsilon$ gives a multilinear map

$$F(\widehat{X}_\epsilon)\colon F\bigl(S_\epsilon(x_1)\bigr) \times \cdots \times F\bigl(S_\epsilon(x_k)\bigr) \longrightarrow \mathbb{C} \tag{3.3}$$

whose inverse limit as $\epsilon \to 0$ is the *Wick-rotated correlation function*. The inverse limit

$$\varprojlim_{\epsilon \to 0} F\bigl(S_\epsilon(x_i)\bigr) \tag{3.4}$$

is the vector space of *observables* at x_i. In the physics literature it is often called the vector space of *local operators*.[30]

In theories which depend on a Riemannian metric Axiom System 3.1 encodes scale-dependent information as well as the dynamics which relates different scales. There is a useful analogy with differential equations. Namely, a differential equation describes a system at short range: it captures the small distance and small time features. One then asks the *well-posedness* of the equation, i.e., the passage *infinitesimal* $\longrightarrow$ *local*. In some cases the well-posedness is very delicate. For well-posed equations we can then ask about the *global* behavior. Here questions center around qualitative features, such as long-time existence, singularity formation, ergodicity, *et cetera*. The situation is similar in quantum field theory: the input to the theory (operator product expansion) describes a system at short range, and the dynamical questions concern the behavior of a solution—an actual theory—at long range. We refer to the discussion in [**W1**, §1.0].

REMARK 3.5. A functor (3.2) encodes both short and long range behavior, but we ought to include a condition analogous to well-posedness. Namely, we should postulate [**Se4**] an auxiliary conformal field theory (but see Remark 3.7) and a statement that it is the short range limit of F. For example, if that auxiliary theory is free,[31] then the theory F is called *asymptotically free*.

REMARK 3.6. Suppose the background fields include a Riemannian metric. Then the *renormalization group* gives a family F_μ of theories parametrized by $\mu \in \mathbb{R}^{>0}$. It is constructed by composing F with the 1-parameter family of automorphisms of $\mathrm{Bord}_{\langle n-1,n\rangle}(\mathcal{X}_n^\nabla)$ that scales the Riemannian metric $g \mapsto \mu^2 g$. The condition in the previous paragraph concerns the short range limit $\mu \to 0$. The global

[30] The author prefers 'point operators' since extended operators—line operators, surface operators, etc.—also obey locality properties.

[31] We do not know a characterization of freeness in the Axiom System.

questions of *dynamics* [**W1**]—including the classification of phases Problem 2.33—concerns the long range limit $\mu \to \infty$.

REMARK 3.7. Segal considers a variation of Axiom System 3.1 in which the theory is only defined for bordisms each component of which has a nonempty incoming boundary. Such theories are termed 'noncompact' and they avoid some the finiteness restrictions that follow from having coevaluation bordisms. Examples include the massless scalar fields with values in a noncompact manifold, such as $\mathbb{R}$. The short range limit of a field theory may be of noncompact type; see Example 4.38.

3.2. Relativistic quantum field theory

First, let us review special relativity. Our account of quantum mechanics in Lecture 2 has a time line $\mathbb{M}^1$ in the axioms; space is not included. In the example of a particle moving on a manifold, space is treated separately from time. If we take physical space to be Euclidean space $\mathbb{E}^d$, then there is a *Galilean spacetime* [**Ar**], [**FM1**, §2] which exhibits the full symmetry of the system. In special relativity one uses instead *Minkowski spacetime*, which is a deformation of the Galilean structure [**DF1**, §1.4]. Standard Minkowski spacetime $\mathbb{M}^n$ $(n = d + 1)$ is the standard real affine space $\mathbb{A}^n$ equipped with a translationally invariant metric of Lorentz signature. Denote the Lorentzian inner product space of translations as $\mathbb{R}^{1,n-1}$. In terms of standard affine coordinates[32] $x^0 = ct, x^1, \dots, x^{n-1}$ on $\mathbb{M}^n$ the metric is

$$(dx^0)^2 - (dx^1)^2 - \cdots - (dx^{n-1})^2. \tag{3.8}$$

The isometry group $\mathcal{I}_{1,n-1} = \mathrm{Iso}(\mathbb{M}^n)$ is a group of affine transformations, an extension

$$1 \longrightarrow \mathbb{R}^{1,n-1} \longrightarrow \mathcal{I}_{1,n-1} \longrightarrow \mathrm{O}_{1,n-1} \longrightarrow 1 \tag{3.9}$$

with kernel the group of translations and quotient the linear orthogonal group. If $n \geqslant 2$ then $\pi_0\,\mathrm{O}_{1,n-1} \cong \{\pm 1\} \times \{\pm 1\}$ with two homomorphisms $\mathrm{O}_{1,n-1} \to \{\pm 1\}$ telling if an orthogonal transformation is (i) orientation-preserving, (ii) time orientation[33] preserving. The identity component of $\mathcal{I}_{1,n-1}$ has a canonical double cover called the *Poincaré group* $\mathcal{P}_n$; it splits over the translation subgroup $\mathbb{R}^{1,n-1} \subset \mathcal{I}_{1,n-1}$. Let $k_0 \in \mathcal{P}_n$ denote the central element of order two.

A relativistic quantum field theory is, first of all, a quantum mechanical system with symmetry group $\mathcal{P}_n$. (We discuss more general symmetry groups in §3.4.) There is a Hilbert space $\mathcal{H}$ which determines the convex space of states as in ordinary quantum mechanics. To allow both bosonic and fermionic states we take $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ to be $\mathbb{Z}/2\mathbb{Z}$-graded. The bosonic states are homogeneous vectors in $\mathcal{H}^0$, the fermionic states homogeneous vectors in $\mathcal{H}^1$; see [**DeM**, §4] for consequences of the Koszul sign rule on the Hilbert space structure and operators. In quantum field theory one usually fixes a particular pure state called the *vacuum*. Typically correlation functions are taken with respect to the vacuum state. In quantum mechanics time evolution of states is encoded by a unitary representation

[32] The constant c is the speed of light.

[33] There are two components of vectors in $\mathbb{R}^{1,n-1}$ of positive norm square—timelike vectors; the components are preserved or exchanged by an orthogonal transformation. Choose the distinguished component $\mathbb{R}^{1,n-1}_+ \subset \mathbb{R}^{1,n-1}$ to consist of vectors with positive zeroth component, and in this way time orient $\mathbb{M}^n$. Dually, this induces a notion of positive energy.

of the time translation group $\mathbb{R}$; in quantum field theory there is a unitary representation $U\colon \mathcal{P}_n \to U(\mathcal{H})$ of the Poincaré group.[34] The vacuum state is assumed to be invariant under U. The restriction $U\colon \mathbb{R}^{1,n-1} \to \mathrm{U}(\mathcal{H})$ has a spectral measure $\sigma(U)$ on $(\mathbb{R}^{1,n-1})^*$; *positivity of energy* is the hypothesis that $\sigma(U)$ is supported on the closure of the dual forward light cone. The observables form a real vector bundle $\mathcal{O} \to \mathbb{M}^n$ with covariant derivative, and the basic quantities in the theory are correlation functions, as in (2.7), now taken at points $x_i \in \mathbb{M}^n$ and with respect to the distinguished vacuum state. Correlation functions are distributions, rather than functions, but nonetheless are called *Wightman functions*. See [**K**, §1] for a careful account of the Wightman Axiom System for relativistic quantum field theory.

In the context of the Wightman axioms there are several important general theorems [**SW**, **GJ**, **K**, **AgVm**]. For example, one version of the *spin-statistics theorem* asserts that $U(k_0)$ is the grading operator on $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$. (Recall that k_0 is the central element of the Poincaré group.) The *spin* of a representation of $\mathrm{Spin}_{1,n-1}$, the double cover of the identity component $\mathrm{O}^0_{1,n-1}$, is integer or half-integer according as the representation drops to $\mathrm{O}^0_{1,n-1}$ or not. The *statistics* of a homogeneous state is its grading: even or odd. Another important result has the malapropos[35] appellation *CPT theorem*. Roughly, it states that the symmetry group[36] of the theory includes all orientation-preserving transformations, whereas we only assume initially that the identity component acts. In [**FH1**, Appendix A] we give a proof, following Jost, for general symmetry types.

3.3. Wick rotation of relativistic quantum field theory

Recall (§2.5) that positivity of energy in quantum mechanics implies that the unitary real time physical quantum theory is the boundary value of a holomorphic theory defined on a complex domain. This applies both to the time translation representation and the correlation functions. Analogous statements hold in a relativistic quantum field theory. Let $\mathbb{R}^{1,n-1}_+ \subset \mathbb{R}^{1,n-1}$ denote the open cone of forward timelike vectors. Then positivity of energy implies that $U\colon \mathbb{R}^{1,n-1} \to \mathrm{U}(\mathcal{H})$ is the boundary value of a bounded representation of the complex semigroup $\mathbb{C}^{1,n-1}_- = \mathbb{R}^{1,n-1} - \sqrt{-1}\mathbb{R}^{1,n-1}_+ \subset \mathbb{C}^{1,n-1}$. The corresponding story for correlation functions is more complicated [**K**, §2.1]. In essence, if we consider a k-point correlation function, then there is a holomorphic function on a complex domain $\mathcal{D}_k$ in the k-fold Cartesian product of the complexification of $\mathbb{M}^n$ such that $(\mathbb{M}^n)^{\times k}$ sits in the boundary of $\mathcal{D}_k$ and the holomorphic function limits on this boundary to the real correlation function. The k-fold Cartesian product $(\mathbb{E}^n)^{\times k}$ lies in $\mathcal{D}_k$. The *Euclidean correlation functions*, or *Schwinger functions*, are the restrictions of the holomorphic correlation functions to $(\mathbb{E}^n)^{\times k}$.

REMARK 3.10. There are many imprecisions here. First, correlation functions are distributions, rather than functions, and we must remove all diagonals to consider them as functions. Next, rather than complexify the affine space $(\mathbb{M}^n)^{\times k}$,

[34] The image consists of *even* unitary transformations, those which preserve the grading $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$. Theories with supersymmetry include odd transformations as well.

[35] See the discussion in [**W2**, footnote 8].

[36] By Wigner's theorem (see [**F9**] and the references therein) symmetries of a quantum system lift to act unitarily or antiunitarily on the Hilbert space $\mathcal{H}$; isometries of $\mathbb{M}^n$ which are time orientation-reversing act antiunitarily in a quantum field theory.

one uses overall $\mathbb{R}^{1,n-1}$-invariance to descend to the quotient $n(k-1)$-dimensional vector space of difference vectors. Also, in the analytic continuation one allows permutations of the arguments. Finally, to embed $\mathbb{E}^n$ in the complexification of $\mathbb{M}^n$ we choose a particular time direction, so a splitting $\mathbb{R}^{1,n-1} \cong \mathbb{R} \oplus \mathbb{R}^{n-1}$ of the real translation group as a sum of time and space translations; the Euclidean translation group is $\sqrt{-1}\mathbb{R} \oplus \mathbb{R}^{n-1}$.

The fundamental *reconstruction theorem*[37] of Osterwalder-Schrauder [**OS**] reverses the Wick oration process: a relativistic quantum field theory can be reconstructed from Euclidean correlation functions. Here is a schematic depiction of the Wick rotation process:

$$\mathbb{M}^n \rightsquigarrow \mathcal{D} \rightsquigarrow \mathbb{E}^n. \tag{3.11}$$

To arrive at the Axiom System outlined in §3.1 it remains to pass from Euclidean space $\mathbb{E}^n$ to compact manifolds and the bordism category. We do so at the end of this lecture, but first discuss more general symmetry groups and their Wick rotations.

3.4. Symmetry groups in quantum field theory

Let $\mathcal{I}^{\uparrow}_{1,n-1} \subset \mathcal{I}_{1,n-1}$ denote the subgroup of time orientation-preserving isometries. The traditional approach is to assume the full symmetry group[38] $\mathcal{G}_{1,n-1}$ of a relativistic quantum field theory is a *subgroup* of a double cover of $\mathcal{I}^{\uparrow}_{1,n-1}$ whose identity component is the Poincaré group. But the more natural hypothesis is that here is a homomorphism[39] $\mathcal{G}_{1,n-1} \to \mathcal{I}^{\uparrow}_{1,n-1}$ whose image contains the identity component. This is natural geometrically: any symmetry of the geometry/physics happening over $\mathbb{M}^n$ induces a symmetry of $\mathbb{M}^n$. The hypothesis about the image captures relativistic invariance. We further assume:

(1) The kernel of $\mathcal{G}_{1,n-1} \to \mathcal{I}^{\uparrow}_{1,n-1}$ is a compact Lie group K,
(2) The translation subgroup $\mathbb{R}^{1,n-1} \subset \mathcal{I}^{\uparrow}_{1,n-1}$ lifts to a normal subgroup of $\mathcal{G}_{1,n-1}$.

The kernel K is the group of *internal symmetries*, symmetries which fix spacetime pointwise. The compactness rules out examples of field theories which may be considered "noncompact" (Remark 3.7), such as a free massless $\mathbb{R}$-valued scalar field (in which K is the noncompact translation group of $\mathbb{R}$), or the topological short range limit of 2-dimensional Yang-Mills theory (Example 4.38). Hypothesis (2) essentially means that $\mathcal{G}_{1,n-1}$ is the symmetry group of a translationally-invariant geometric structure on $\mathbb{M}^n$; see [**FM1**, Remark 2.13] for further comments.

Define $H_{1,n-1} = \mathcal{G}_{1,n-1}/\mathbb{R}^{1,n-1}$, the *vector group* of global symmetries. In (3.11) the complexification of $H_{1,n-1}$ acts as symmetries on the complex correlation

[37] This is a theorem in the Wightman framework for quantum field theory; see [**K**] for an exposition.

[38] $\mathcal{G}_{1,n-1}$ is a Lie group which consists of "unbroken global" symmetries. In general, we should include odd symmetries (supersymmetries) as well as higher symmetries [**GKSW**], but we will not do so in these lectures.

[39] Conformal field theories have a different structure in which $\mathcal{I}^{\uparrow}_{1,n-1}$ is replaced by a larger group of conformal symmetries.

functions and a different real form H_n acts on the Wick-rotated Euclidean theory. There is a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, hook] & H_{1,n-1} \arrow[r] \arrow[d, hook] & O^{\uparrow}_{1,n-1} \arrow[d, hook] \\ 1 \arrow[r] & K(\mathbb{C}) \arrow[r] & H_n(\mathbb{C}) \arrow[r] & \mathrm{O}_n(\mathbb{C}) \\ 1 \arrow[r] & K \arrow[r] \arrow[u, hook] & H_n \arrow[r, "\rho_n"] \arrow[u, hook] & \mathrm{O}_n \arrow[u, hook] \end{tikzcd} \tag{3.12}$$

of Lie groups. We have included the hypothesis:

(3) The kernel of $\rho_n\colon H_n \to \mathrm{O}_n$ is the same compact real form $K \subset K(\mathbb{C})$ as the kernel of $H_{1,n-1} \to O^{\uparrow}_{1,n-1}$.

In other words, the internal symmetry group does not change under Wick rotation.

Remark 3.13. Since the image of $H_{1,n-1} \to O^{\uparrow}_{1,n-1}$ includes the identity component, the image of ρ_n is either SO_n or O_n. In the latter case the theory has time-reversing symmetries, whereas in the former case it does not.

The Lie group H_n is the vector symmetry group of the Euclidean theory. It is a *compact* Lie group. The rigidity of compact Lie groups leads to structure theorems which are important in quantum field theory generally and here in subsequent lectures. Proofs are in [**FH1**, §2]. Define $SH_n = \rho_n^{-1}(\mathrm{SO}_n)$ and let $\widetilde{SH}_n$ be the double cover of SH_n constructed from the spin double cover of SO_n. These compact Lie groups are usefully encoded in the pullback diagram

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, equal] & \widetilde{SH}_n \arrow[r] \arrow[d, "2:1"] & \mathrm{Spin}_n \arrow[r] \arrow[d, "2:1"] & 1 \\ 1 \arrow[r] & K \arrow[r] \arrow[d, equal] & SH_n \arrow[r] \arrow[d, hook, "1:2"] & \mathrm{SO}_n \arrow[r] \arrow[d, hook, "1:2"] & 1 \\ 1 \arrow[r] & K \arrow[r] & H_n \arrow[r, "\rho_n"] & \mathrm{O}_n & \end{tikzcd} \tag{3.14}$$

If $\rho_n\colon H_n \to \mathrm{O}_n$ is surjective, define $\widetilde{H}_n$ as the pullback[40]

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, equal] & \widetilde{H}_n \arrow[r, "\rho_n"] \arrow[d] & \mathrm{Pin}^+_n \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & K \arrow[r] & H_n \arrow[r, "\rho_n"] & \mathrm{O}_n \arrow[r] & 1 \end{tikzcd} \tag{3.15}$$

Let $\mathfrak{k}, \mathfrak{h}_n, \mathfrak{o}_n$ denote the Lie algebras of K, H_n, O_n.

Proposition 3.16. *Assume $n \geqslant 3$.*

(1) *There is a splitting $\mathfrak{h}_n \cong \mathfrak{o}'_n \oplus \mathfrak{k}$, and ρ_n induces an isomorphism of Lie algebras $\mathfrak{o}'_n \xrightarrow{\cong} \mathfrak{o}_n$.*

[40]For a quick review of pin groups and pin manifolds, see [**FH1**, §A.1].

(2) *If $n \geqslant 3$ there is an isomorphism $\widetilde{SH}_n \cong \mathrm{Spin}_n \times K$. Hence there exists a central element $k_0 \in K$ with $(k_0)^2 = 1$ and an isomorphism*

$$SH_n \cong \mathrm{Spin}_n \times K \,/\, \langle(-1, k_0)\rangle, \tag{3.17}$$

where $\langle(-1, k_0)\rangle$ is the cyclic group generated by $(-1, k_0)$.

(3) *There is a canonical homomorphism $\mathrm{Spin}_n \to H_n$ under which the image of the central element $-1 \in \mathrm{Spin}_n$ is $k_0 \in K$.*

(4) *If $n \geqslant 3$ and $\rho_n \colon H_n \to \mathrm{O}_n$ is surjective, then there exists a group extension*

$$1 \longrightarrow K \longrightarrow J \longrightarrow \{\pm 1\} \longrightarrow 1 \tag{3.18}$$

and a pullback diagram of group extensions

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, equal] & \widetilde{H}_n \arrow[r, "\rho_n"] \arrow[d] & \mathrm{Pin}_n^+ \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & K \arrow[r] & J \arrow[r] & \{\pm 1\} \arrow[r] & 1 \end{tikzcd} \tag{3.19}$$

There is an isomorphism

$$H_n \cong \widetilde{H}_n \,/\, \langle(-1, k_0)\rangle. \tag{3.20}$$

Part (1) is the analog in our approach of the Coleman-Mandula theorem [**CM**]; Part (2) a splitting at the level of Lie groups rather than Lie algebras. Part (3), anti Wick-rotated back to Minkowski spacetime, produces a homomorphism $\mathcal{P}_n \to \mathcal{G}_{1,n-1}$ of the Poincaré group into the total symmetry group of the theory, the traditional starting point. The pullback (3.19) in Part (4) shows that the failure of $\widetilde{H}_n$ to be a product is encoded in the group extension (3.18), which is independent of n.

Remark 3.21. Let $n = 3$ and $H_3 = \mathbb{Z}/2\mathbb{Z} \ltimes (\mathrm{SO}_3 \times \mathrm{SO}_3)$, where the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ acts by shearing $(g_1, g_2) \mapsto (g_1, g_1 g_2)$. Define $\rho_3 \colon H_3 \to \mathrm{O}_3$ to kill the last factor $K = \mathrm{SO}_3$ and send the generator of $\mathbb{Z}/2\mathbb{Z}$ to the central element $-1 \in \mathrm{O}_3$. Then H_3 is not a possible symmetry group; it violates Proposition 3.16(4). For $n = 2$ we are allowed $SH_2 = \mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{T} \times \mathbb{T})$ with involution $(\lambda_1, \lambda_2) \mapsto (\lambda_1, \lambda_1^{-1}\lambda_2^{-1})$ on $\mathbb{T} \times \mathbb{T}$. This illustrates the dimension restriction in Part (2) of Proposition 3.16.

The rigidity of compact Lie groups also leads to easy classification theorems.

Example 3.22. If $K = \{1\}$ is trivial, then $H_n \cong \mathrm{SO}_n$ or O_n. Let $\mu_4 \subset \mathbb{T}$ be the group of 4^{th} roots of unity. Define $E_n \subset \mathrm{O}_n \times \mu_4$ as the subgroup of (A, λ) such that $\det A = \lambda^2$. Then if $K \cong \{\pm 1\}$ is cyclic of order two, there are 6 possibilities for H_n up to isomorphism: $\mathrm{SO}_n \times \{\pm 1\}$, Spin_n, $\mathrm{O}_n \times \{\pm 1\}$, E_n, Pin_n^+, Pin_n^-. If we demand that k_0 be the non-identity element of K, then we are left with 3 possibilities: Spin_n, Pin_n^+, Pin_n^-.

The following chart summarizes the most basic symmetry groups:

(3.23)

states/symmetry	H_n	K	k_0
bosons only	SO_n	$\{1\}$	1
fermions allowed	Spin_n	$\{\pm 1\}$	-1
bosons, time-reversal (T)	O_n	$\{1\}$	1
fermions, $T^2 = (-1)^F$	Pin_n^+	$\{\pm 1\}$	-1
fermions, $T^2 = \mathrm{id}$	Pin_n^-	$\{\pm 1\}$	-1

That the last two entries are Wick-rotated symmetry groups of systems with time-reversal symmetry whose square is the indicated endomorphism of the $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space of states is a consequence of the circle of ideas around the CPT theorem; see [**FH1**, Appendix A].

The main structure result we need about symmetry groups in quantum field theory is a stabilization of H_n.

THEOREM 3.24. *Assume $n \geqslant 3$. There exist compact Lie groups H_m, $m > n$, and homomorphisms i_n, ρ_n which fit into the commutative diagram*

$$\begin{tikzcd} H_n \arrow[r, hook, "i_n"] \arrow[d, "\rho_n"] & H_{n+1} \arrow[r, hook, "i_{n+1}"] \arrow[d, "\rho_{n+1}"] & H_{n+2} \arrow[r, hook] \arrow[d, "\rho_{n+2}"] & \cdots \\ \mathrm{O}_n \arrow[r, hook] & \mathrm{O}_{n+1} \arrow[r, hook] & \mathrm{O}_{n+2} \arrow[r, hook] & \cdots \end{tikzcd} \tag{3.25}$$

in which squares are pullbacks. Furthermore, any two choices of stabilization (3.25) *are isomorphic.*

The internal symmetry group $K = \ker \rho_n$ is independent of n.

Theorem 3.24 allows us to speak about symmetry groups in quantum field theory independent of dimension. Set

$$H = \operatorname*{colim}_{n \to \infty} H_n. \tag{3.26}$$

For $H_n = \mathrm{SO}_n$ we obtain $H = \mathrm{SO}_\infty = \mathrm{SO}$. Thus we can speak of 'oriented theories'='SO theories', 'Spin theories', 'Pin^+ theories', etc. Set $\mathrm{O} = \mathrm{O}_\infty = \operatorname*{colim}_{n\to\infty} \mathrm{O}_n$. The colimit of (3.25) is a homomorphism $\rho\colon H \to \mathrm{O}$ with kernel K.

DEFINITION 3.27. The homomorphism $\rho\colon H \to \mathrm{O}$ is called the *symmetry type* and is denoted (H, ρ).

By pullback (3.25) we obtain (H_n, ρ_n) for all $n \in \mathbb{Z}^{>0}$. Theorem 3.24 lets us reconstruct (H, ρ) from (H_n, ρ_n), hence we also use the term 'symmetry type' for the pair (H_n, ρ_n).

REMARK 3.28. The proof uses Hypothesis (3) following (3.12) above to rule out some groups which do not admit stabilizations. As an example of a group which is excluded, let $n = 3$ and $H_3 = \mathbb{Z}/2\mathbb{Z} \ltimes (\mathrm{SO}_3 \times \mathrm{SO}_3)$, where the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ acts by shearing $(g_1, g_2) \mapsto (g_1, g_1 g_2)$; the homomorphism ρ_3 which kills the second factor $K = \mathrm{SO}_3$ maps $H_3 \to \mathrm{O}_3$ and sends the generator of $\mathbb{Z}/2\mathbb{Z}$ to the central element $-1 \in \mathrm{O}_3$.

3.5. Interlude on differential geometry

The last stage in the progression from relativistic quantum field theory on Minkowski spacetime $\mathbb{M}^n$ to the Axiom System 3.1 is the passage from Euclidean space $\mathbb{E}^n$ to curved manifolds. This is familiar in differential geometry—the passage from Klein's *Erlangen Programm* [**BB**] to Cartan's H-structures.[41] We digress briefly to explain this perspective.

Affine geometry on the model affine space $\mathbb{A}^n$ has symmetry group

$$1 \longrightarrow \mathbb{R}^n \longrightarrow \mathrm{Aff}_n \longrightarrow \mathrm{GL}_n\,\mathbb{R} \longrightarrow 1 \tag{3.29}$$

A translationally-invariant geometric structure on $\mathbb{A}^n$ has a symmetry Lie group which fits into the diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{R}^n \arrow[r] \arrow[d, equal] & \mathcal{H}_n \arrow[r] \arrow[d] & H_n \arrow[r] \arrow[d, "\rho_n"] & 1 \\ 1 \arrow[r] & \mathbb{R}^n \arrow[r] & \mathrm{Aff}_n \arrow[r] & \mathrm{GL}_n\,\mathbb{R} \arrow[r] & 1 \end{tikzcd} \tag{3.30}$$

The pair (H_n, ρ_n) encodes the *Cartan*[41] *symmetry type* of the geometric structure. Notice that in contrast to the situation in quantum field theory, the Lie group H_n may be noncompact and the codomain of ρ_n is $\mathrm{GL}_n\,\mathbb{R}$ in place of O_n.

EXAMPLE 3.31. We give the group H_n and leave the reader to construct the homomorphism ρ_n.

(3.32)

H_n	affine geometry	curved geometry
O_n	Euclidean	Riemannian
$\mathrm{O}_{1,n-1}$	Minkowskian	Lorentzian
$\mathrm{Sp}_{n/2}\,\mathbb{R}$	symplectic	symplectic
$\mathrm{GL}_{n/2}\,\mathbb{C}$	complex	complex
$\mathrm{U}_{n/2}$	Hermitian	Kähler
Spin_n	spin	spin

There are many other Cartan symmetry types which lead to other geometries: foliation geometry, Galilean geometry, etc. We can include internal symmetry groups as well.

Affine geometry has a global parallelism encoded in the simply transitive action of the translation group. The curved analog of affine geometry is the infinitesimal parallelism of an affine connection on a smooth manifold [**KN**]. It is natural[42] to demand that the affine connection be *torsionfree*, roughly the condition that "parallelograms" close to leading order. (Recall that an affine connection on a smooth n-manifold M is a connection on the principal $\mathrm{GL}_n\,\mathbb{R}$-bundle of frames $\mathcal{B}(M) \to M$.)

[41] Cartan considers only pairs (H_n, ρ_n) in which $\rho_n \colon H_n \to \mathrm{GL}_n\,\mathbb{R}$ is an *injective* homomorphism.

[42] Well, certainly convenient. But the Cartan theory of H-structures is much, much richer [**S**] and the torsionfree requirement limits the examples substantially. In quantum field theory the image of ρ_n lies in O_n and we are in the realm of Riemannian geometry and the unique torsionfree Levi-Civita connection.

DEFINITION 3.33. Let (H_n, ρ_n) be a Cartan symmetry type. A *differential H_n-structure* on a smooth n-manifold M is a triple (P, Θ, φ), where $P \to M$ is a principal H_n-bundle with connection Θ and[43] $\varphi\colon \mathcal{B}(M) \to \rho_n(P)$ is an isomorphism of principal $\mathrm{GL}_n\,\mathbb{R}$-bundles such that $\varphi^*\rho_n(\Theta)$ is torsionfree.

There is a category of manifolds equipped with differential H_n-structure and their diffeomorphisms. H_n-geometry is the study of invariants in this category.

3.6. Wick-rotated field theory on compact manifolds

Just as we pass from affine geometry to curved manifolds, and with O_n symmetry from Euclidean geometry to Riemannian geometry, so too we pass from Euclidean field theory to Axiom System 3.1: Wick-rotated field theory on compact manifolds.

REMARK 3.34. The restriction to *compact* manifolds is potentially a radical step if one is interested in long range behavior, since the "range" of a compact manifold is bounded above. Yet we will see that in at least one class of examples the information about the long range theory is retained.

For a theory with symmetry type (H_n, ρ_n) the domain of the Wick-rotated theory is the bordism category $\mathrm{Bord}_{\langle n-1,n\rangle}(H_n^\nabla)$ of $(n-1)$- and n-dimensional manifolds with differential H_n-structure. The theory itself is then a symmetric monoidal functor

$$F\colon \mathrm{Bord}_{\langle n-1,n\rangle}(H_n^\nabla) \longrightarrow t\mathrm{Vect}_{\mathbb{C}}, \tag{3.35}$$

as in (3.2). As discussed in §3.1 the functor F encodes the states spaces, observables, and correlation functions of the Wick-rotated theory.

The passage to compact manifolds completes the procession (3.11):

$$\mathbb{M}^n \rightsquigarrow \mathcal{D} \rightsquigarrow \mathbb{E}^n \rightsquigarrow X^n. \tag{3.36}$$

One can hope for a reconstruction theorem which reverses (3.36), and perhaps such structural theorems lie in the future, after a precise formulation of the Axiom System (see Remark 3.5).

A major challenge is to construct examples. In the topological case the future is now: there is a rigorous theory with abundant examples and no issue about reconstruction. The twin pillars of quantum field theory—*locality* and *unitarity*—have been developed beyond what is traditionally discussed for physical theories, and these extensions play a crucial role in what follows; see Lecture 5 and Lecture 7. The view of non-topological field theories through the lens of the Axiom System is important for our story—and, experience teaches, provides a powerful tool for exploring quantum field theory more generally.

[43] $\rho_n(P) \to M$ is the associated principal $\mathrm{GL}_n\,\mathbb{R}$-bundle with connection $\rho_n(\Theta)$.

LECTURE 4

Classification Theorems

It is difficult to contemplate attacking a classification problem in mathematics without a definition, or axiom system, for the object being classified. While that may sound anodyne or tautological, it is hardly so away from mathematics. Indeed our central Problem 2.33 is a classification problem in physics without a definition or axiom system (for lattice systems). Returning to mathematics, sometimes there is uniqueness in which case an axiom system is a characterization: the real numbers form the unique complete ordered field. Other times one extracts invariants to distinguish inequivalent objects and, in some cases, a complete invariant. For example, the fundamental group of the complement of a knot in S^3 is a powerful, but not complete, invariant. In any case it is the definition of a knot—a connected 1-dimensional closed submanifold of S^3—that enables us to classify at all.

In this lecture we illustrate how Axiom System 1.17 for topological field theory enables us to prove classification theorems in low dimensions. We also state classification theorems for non-topological field theories, based on variations of Axiom System 3.1. The proofs use Morse and Cerf theory, so illustrate techniques used in topological field theory, in particular those which enter at least one approach to the cobordism hypothesis (§5.6). In §4.3 we study 2-dimensional field theories of manifolds with an area form. These are perhaps the simplest non-topological field theories and provide good examples and illustrations of the ideas in Lecture 3. They also provide an explicit example of short range and long range behavior of a scale-dependent physical system. We conclude with another classification theorem: the 1-dimensional cobordism hypothesis for oriented manifolds.

Stolz-Teichner and their collaborators have proved classification theorems for non-topological field theories in low dimensions; see [**ST**] for a sample of their results.

4.1. Review of Morse and Cerf theory

Let M be a smooth manifold and $f\colon M \to \mathbb{R}$ a smooth function. Recall that $p \in M$ is a *critical point* if $df_p = 0$. A number $c \in \mathbb{R}$ is a *critical value* if $f^{-1}(c)$ contains a critical point. At a critical point p the second differential, or Hessian,

$$d^2 f_p\colon T_pM \times T_pM \longrightarrow \mathbb{R} \tag{4.1}$$

is a well-defined symmetric bilinear form. To evaluate it on $\xi_1, \xi_2 \in T_pM$ extend ξ_2 to a vector field to near p, and set $d^2f_p(\xi_1,\xi_2) = \xi_1\xi_2 f(p)$, the iterated directional derivative. We say p is a *nondegenerate critical point* if the Hessian (4.1) is a nondegenerate symmetric bilinear form.

Lemma 4.2 (Morse). *If p is a nondegenerate critical point of the function $f\colon M \to \mathbb{R}$, then there exists a local coordinate system $x^1, \dots, x^n$ about p such* *that*

$$f = (x^1)^2 + \cdots + (x^r)^2 - (x^{r+1})^2 - \cdots - (x^n)^2 + c \tag{4.3}$$

for some p.

The number $n - r$ of minus signs in (4.3) is the *index* of the critical point p. A function is *Morse* if each of its critical points is nondegenerate. An application of Sard's theorem proves that Morse functions exist, and in fact are open and dense in the space of C^∞ functions [**H**, §6.1].

If X is a manifold with boundary we consider smooth functions which are constant on ∂X and have no critical points on ∂X. The following terminology is apparently due to Thom.

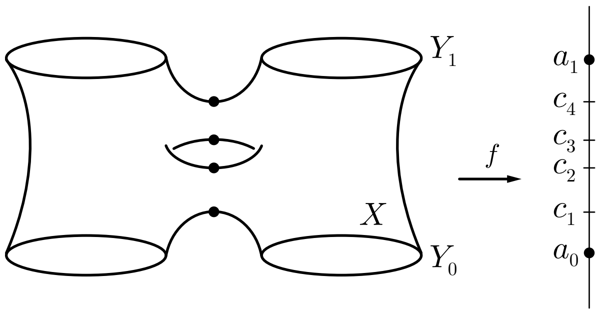

FIGURE 4.1. An excellent Morse function on a bordism

DEFINITION 4.4. Let $X\colon Y_0 \to Y_1$ be a bordism. An *excellent Morse function* $f\colon X \to \mathbb{R}$ satisfies

(1) $f(Y_0) = a_0$ is constant;
(2) $f(Y_1) = a_1$ is constant; and
(3) the critical points $x_1, \dots, x_N$ are nondegenerate with distinct critical values $c_1, \dots, c_N$ which satisfy

$$a_0 < c_1 < \cdots < c_N < a_1. \tag{4.5}$$

We depict an excellent Morse function on a bordism in Figure 4.1. The space of excellent Morse functions on a bordism is dense in the space of C^∞ functions.

The basic theorems of Morse theory tell the structure of $X_{a',a''} = f^{-1}([a', a''])$ if a', a'' are regular values. If there are no critical values in $[a', a'']$, then $X_{a',a''}$ is diffeomorphic to the Cartesian product of $[a', a'']$ and $Y = f^{-1}(a)$ for any $a \in [a', a'']$. If there is a single critical value $c \in [a', a'']$ and $f^{-1}(c)$ contains a single critical point of index r, then $X_{a',a''}$ is obtained from $X_{a',c-\epsilon}$ by attaching an n-*dimensional r-handle*. We can also deduce the *Morse surgery* which constructs the level set $X_{a''} = f^{-1}(a'')$ from $X_{a'} = f^{-1}(a')$. A neighborhood of the critical point is the closed n-ball $D^r \times D^{n-r}$, and the surgery is the transition

$$S^{r-1} \times D^{n-r} \rightsquigarrow D^r \times S^{n-r-1} \tag{4.6}$$

which happens in a neighborhood of the critical point in passing from $X_{a'}$ to $X_{a''}$. See [**Mi2**, **H**, **PT**] for detailed treatments and many applications, including classification theorems for manifolds of dimension $\leqslant 2$.

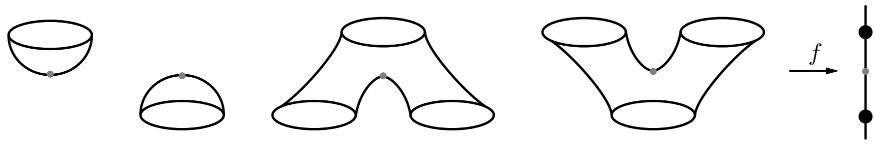

FIGURE 4.2. Some elementary 2-dimensional bordisms

DEFINITION 4.7. A bordism $X\colon Y_0 \to Y_1$ is an *elementary bordism* if it admits an excellent Morse function with a single critical point.

The elementary 2-dimensional bordisms are depicted in Figure 4.2. Notice that the Euler number of an elementary bordism is $(-1)^r$, where r is the index of the critical point.

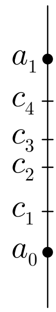

FIGURE 4.3. The Kirby graphic of Figure 4.1

An excellent Morse function on *any* bordism $X\colon Y_0 \to Y_1$ expresses it as a composition of elementary bordisms

$$X = X_N \circ \cdots \circ X_1 \tag{4.8}$$

where $X_1 = f^{-1}([a_0, c_1{+}\epsilon])$, $X_2 = f^{-1}([c_1{+}\epsilon, c_2{+}\epsilon]), \dots, X_N = f^{-1}([c_{N-1}{+}\epsilon, a_1])$ for sufficiently small $\epsilon > 0$. Excellent functions connected by a path of excellent Morse functions lead to an equivalent decomposition: corresponding elementary bordisms are diffeomorphic. Track the equivalence class by means of a *Kirby graphic* (Figure 4.3) which indicates the distribution of critical points and their indices. The space of excellent Morse functions is not connected; a bordism has (infinitely) many decompositions with different Kirby graphics.

Jean Cerf [**Ce**] studied a filtration on the space of smooth functions. The subleading part of the filtration connects different components of excellent Morse functions.

DEFINITION 4.9. A smooth function $f\colon M \to \mathbb{R}$ on an n-manifold M has a *birth-death singularity* at $p \in M$ if there exist local coordinates $x^1, \dots, x^n$ in which

$$f = (x^1)^3 + (x^1)^2 + \cdots + (x^r)^2 - (x^{r+1})^2 - \cdots - (x^n)^2 + c \tag{4.10}$$

The *index* of p is defined to be $n - r$.

There is an intrinsic definition: p is a degenerate critical point, the null space $N_p \subset T_pM$ of d^2f_p has dimension one, and the third differential d^3f_p is nonzero on N_p.

DEFINITION 4.11. Let $X\colon Y_0 \to Y_1$ be a bordism and $f\colon X \to \mathbb{R}$ a smooth function.

(1) f is *good of Type* α if f is excellent Morse except at a single point at which f has a birth-death singularity.
(2) f is *good of Type* β if f is excellent Morse except that there exist exactly two critical points x_i, x_{i+1} with the same critical value $f(x_i) = f(x_{i+1})$.

We say f is *good* if it is either excellent Morse or good of Type α or good of Type β.

THEOREM 4.12 (Cerf [**Ce**]). *Let $X\colon Y_0 \to Y_1$ be a bordism. Then the space of good functions is connected. More precisely, if f_0, f_1 are excellent Morse, then there exists a path f_t of good functions such that f_t is excellent Morse except at finitely many values of t.*

There is an even more precise statement. The space of good functions is an infinite dimensional manifold, the space of good functions which are not excellent Morse is a codimension one submanifold, and we can arrange that the path $t \mapsto f_t$ cross this submanifold transversely, hence at finitely many values of t.

A path of good functions has an associated Kirby graphic which encodes the excellent chambers and wall crossings of the path. The horizontal variable is t and the vertical is the critical value. The curves in the graphic are labeled by the index of the critical point in the preimage. Birth-death singularities occur with critical points of neighboring indices. Paths can also pass through a wall of Type β in which the critical values of two critical points exchange which is larger. Kirby uses these graphics in his calculus [**Ky**]. Figure 4.4 shows some simple Kirby graphics.

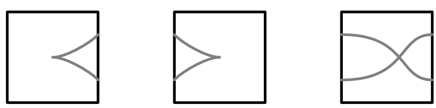

FIGURE 4.4. Kirby graphics of a birth, death, and exchange

EXAMPLE 4.13. The prototype for crossing a wall of Type α is the path of functions

$$f_t(x) = \frac{x^3}{3} - tx \tag{4.14}$$

defined for $x \in \mathbb{R}$. Then f_t is Morse for $t \neq 0$, has no critical points if $t < 0$, and has two critical points $x = \pm\sqrt{t}$ for $t > 0$. As t increases through $t = 0$ the two critical points are born; as t decreases through $t = 0$ they die. The critical values are $\pm t^{3/2}$, up to a multiplicative constant, which explains the shape of the Kirby graphic, the first picture in Figure 4.4.

4.2. Classification of 2-dimensional topological field theories

We begin with some algebra.

DEFINITION 4.15. Let k be a field. A *commutative Frobenius algebra* (A, τ) over k is a finite dimensional unital commutative associative algebra A over k and a linear map $\tau\colon A \to k$ such that

$$\begin{aligned} A \times A &\longrightarrow k \\ x, y &\longmapsto \tau(xy) \end{aligned} \tag{4.16}$$

is a nondegenerate pairing.

EXAMPLE 4.17 (Frobenius). Let G be a finite group. Let A be the vector space of functions $f\colon G \to \mathbb{C}$ which are *central*:

$$f(gxg^{-1}) = f(x) \qquad \text{for all } x, g \in G. \tag{4.18}$$

Define multiplication as convolution:

$$(f_1 * f_2)(x) = \sum_{x_1 x_2 = x} f_1(x_1) f_2(x_2). \tag{4.19}$$

A straightforward check shows $*$ is commutative and associative and the unit is the "δ-function at e", which is 1 at the identity $e \in G$ and 0 elsewhere. The trace is

$$\tau(f) = \frac{f(e)}{\#G}. \tag{4.20}$$

If we remove the central condition (4.18), then we obtain the *noncommutative* Frobenius algebra of all complex-valued functions on G.

EXAMPLE 4.21. Let M be a closed oriented n-manifold. Then $H^\bullet(M;\mathbb{C})$ is a super commutative Frobenius algebra. Multiplication is cup product and the trace is evaluation on the fundamental class. 'Super' reflects the sign in the cup product. For $M = S^2$ we obtain an ordinary commutative Frobenius algebra since there is no odd cohomology. The commutative Frobenius algebra $H^\bullet(S^2;\mathbb{C})$, or rather the commutative Frobenius ring $H^\bullet(S^2;\mathbb{Z})$, is a key ingredient in the construction of Khovanov homology [**Kh**].

The following classification theorem was well-known by the late 1980s. It appears in Dijkgraaf's thesis [**Dij**]. Mathematical treatments are given in [**Ab**, **Ko**]. The Morse theory proof we give here is taken from [**MS**, Appendix].

THEOREM 4.22. *Let $F\colon \mathrm{Bord}_{\langle 1,2\rangle}(\mathrm{SO}_2) \to \mathrm{Vect}_k$ be a topological field theory. Then $F(S^1)$ is a commutative Frobenius algebra. Conversely, if A is a commutative Frobenius algebra, then there exists a 2-dimensional topological field theory $F_A\colon \mathrm{Bord}_{\langle 1,2\rangle}(\mathrm{SO}_2) \to \mathrm{Vect}_k$ such that $F_A(S^1) = A$.*

The theory F_A is unique up to isomorphism.

REMARK 4.23. The 2-dimensional field theory constructed from the Frobenius algebra in Example 4.17 is the finite gauge theory of Example 1.23. Recall that it has a "classical" description: it counts principal G-bundles. The invariant $F(X)$ of a closed surface of genus g is given by a classical formula of Frobenius. (For a lattice approach, see [**Sny**].) Topological field theory provides a proof of that formula by cutting a surface of genus g into elementary pieces.

PROOF. Given $F\colon \mathrm{Bord}_{\langle 1,2\rangle}(\mathrm{SO}_2) \to \mathrm{Vect}_k$ define the vector space $A = F(S^1)$. The elementary bordisms in Figure 4.5 define a unit $u\colon k \to A$, a trace $\tau\colon A \to k$, and a multiplication $m\colon A \otimes A \to A$. (We read "time" as flowing up in these bordisms; the bottom boundaries are incoming and the top boundaries are outgoing.) The bilinear form (4.16) is the composition in Figure 4.6, and it has an inverse given by the cylinder with both boundary components outgoing, as is proved by the S-diagram argument (Lemma 1.20). Therefore, it is nondegenerate. This proves that (A, u, m, τ) is a commutative Frobenius algebra.

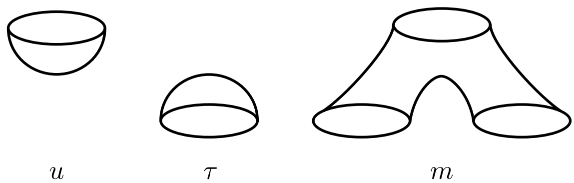

FIGURE 4.5. Elementary bordisms which define the Frobenius structure

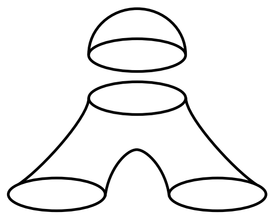

FIGURE 4.6. The bilinear form

Next, we compute the map m^* defined by time-reversal[44] Let $x_1, \dots, x_n$ and $x^1, \dots, x^n$ be dual bases of A relative to (4.16): $\tau(x^i x_j) = \delta^i_j$. Then

$$\begin{aligned} m^* \colon A &\longrightarrow A \otimes A \\ x &\longmapsto x x_i \otimes x^i \end{aligned} \tag{4.24}$$

This is the adjoint of multiplication relative to the pairing (4.16). Similarly, note that the unit $u = \tau^*$ is adjoint to the trace. These adjunctions follow from general duality in symmetric monoidal categories (§A.2): duality in the bordism category is effected by time-reversal (see Lemma 7.26), and the dual of an object in the category of vector spaces is the dual vector space. A symmetric monoidal functor, such as F_A, maps duals to duals.

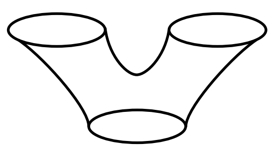

FIGURE 4.7. The adjoint m^*

For the converse, suppose A is a commutative Frobenius algebra. We construct a 2-dimensional topological field theory F_A.

It is easy to prove that the topological group $\mathrm{Diff}^{\mathrm{SO}}(S^1)$ of orientation-preserving diffeomorphisms deformation retracts onto the group of rotations, which is

[44]The time-reversed bordism replaces $p\colon \partial X \to \{0,1\}$ with $1-p$: it exchanges incoming and outgoing.

connected. Since diffeomorphisms act on A through their isotopy class, the action is trivial. Thus if Y is any oriented manifold diffeomorphic to a circle, there is up to isotopy a unique orientation-preserving diffeomorphism $Y \to S^1$. For any closed oriented 1-manifold Y define $F_A(Y) = A^{\otimes(\#\pi_0 Y)}$; orientation-preserving diffeomorphisms of closed 1-manifolds act as the identity.

The values of F_A on elementary 2-dimensional bordisms (Figure 4.2) are given by the structure maps $u = \tau^*, \tau, m, m^*$ of the Frobenius algebra. An arbitrary bordism is a composition of elementary bordisms (tensor identity maps) via an excellent Morse function, and we use such a decomposition to define F_A. However we must check that the value is independent of the excellent Morse function. For that we use Cerf's Theorem 4.12. It suffices to check what happens when we cross a wall of Type α or of Type β.

First, a simplification. Since time-reversal implements duality, if an equality of maps holds for a wall-crossing it also holds for its time-reversal. This cuts down the number of diagrams one needs to consider.

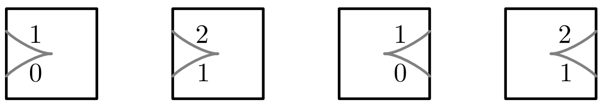

FIGURE 4.8. The four Type α wall-crossings

There are four Type α wall-crossings, as indicated by their Kirby graphics in Figure 4.8. The numbers indicate the indices of the critical points. If f_t is a path of Morse functions with the first Kirby graphic, then the three subsequent ones may be realized by $-f_t$, f_{1-t}, and $-f_{1-t}$, respectively. (Here $0 \leqslant t \leqslant 1$.) It follows that we need only check the first. The corresponding transition of bordisms is indicated in Figure 4.9. These bordisms both map to $\mathrm{id}_A \colon A \to A$: for the first this expresses that u is an identity for multiplication m.

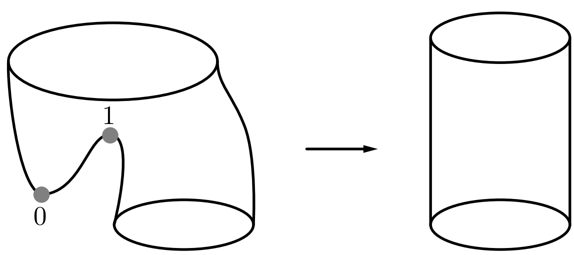

FIGURE 4.9. Crossing a birth-death singularity

In a Type β wall-crossing there are two critical points and the critical levels cross. So on either side of the wall the bordism X is a composition of two elementary bordisms. Assume X is connected or else there is nothing to prove. Furthermore, if the indices of the critical points are r_1, r_2, then the Euler number of the bordism is $(-1)^{r_1} + (-1)^{r_2}$, by elementary Morse theory. Let C denote the critical contour at the critical time t_{crit}, when the two critical levels cross. Since the bordism is connected there are two possibilities: either C is connected or it consists of two components, each with a single critical point. In the latter case there would have to be another critical point in the bordism to connect the two components, else the bordism would not be connected. Therefore, C is connected and it follows easily that both critical points have index 1, hence X has Euler number -2.

Now in each elementary bordism (Figure 4.2) the number of incoming and outgoing circles differs by one, so in a composition of two elementary bordisms the number of circles changes by two or does not change at all. This leads to four possibilities for the number of circles: $1 \to 1$, $2 \to 2$, $3 \to 1$, or $1 \to 3$. The last is the time-reversal of the penultimate, so we have three cases to consider.

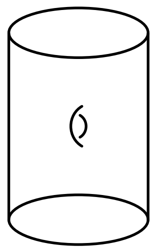

FIGURE 4.10. $1 \to 1$

The first, $1 \to 1$, is a torus with two disks removed. Figure 4.10 is not at the critical time—the two critical levels are distinct. Note that at a regular value between the two critical values, the level curve has two components, by the classification of elementary bordisms (Figure 4.2). So the composition is

$$A \xrightarrow{m^*} A \otimes A \xrightarrow{m} A \tag{4.25}$$

Passing through the Type β wall we emerge to an isomorphic bordism, so to the same linear map (4.25).

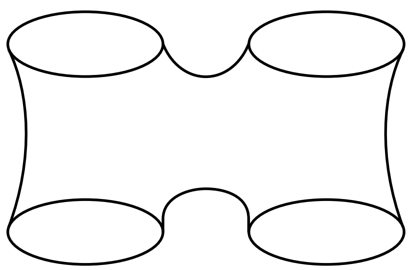

FIGURE 4.11. $2 \to 1 \to 2$

The second case, $2 \to 2$, is somewhat more complicated than the others. The numbers of circles in the composition on the two sides of the wall are $2 \to 1 \to 2$ and $2 \to 3 \to 2$. The $2 \to 1 \to 2$ composition, depicted in Figure 4.11, is $m^* \circ m$, which is the map

$$x \otimes y \longmapsto xy \longmapsto xyx_i \otimes x^i, \tag{4.26}$$

using the dual bases introduced above. The $2 \to 3 \to 2$ composition, depicted in Figure 4.12, is either $(m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes m^*)$ or $(\mathrm{id} \otimes m) \circ (m^* \otimes \mathrm{id})$, so either

$$x \otimes y \longmapsto x \otimes yx_i \otimes x^i \longmapsto xyx_i \otimes x^i \tag{4.27}$$

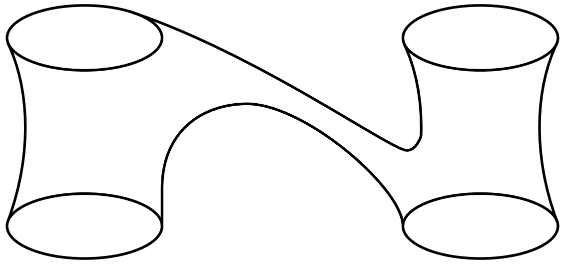

FIGURE 4.12. $2 \to 3 \to 2$

or

$$x \otimes y \longmapsto xx_i \otimes x^i \otimes y \longmapsto xx_i \otimes x^i y. \tag{4.28}$$

To see that these are equal, use the identity $z = \tau(zx_j)x^j$ for all $z \in A$. Thus

$$xx_i \otimes x^i y = \tau(x^i y x_j) xx_i \otimes x^j = yx_j x \otimes x^j = xyx_i \otimes x^i. \tag{4.29}$$

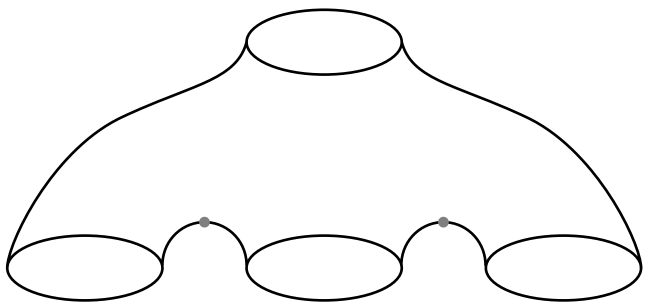

FIGURE 4.13. $3 \to 1$

The last case, $3 \to 1$, is depicted in Figure 4.13 at the critical time. On either side of the wall we have a composition $3 \to 2 \to 1$, and the two different compositions $A^{\otimes 3} \longrightarrow A^{\otimes 2} \longrightarrow A$ are equal by the associative law for m. ☐

EXERCISE 4.30. The commutative algebra $A = H^\bullet(S^2; \mathbb{C}) \cong \mathbb{C}[x]/(x^2)$ is not semisimple. It has a Frobenius trace τ defined by evaluation against the fundamental class, so $\tau(1) = 0$ and $\tau(x) = 1$. Theorem 4.22 produces an associated 2-dimensional topological field theory F_A. Evaluate $F_A(X_g)$ for X_g a closed connected surface of genus g. Observe that this partition function vanishes for some g (which?); vanishing partition functions can occur in non-invertible theories.

EXERCISE 4.31. A semisimple commutative algebra A over $\mathbb{C}$ is isomorphic to a product $\mathbb{C} \times \cdots \times \mathbb{C}$ (N times). Show that a Frobenius structure on A is determined by N numbers $\lambda_1, \dots, \lambda_N$. What constraints to λ_i obey? Let F_A denote the associated field theory. Determine $F_A(X_g)$ as a function of g and λ_i; you will derive the *Verlinde formula* [**V**]. Do there exist λ_i such that $F_A(X_g)$ in Exercise 4.30 is computed by a formula of this shape?

I do not know a "classical model" for the theory F_A in Exercise 4.30, as opposed, say, to finite gauge theory (Example 1.23). Thus Theorem 4.22 is a direct quantum construction of a field theory.

Theorem 4.22 has many generalizations. What follows is an incomplete list and a small sample of references. The spin case is treated in [**MS**, **Gu**]. The unoriented case is in [**AlNa**, **TT**, **S-P1**]. The oriented case with a background principal G-bundle is treated in many references, among them [**Tu**, **MS**, **Dav**].

4.3. 2-dimensional area-dependent theories

In general, as described in Lecture 3, a Wick-rotated quantum field theory is defined on a bordism category of Riemannian manifolds. A Riemannian metric can be regarded as two pieces of data: a conformal structure and a volume form. There are special quantum field theories which depend on just one part of the metric.[45] Many interesting and important field theories just depend on the conformal structure, and indeed conformal field theories appear as short range or long range limits of general theories. In this section we study the opposite case of theories which depend only on a volume form. The simplest case, beyond quantum mechanics, is the 2-dimensional case of a theory depending on an area form. We follow the lecture notes [**Se2**, §1.4].

First, a theorem of Moser states that up to diffeomorphism the only invariant of a volume form on a compact connected manifold is the total volume.

Theorem 4.32 ([**Mos**]). *Let M be a compact connected oriented n-manifold with boundary, and suppose $\omega_0, \omega_1 \in \Omega^n_M$ satisfy $\int_M \omega_0 = \int_M \omega_1$. Then there exists a diffeomorphism $\varphi\colon M \to M$ such that $\varphi^*\omega_1 = \omega_0$.*

Proof. Choose $\eta \in \Omega^{n-1}_M$ such that $\omega_1 - \omega_0 = d\eta$. Set $\omega_t = \omega_0 + t\,d\eta$, $0 \leqslant t \leqslant 1$. Then ω_t is a volume form, so there exists a unique vector field ξ_t such that $\iota_{\xi_t}\omega_t + \eta = 0$. Let φ be the flow generated by the time-varying vector field ξ_t. Then

$$\frac{d}{dt}(\varphi_t^*\omega_t) = \varphi_t^*(\mathcal{L}_{\xi_t}\omega_t + \frac{d}{dt}\omega_t) = \varphi_t^* d(\iota_{\xi_t}\omega_t + \eta) = 0, \tag{4.33}$$

hence $\varphi_1^*\omega_1 = \omega_0$. This completes the proof if M is closed. If $\partial M \neq 0$ we must be careful to choose η so that ξ_t is tangent to ∂M. In fact, we can and do arrange that η, hence ξ_t, vanishes on ∂M. Namely, if $\lambda \in \Omega^{n-1}_{\partial M}$ then the pair (ω, λ) represents an element in the relative de Rham cohomology group $H^n_{dR}(M, \partial M)$. Recall [**BT**, p. 78] the isomorphism

$$\begin{aligned} H^n_{dR}(M, \partial M) &\longrightarrow \mathbb{R} \\ (\omega, \lambda) &\longmapsto \int_M \omega - \int_{\partial M} \lambda. \end{aligned} \tag{4.34}$$

Thus $(\omega_0, 0)$ and $(\omega_1, 0)$ are cohomologous in $H^n_{dR}(M, \partial M)$, so we can find $(\tilde\eta, \mu) \in \Omega^{n-1}_M \times \Omega^{n-2}_{\partial M}$ such that $d\tilde\eta = \omega_1 - \omega_0$ and $d\mu + i^*\tilde\eta = 0$, where $i\colon \partial M \hookrightarrow M$ is the inclusion. Choose a collar neighborhood U of ∂M and a cutoff function $\rho\colon U \to [0,1]$ such that $\rho = 1$ in a neighborhood of ∂M and vanishes outside of U. Set $\eta = \tilde\eta + d(\rho\mu)$. ☐

Exercise 4.35. Give a proof of the Morse lemma using the Moser technique.

As a consequence of Theorem 4.32, in the bordism category of oriented n-manifolds with volume form, the non-topological information in a morphism—an n-dimensional bordism up to diffeomorphism—is the total volume of each component.

[45] Topological field theories depend on neither.

In that sense a field theory F depending only on a volume form is just one step removed from a topological theory.

Next, we elaborate on the topological vector space assigned to an object in a field theory (3.2), at least in theories which depend only on a volume form. Let Y^{n-1} be a closed manifold. The cylinder $[0,1] \times Y$ with total volume $t \in \mathbb{R}^{>0}$ induces an endomorphism U_t of the topological vector space $F(Y)$. With suitable hypotheses on the category $t\mathrm{Vect}_{\mathbb{C}}$ we can deduce that $t \mapsto U_t$ is a continuous semigroup of trace class operators. We obtain a directed system of topological vector spaces parametrized by $\mathbb{R}^{>0}$; let $\check{F}(Y), \widehat{F}(Y)$ denote the colimit as $t \to \infty$ and limit as $t \to 0$, respectively.[46] We can replace $F(Y)$ with $\check{F}(Y)$. Then a vector $\xi \in F(Y)$ can be expressed as $U_t\xi_t$ for any $t > 0$.

Segal [**Se2**, Proposition 1.4.3] states a classification theorem in the 2-dimensional case. Suppose F is a 2-dimensional theory of oriented manifolds with area form. Then $A = F(S^1)$ does not depend on an area form; all area forms on a germ of S^1 are equivalent. Define a multiplication on A as follows (see Figure 4.5): choose $s, t \in \mathbb{R}^{>0}$ and let X be the pair of pants of total area $s + t$. Then the product of $U_s\xi, U_t\eta \in A$ is $F(X)(\xi \otimes \eta)$. The disk of area $t \in \mathbb{R}^{>0}$ with outgoing boundary gives a vector $\varepsilon_t \in A$, and there is a limit $\lim_{t\to 0} \varepsilon_t$. It follows from the usual gluing argument that U_t is multiplication by ε_t. The trace $\tau\colon A \to \mathbb{C}$ is the value of F on the disk of unit area with incoming boundary.

PROPOSITION 4.36 ([**Se2**]). *There is an equivalence between 2-dimensional theories of oriented manifolds with area form and triples (A, τ, ε_t) in which A is a commutative topological algebra; $\tau\colon A \to \mathbb{C}$ a nondegenerate trace; and $\varepsilon_t \in A$ satisfy $\varepsilon_t \to 1$ as $t \to 0$, $\varepsilon_{t_1}\varepsilon_{t_1} = \varepsilon_{t_1+t_2}$, and multiplication by ε_t is a trace class operator on A.*

The recent paper [**RS**] by Runkel-Szegedy has a careful treatment of this theorem.

REMARK 4.37. The renormalization group acts on the space of these theories by scaling the total area of bordisms; see Remark 3.6. The short range "limit" takes the area to 0, the long range "limit" takes the area to ∞. In general there is no limit, but for theories which depend only on a volume form if the limit exists it is a topological theory, hence by Theorem 4.22 is given by a commutative Frobenius algebra. In the context of Proposition 4.36 short range is $t \to 0$ and long range is $t \to \infty$. If A is infinite dimensional, then $t \to 0$ does not lead to a finite theory: in a 2-dimensional *topological* theory the vector space attached to S^1 is finite dimensional (Corollary 1.21). We can hope, though, for a theory of noncompact type (Remark 3.7). On the other hand, if U_t converges to a finite rank projection operator as $t \to \infty$, then there is a topological field theory which is the long range limit.

EXAMPLE 4.38. Let G be a *compact* Lie group. If G is finite, then the associated gauge theory is topological and was already treated in Example 1.23. The corresponding Frobenius algebra is described in Example 4.17. For $\dim G > 0$ we cannot simply count G-bundles. The standard procedure in physics introduces a connection and sums (path integrates) over connections. The standard weighting

[46] An analogy to keep in mind: Let M be a compact Riemannian manifold and $U_t = e^{-t\Delta}$ the heat operator acting on L^2 functions. Then $\check{F}$ consists of C^∞ functions whereas $\widehat{F}$ consists of distributions.

factor is the Yang-Mills action, which in 2 dimensions depends on an area form but not on the conformal structure. See [**W3**], [**Mig**], [**W1**, Lecture 11] for more details about this physically defined theory. Segal [**Se2**, §1.5] argues that under the equivalence in Proposition 4.36, this 2-dimensional Yang-Mills theory corresponds to the triple

$$\begin{aligned} A &= \text{center of } C^\infty(G) \text{ under convolution,} \\ \tau(f) &= f(1), \\ \varepsilon_t &= e^{-t\Delta}(\delta_1). \end{aligned} \tag{4.39}$$

Here Δ is the Laplace operator with respect to a bi-invariant metric[47] on G, and δ_1 is Dirac's δ-distribution supported at the identity $1 \in G$.

We make two observations.

- $\lim_{t\to\infty} \varepsilon_t$ exists and is a constant function on G. Therefore, the long range limit exists and is an invertible theory: $\dim A_\infty = 1$. One-dimensional Frobenius algebras have a single parameter, $\tau(1)$, which in this case is determined by the total volume of G. This invertible theory is an Euler theory ϵ_μ (1.53) for some μ determined by $\mathrm{Vol}(G)$.
- The invariant in 2-dimensional Yang-Mills theory of a closed oriented surface of genus g and area t is

$$\sum_V \frac{e^{-t\lambda_V}}{(c \dim V)^{2g-2}} \tag{4.40}$$

  for some $c > 0$, where the sum runs over isomorphism classes of irreducible representations of G. The limit $t \to 0$ exists if $2g-2 > 0$, i.e., for surfaces of negative Euler number, so the short range topological theory is only partially defined: it is a noncompact theory. As $t \to 0$ only the trivial representation survives in the sum (4.40).

4.4. Classification of 1-dimensional field theories

Of course, 1-dimensional field theories have a simpler structure than 2-dimensional theories, but we take a different approach than above to illustrate different aspects. We begin with topological theories with a general codomain category, which foreshadows the setup of extended field theories in Lecture 5. The classification result Theorem 4.46 is a special case of the much more general cobordism hypothesis (Theorem 5.34).

The oriented bordism group in dimension zero is the free abelian group on one generator: $\Omega_0(\mathrm{SO}) \cong \mathbb{Z}$. This simple theorem can be restated in terms of bordism invariants as follows. Let M be any commutative monoid. Then the set of 0-dimensional bordism invariants with values in M is isomorphic to the commutative monoid $\mathrm{Hom}(\Omega_0(\mathrm{SO}), M)$, where the sum $F+G$ of two bordism invariants is computed element-wise: $(F+G)(Y) = F(Y)+G(Y)$ for all compact 0-manifolds Y. (See Remark 1.11.) Then $F(Y)$ is automatically invertible, since $\Omega_0(\mathrm{SO})$ is a group. Let $M^\sim \subset M$ be the group of units (invertible elements).

[47]The metric is needed to define the action of 2-dimensional Yang-Mills.

THEOREM 4.41 (Cobordism hypothesis—set version). *The map*

$$\begin{aligned} \Phi\colon \operatorname{Hom}(\Omega_0(\mathrm{SO}), M) &\longrightarrow M^{\sim} \\ F &\longmapsto F(\mathrm{pt}_+) \end{aligned} \tag{4.42}$$

is an isomorphism *of abelian groups.*

Now categorify. Fix a symmetric monoidal category C and let

$$\mathrm{TFT}^{\mathrm{SO}}_{\langle 0,1\rangle}(C) = \operatorname{Hom}^{\otimes}(\mathrm{Bord}_{\langle 0,1\rangle}(\mathrm{SO}_1), C) \tag{4.43}$$

be the category of topological 1-dimensional oriented field theories with values in C, so (Axiom System 1.17) symmetric monoidal functors from the bordism category to C.

LEMMA 4.44. *A morphism*

$$(\eta\colon F \to G) \in \mathrm{TFT}^{\mathrm{SO}}_{\langle 0,1\rangle}(C)$$

is invertible: $\mathrm{TFT}^{\mathrm{SO}}_{\langle 0,1\rangle}(C)$ *is a groupoid.*

This follows from Proposition A.44 in the appendix.

We need the following categorical notion of finiteness in the symmetric monoidal situation. Recall Definition A.14 of a full subcategory.

DEFINITION 4.45. Let C be a symmetric monoidal category. Define $C^{\mathrm{fd}} \subset C$ as the full subcategory whose objects are the dualizable objects of C.

The notation 'fd' puts in mind 'finite dimensional', which is correct for the category Vect: dualizable vector spaces are those which are finite dimensional (Exercise 1.22). It also stands for 'fully dualizable', though in the 1-dimensional case 'fully' is not relevant: fully dualizable = dualizable. (See §5.6 for a discussion of full dualizability.) Let $(C^{\mathrm{fd}})^{\sim} \subset C^{\mathrm{fd}}$ be the groupoid of units (Definition A.17).

THEOREM 4.46 (Cobordism hypothesis—1-categorical version). *Let C be a symmetric monoidal category. Then the map*

$$\begin{aligned} \Phi\colon \mathrm{TFT}^{\mathrm{SO}}_{\langle 0,1\rangle}(C) &\longrightarrow (C^{\mathrm{fd}})^{\sim} \\ F &\longmapsto F(\mathrm{pt}_+) \end{aligned} \tag{4.47}$$

is an equivalence of groupoids.

The map Φ is well-defined since $F(\mathrm{pt}_+)$ is dualizable; see Proposition A.44(1).

The proof relies on the classification of closed 0-manifolds and compact 1-manifolds with boundary [**Mi1**]. Note that if Y_0, Y_1 are closed 0-manifolds which are diffeomorphic, then the set of diffeomorphisms $Y_0 \to Y_1$ is a torsor for the group of permutations (of, say, Y_0). A connected compact 1-manifold with boundary is diffeomorphic to a circle or a closed interval, which immediately leads to the classification of connected morphisms in $\mathrm{Bord}_{\langle 0,1\rangle}(\mathrm{SO}_1)$, as illustrated in Figure 4.14: every connected oriented bordism is diffeomorphic to one of the five illustrations.

PROOF. We must show that Φ is fully faithful and essentially surjective; see Proposition A.11.

First, let F, G be field theories, $\eta_1, \eta_2\colon F \to G$ isomorphisms, and suppose $\eta_1(\mathrm{pt}_+) = \eta_2(\mathrm{pt}_+)$. Since $\mathrm{pt}_- = \mathrm{pt}_+^{\vee}$, according to the formula proved in Proposition A.44 we have $\eta(\mathrm{pt}_-) = \big(\eta(\mathrm{pt}_+)^{\vee}\big)^{-1}$ for any natural isomorphism η. It follows

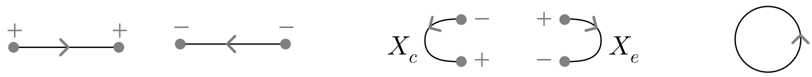

FIGURE 4.14. The five connected oriented bordisms in $\mathrm{Bord}_{\langle 0,1\rangle}(\mathrm{SO}_1)$

that $\eta_1(\mathrm{pt}_-) = \eta_2(\mathrm{pt}_-)$. Since any compact oriented 0-manifold Y is a finite disjoint union of copies of pt_+ and pt_-, it follows that $\eta_1(Y) = \eta_2(Y)$ for all Y, hence $\eta_1 = \eta_2$. This shows that Φ is faithful.

To show Φ is full, given F, G and an isomorphism $f\colon F(\mathrm{pt}_+) \to G(\mathrm{pt}_+)$ we must construct $\eta\colon F \to G$ such that $\eta(\mathrm{pt}_+) = f$. So define $\eta(\mathrm{pt}_+) = f$ and $\eta(\mathrm{pt}_-) = (f^\vee)^{-1}$. Extend using the monoidal structure in C to define $\eta(Y)$ for all compact oriented 0-manifolds Y. This uses the statement given before the proof that any such Y is diffeomorphic to $(\mathrm{pt}_+)^{\amalg n_+} \amalg (\mathrm{pt}_-)^{\amalg n_-}$ for unique $n_+, n_- \in \mathbb{Z}^{\geqslant 0}$. Also, the diffeomorphism is determined up to permutation, but because of coherence the resulting map $\eta(Y)$ is independent of the chosen diffeomorphism. It remains to show that η is a natural isomorphism, so to verify (A.8) for each morphism in $\mathrm{Bord}^{\mathrm{SO}}_{\langle 0,1\rangle}$. It suffices to consider connected bordisms, so each of the morphisms in Figure 4.14. The first two are identity maps, for which (A.8) is trivial. The commutativity of the diagram

$$\begin{tikzcd} & F(-)F(+) \arrow[dd, "(f^\vee)^{-1}f"] \\ 1 \arrow[ur, "F(X_c)"] \arrow[dr, "G(X_c)"'] & \\ & G(-)G(+) \end{tikzcd} \tag{4.48}$$

for coevaluation X_c follows from the commutativity of

$$\begin{tikzcd} 1 \arrow[r, "F(X_c)"] \arrow[d, "G(X_c)"'] & F(-)F(+) \arrow[r, "1f"] \arrow[d, "11G(X_c)"] & F(-)G(+) \\ G(-)G(+) \arrow[r, "F(X_c)11"] & F(-)F(+)G(-)G(+) \arrow[r, "1f11"] & F(-)G(+)G(-)G(+) \arrow[u, "1G(X_e)1"'] \end{tikzcd} \tag{4.49}$$

In these diagrams we use '+' and '−' for 'pt_+' and 'pt_-', and also denote identity maps as '1'. The argument for evaluation X_e is similar, and that for the circle follows since the circle is $X_e \circ \sigma \circ X_c$ for σ the symmetry. Notice that the commutative diagram (A.8) for the circle S^1 asserts $F(S^1) = G(S^1)$.

Finally, we must show that Φ is essentially surjective. Given $y \in C$ dualizable, we must[48] construct a field theory F with $F(\mathrm{pt}_+) = y$. Let $(y^\vee, c, e)$ be duality data for y. Define $F(\mathrm{pt}_+) = y$, $F(\mathrm{pt}_-) = y^\vee$, and

$$F\big((\mathrm{pt}_+)^{\amalg n_+} \amalg (\mathrm{pt}_-)^{\amalg n_-}\big) = y^{\otimes n_+} \otimes (y^\vee)^{\otimes n_-}. \tag{4.50}$$

Any compact oriented 0-manifold Y is diffeomorphic to some $(\mathrm{pt}_+)^{\amalg n_+} \amalg (\mathrm{pt}_-)^{\amalg n_-}$, and again by coherence the choice of diffeomorphism does not matter. Now any

[48] In fact, we only need construct F with $F(\mathrm{pt}_+) \cong y$, but we will construct F such that the stronger condition $F(\mathrm{pt}_+) = y$ holds.

oriented bordism $X\colon Y_0 \to Y_1$ is diffeomorphic to a disjoint union of the bordisms in Figure 4.14, and for these standard bordisms we define $F(X_c) = c$, $F(X_e) = e$, and $F(S^1) = e \circ \sigma \circ c$; the first two bordisms in the figure are identity maps, which necessarily map to identity maps. Define F to map X to the corresponding tensor product of c, e, and $e \circ \sigma \circ c$. It remains to check that F is a functor, i.e., that compositions map to compositions. When composing in $\mathrm{Bord}^{\mathrm{SO}}_{\langle 0,1\rangle}$ the only nontrivial compositions are those indicated in Figure 4.15. The first composition is what we use to define $F(S^1)$. The S-diagram relations (A.29) show that the last compositions are consistent under F. $\square$

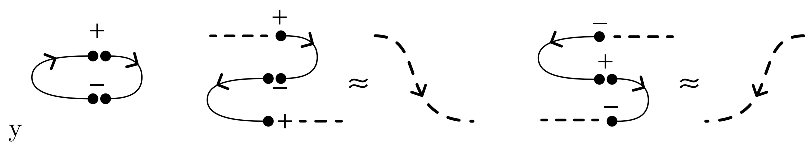

FIGURE 4.15. Nontrivial compositions in $\mathrm{Bord}^{\mathrm{SO}}_{\langle 0,1\rangle}$

LECTURE 5

Extended Locality

The twin pillars of quantum field theory are locality and unitarity. In this lecture we focus on locality; we take up unitarity in Lecture 7. Axiom System 3.1 encodes a *limited* version of locality: the partition function and correlation functions in a theory F can be computed by cutting a manifold in codimension one, computing on the resulting pieces, and combining the results algebraically. The proof of Theorem 4.22 proceeds along those lines, for example. But *full* locality on an n-dimensional manifold requires cutting in n independent directions, and then assembling the manifold from balls. This leads to n composition laws, one for each direction of cutting, and so naturally to n-categories. When full locality in the form of *extended field theory* was first introduced [**F3**, **La**, **F4**] the theory of higher categories was in a primitive state. By now there are many complete theories as well as detailed comparisons among them; see [**BR1**, **BR2**] and the references therein. For the purposes of these lectures we work with higher categories heuristically. Our technical results pertain to *invertible* field theories, and as we will see in Lecture §6 these can be modeled using maps of spaces rather than higher categories.

The notion of an extended field theory applies to both topological and non-topological theories, though the non-topological case requires a mix of higher categorical and (soft) analytical ideas that has not been fully developed. We focus here on the topological case.

We give some motivation for higher categories in §5.1, with an emphasis on higher groupoids as those are most important for us. Higher categorical versions of both the domain and codomain of a field theory are discussed in §5.2. We give some motivation and examples of extended field theories in §5.3. Another route to extended field theory is through extended operators—line operators, surface operators, etc.—as we describe in §5.4, and in this form the idea has been in physics since the 1970s. In §5.5 we briefly describe the algebra structures on the images of spheres under a field theory. The central theorem of extended field theory is the *cobordism hypothesis*, which we introduce in §5.6. Finite gauge theory in 3 dimensions provides an illustration of many ideas in this lecture, and we give an account in §5.7.

5.1. Higher categories

Let S be a topological space. The simplest algebraic invariant we can extract from S is the set $\pi_0 S$ of path components. Namely, define an equivalence relation—$s_0, s_1 \in S$ are equivalent if there exists a continuous path from s_0 to s_1—for which $\pi_0 S$ is the set of equivalence classes. The next step is a categorification. Instead of using paths as a *condition*, incorporate paths as *data*. Thus define a category $\pi_{\leqslant 1} S$ whose set of objects is S and where the morphism set $(\pi_{\leqslant 1} S)(s_0, s_1)$ is the set of homotopy classes of paths from s_0 to s_1. Homotopy classes have an associative composition law defined by juxtaposing paths, and the homotopy class of the constant path at s_0 is the identity for composition. Furthermore, paths are reversible. The reverse path is, up to homotopy, inverse to the original path. Therefore, $\pi_{\leqslant 1}S$ is a groupoid, the *fundamental groupoid* of S. We can recover $\pi_0 S$ as the set of equivalence classes of objects in the fundamental groupoid $\pi_{\leqslant 1}S$. The homotopy groups $\pi_1(S,s) = (\pi_{\leqslant 1}S)(s,s)$ for all basepoints $s \in S$ are also part of the data of the fundamental groupoid.

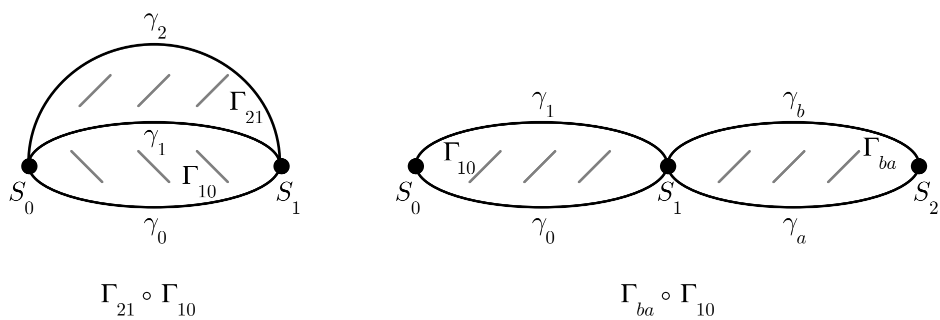

FIGURE 5.1. 2 compositions of 2-morphisms

Let us categorify again to incorporate π_2. Thus we envision a gadget $\pi_{\leqslant 2}S$ with set of objects S and for $s_0, s_1 \in S$ the set of *1-morphisms* $(\pi_{\leqslant 2}S)(s_0,s_1)$ is the set of paths from s_0 to s_1. But now there needs to be another layer of structure. Note too that there is a composition law of concatenation of paths, but it is neither associative nor unital. Both properties only hold up to homotopy, and they are witnessed by an additional layer of structure: *2-morphisms*. If $\gamma_0, \gamma_1 \in (\pi_{\leqslant 2}S)(s_0,s_1)$ are paths from s_0 to s_1, then a 2-morphism is a homotopy class of homotopies $\Gamma_{10}\colon \gamma_0 \to \gamma_1$. There are 2 composition laws on 2-morphisms, as depicted in Figure 5.1. They are associative, since 2-morphisms are homotopy classes of Γ's, and there are units and inverses. In Figure 5.2 we depict a more symmetric view of the two composition laws. The 2-category $\pi_{\leqslant 2}S$ is a *2-groupoid* in the sense that all 1- and 2-morphisms are invertible up to higher morphisms.

These constructions continue: for any $n \in \mathbb{Z}^{\geqslant 0}$ there is an n-groupoid $\pi_{\leqslant n}S$.

REMARK 5.1. The sequence $\pi_0 S, \pi_{\leqslant 1}S, \pi_{\leqslant 2}S, \dots$ of n-groupoids, $n=0,1,2,\dots$, is subsumed by a single ∞-*groupoid* $\pi_{\leqslant \infty}S$ which has k-morphisms for all $k \in \mathbb{Z}^{\geqslant 1}$. It is complicated to spell out the algebraic structure—data and conditions—of an n-groupoid, much less an ∞-groupoid. There is an inverse (up to homotopy) of the algebraic extraction $S \rightsquigarrow \{\pi_0 S, \pi_{\leqslant 1}S, \pi_{\leqslant 2}S, \dots\}$ which from an n-groupoid $\mathcal{G}$ constructs a topological space $|\mathcal{G}|$ with $\pi_{\leqslant n}|\mathcal{G}| \simeq \mathcal{G}$. Grothendieck's *homotopy hypothesis* states that we can pass between spaces and higher groupoids. Indeed, the simplest models for higher groupoids are based on spaces. See [**L1**, §1.3] for further discussion. We will use this equivalence of groupoids and spaces to pass from the general algebraic formulation of field theories to a topological formulation in the invertible case.

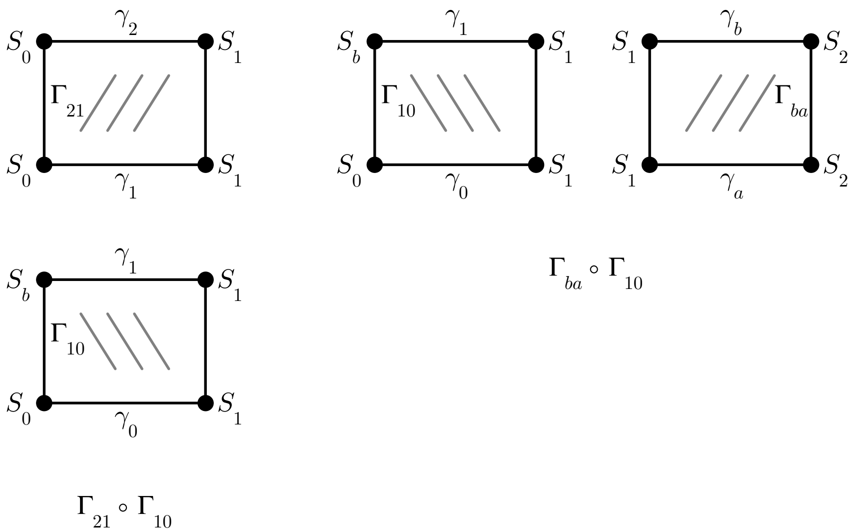

FIGURE 5.2. 2 compositions of 2-morphisms redux

In a general higher category morphisms are not invertible, even up to higher morphisms. Spaces no longer provide a good model.

REMARK 5.2. Fix nonnegative integers m, n with $m \geqslant n$. An m-category in which all k-morphisms for $k \geqslant n$ are invertible (up to higher morphisms) is called an *(m, n)-category*. Allow $m = \infty$ as well. Thus an $(m, 0)$-category is an m-groupoid. There is an elaborate theory of $(\infty, 1)$-categories (see [**L1**, **L2**] and the references therein), and by now too a well-developed theory of (∞, n)-categories for all n (see [**BR1**, **BR2**] and the references therein), with different sorts of models and comparisons between them. We use 'n-category' as shorthand for '(n, n)-category'.

REMARK 5.3. There is a notion of *symmetric monoidal structure* for higher categories. In the case of ∞-groupoids a symmetric monoidal structure corresponds to a topological space with an *∞-loop space* structure, as we describe in §6.6.

5.2. Examples of higher categories

EXAMPLE 5.4 (Higher bordism categories). The bordism 1-category $\mathrm{Bord}_{\langle 1,2\rangle}$ has objects closed 1-manifolds and morphisms 2-dimensional bordisms between them. Introduce a Morse function $t^1 \colon X \to \mathbb{R}$ on a bordism with, say $(t^1)^{-1}(0)$ the incoming boundary and $(t^1)^{-1}(1)$ the outgoing boundary. Composition of bordisms maps to concatenation of intervals under the Morse functions; see Figure 5.3.

Chopping up a 2-manifold at regular values of a single Morse function does not give a fully local picture—we should chop using a generic pair (t^1, t^2) of Morse functions. Then we can write a 2-manifold as a union of balls, presented as 2-manifolds with corners; see Figure 5.4. There are 2 composition laws, as in Figure 5.2, the hallmark of a 2-category. The global Morse functions t^1, t^2 are not part of the data[49]—only near the boundaries and corners do we need these local

[49]In some models, e.g., [**L1**, §2.2], they are in fact included.

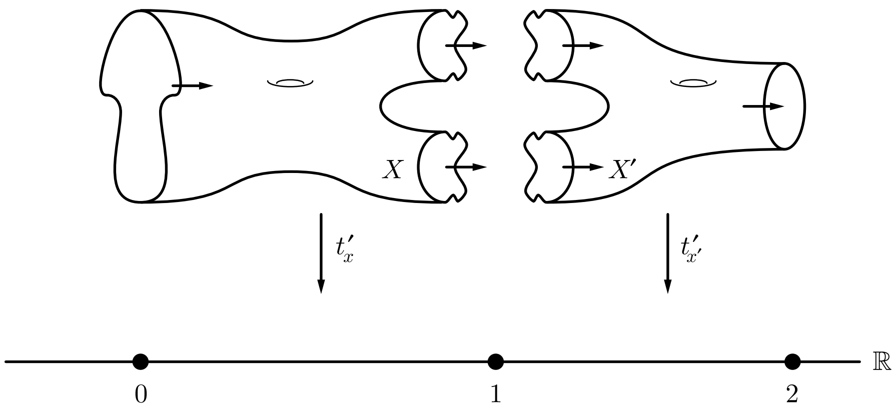

FIGURE 5.3. Chopping a bordism with a single Morse function

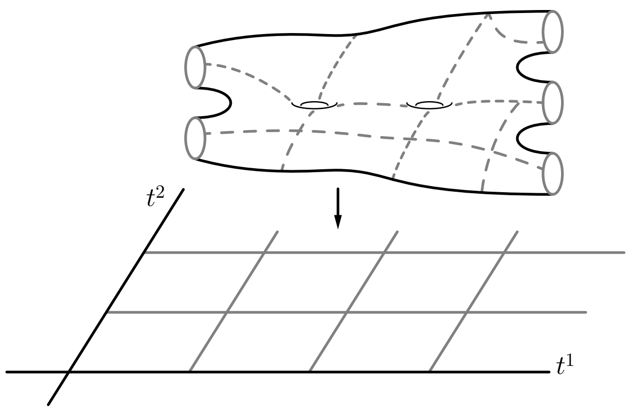

FIGURE 5.4. Chopping a bordism with two Morse functions

"time functions" to define compositions, or, more crudely, to know what is incoming and outgoing for each composition law. In other terms, $\mathrm{Bord}_{\langle 0,1,2\rangle} = \mathrm{Bord}_2$ is the 2-category whose objects are compact 0-manifolds; whose 1-morphisms are 1-dimensional bordisms; and whose 2-morphisms are 2-dimensional bordisms between 1-dimensional bordisms, up to diffeomorphism. Disjoint union is defined at all levels—on objects, 1-morphisms, and 2-morphisms—and endows Bord_2 with a symmetric monoidal structure. See [**S-P1**] for details in this 2-dimensional case and [**L1**, **CaSc**] for higher dimensions.

REMARK 5.5. We can construct an $(\infty, 2)$-category in place of the $(2, 2)$-category Bord_2. In that case the 2-morphisms between two 1-morphisms form the topological space of 2-dimensional bordisms between two 1-dimensional bordisms. The field theories we consider map closed 2-manifolds to complex numbers, so any potential k-morphism, $k > 2$, in the codomain higher category is the identity. Therefore, for these lectures the $(2, 2)$-category version of Bord_2 (and so the (n, n)-category version of Bord_n) suffices.

EXAMPLE 5.6 (Morita 2-category of algebras). Fix a field k. (Usually $k = \mathbb{C}$ in applications to physical field theories.) Let A_0, A_1 be k-algebras. An (A_1, A_0)-*bimodule* B is a left $(A_1 \otimes A_0^{\mathrm{op}})$-module, where A_0^{op} is the opposite algebra. The *Morita 2-category* Alg_k has k-algebras as objects. A 1-morphism $B\colon A_0 \to A_1$ is an (A_1, A_0)-bimodule B. A 2-morphism $f\colon B \to B'$ between 1-morphisms $B, B'\colon A_0 \to A_1$ is a linear map f which intertwines the action of $A_1 \otimes A_0^{\mathrm{op}}$. The composition of 1-morphisms $B_{10}\colon A_0 \to A_1$ and $B_{21}\colon A_1 \to A_2$ is the (A_2, A_0)-bimodule $B_{21} \otimes_{A_1} B_{10}$. The symmetric monoidal structure at all levels is tensor product over k. The tensor unit object is the field k as a k-algebra.

EXERCISE 5.7. Write explicitly the 2 composition laws on 2-morphisms.

EXAMPLE 5.8 ($\mathbb{Z}/2\mathbb{Z}$-graded algebras). The Morita 2-category Alg_k is a *delooping* of Vect_k in the sense that the category of 1-morphisms $\mathrm{Alg}_k(k,k)$ from the tensor unit to itself is isomorphic to Vect_k. There is a delooping $s\mathrm{Alg}_k$ of the category $s\mathrm{Vect}_k$ of super vector spaces (§1.4) whose objects are *superalgebras*—algebras in the symmetric monoidal category $s\mathrm{Vect}_k$. 1-morphisms are $\mathbb{Z}/2\mathbb{Z}$-graded bimodules, and 2-morphisms are even intertwiners between $\mathbb{Z}/2\mathbb{Z}$-graded bimodules.

Just as a 1-category has a maximal subgroupoid (Definition A.17), so too do higher categories. Recall Definition A.46 of a *Picard 1-groupoid.* For symmetric monoidal (higher) categories there is a maximal sub Picard groupoid, or *underlying Picard groupoid*, obtained by discarding non-invertible objects and non-invertible morphisms; see Definition A.48. For example, the Picard 1-groupoid underlying $s\mathrm{Vect}_k$ is $s\mathrm{Line}_k$, the groupoid whose objects are 1-dimensional $\mathbb{Z}/2\mathbb{Z}$-graded k-vector spaces and whose morphisms are invertible even maps between them. The group of isomorphism classes is $\mathbb{Z}/2\mathbb{Z}$—the grading is a complete invariant—and the automorphism group of any object is isomorphic to the units $k^\times$ in k. The Picard 2-groupoid underlying the 2-category $s\mathrm{Alg}_k$ has objects central simple superalgebras.

EXERCISE 5.9. Let $\mathcal{G}$ be a Picard 1-groupoid. Let $\pi_0\mathcal{G}$ denote the group of isomorphism classes of objects and $\pi_1\mathcal{G} = \mathcal{G}(1,1)$ the automorphism group of the tensor unit.

(1) Construct an isomorphism $\pi_1\mathcal{G} \to \mathcal{G}(x,x)$ for all objects $x \in \mathcal{G}$.
(2) Use the symmetry of the symmetric monoidal structure to construct a homomorphism

$$k\colon \pi_0\mathcal{G} \otimes \mathbb{Z}/2\mathbb{Z} \longrightarrow \pi_1\mathcal{G}. \tag{5.10}$$

(3) Show that the triple $(\pi_0\mathcal{G}, \pi_1\mathcal{G}, k)$ determines $\mathcal{G}$ up to equivalence.
(4) Compute $(\pi_0\mathcal{G}, \pi_1\mathcal{G}, k)$ for $\mathcal{G} = \mathrm{Line}_k$. For $\mathcal{G} = s\mathrm{Line}_k$. (You can restrict to $k = \mathbb{R}, \mathbb{C}$.)

EXERCISE 5.11. What is the underlying Picard groupoid of Bord_n?

EXAMPLE 5.12 (2-category of categories). There is a 2-category Cat_k whose objects are k-linear categories, whose 1-morphisms are linear functors between them, and whose 2-morphisms are natural transformations of functors. It is another delooping of Vect_k.

EXERCISE 5.13.

(1) Construct a functor $\mathrm{Alg}_k \to \mathrm{Cat}_k$.
(2) Can you find a delooping $\mathcal{C}$ of $s\mathrm{Vect}_k$ which receives functors $s\mathrm{Alg}_k \to \mathcal{C}$ and $\mathrm{Cat}_k \to \mathcal{C}$, both suitably "injective"?

5.3. Extended field theories

The two pillars of quantum field theory are *locality* and *unitarity*. As explained in §5.1 the domain $\mathrm{Bord}_{\langle n-1,n\rangle}(\mathcal{X}_n^{\nabla})$ of an n-dimensional field theory F (Axiom System 3.1) encodes gluing laws along codimension 1 cuts, but this is not full locality. We would like to cut n times so as to decompose Wick-rotated spacetime into pieces diffeomorphic to balls, as in Figure 5.4. For example, cutting in codimension 2 allows us to formulate a local dependence of the state space attached to an $(n-1)$-manifold Y. We remark that this level of locality is built into lattice models—see the tensor product (2.21) over local pieces—but is more subtle in field theories. We do not expect such a simple tensor product over $\mathbb{C}$, but rather expect data associated to $Z^{n-2} \subset Y^{n-1}$ and a more elaborate expression of the local dependence of $F(Y)$. Extended field theories are much more developed in the topological case, and that is where we focus our attention.

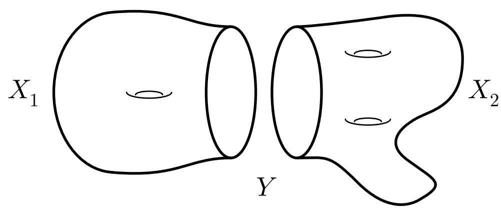

FIGURE 5.5. Factoring the numerical invariant $F(X)$

We can also motivate the extended notion of a field theory as a factorization of invariants. Let

$$F\colon \mathrm{Bord}_{\langle n-1,n\rangle}(\mathcal{X}_n) \to \mathrm{Vect}_k \tag{5.14}$$

be a topological field theory with values in the symmetric monoidal category of vector spaces over k. Thus the theory assigns a number in k to every closed n-manifold X (with $\mathcal{X}_n$-structure, which we do not mention in the sequel). Suppose X is cut in two by a codimension one submanifold Y, as indicated in Figure 5.5. View $X_1\colon \varnothing^{n-1} \to Y$ and $X_2\colon Y \to \varnothing^{n-1}$, so that $F(X_1)\colon k \to F(Y)$ and $F(X_2)\colon F(Y) \to k$. Let $\xi_1, \dots, \xi_k$ be a basis of $F(Y)$ and $\xi^1, \dots, \xi^k$ the dual basis of the dual vector space $F(Y)^\vee$. Write

$$\begin{aligned} F(X_1) &= a^i \xi_i \\ F(X_2) &= b_i \xi^i \end{aligned} \tag{5.15}$$

for some $a^i, b_i \in k$. Then the fact that $F(X) = F(X_2) \circ F(X_1)$ means

$$F(X) = a^i b_i. \tag{5.16}$$

In other words, the field theory allows us to factorize the numerical invariant of a closed n-manifold into a sum of products of numbers. An n-manifold with boundary has an invariant which is not a single number, but rather a vector of numbers.

We ask: Can we factor the vector space $F(Y)$? If so, what kind of equation replaces (5.16)? Well, it must be an equation of sets rather than numbers, and more precisely an equation for vector spaces. Our experience teaches us we should not write an *equality* but rather an *isomorphism*, and that isomorphism takes place in the *category* Vect_k. (Compare: the *equation* (5.16) takes place in the *set* k.)

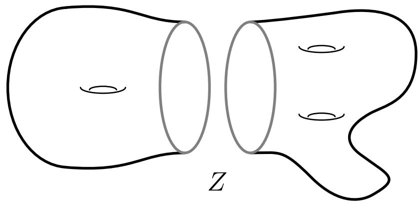

FIGURE 5.6. Factoring the vector space $F(Y)$

Thus given a decomposition of the closed $(n-1)$-manifold Y, as in Figure 5.6, we might by analogy with (5.15) write

$$\begin{aligned} F(Y_1) &= V^i c_i \\ F(Y_2) &= W_i c^i \end{aligned} \tag{5.17}$$

for vector spaces $V^i, W_i \in \mathrm{Vect}_k$, and by analogy with (5.16) write

$$F(Y) \cong \bigoplus_i V^i \otimes W_i \tag{5.18}$$

In these expressions $V^i, W_i \in \mathrm{Vect}_k$. But what are c_i, c^i? By analogy they should be dual bases of a Vect_k-*module* $F(Z)$ which is associated to the closed $(n-2)$-manifold Z. Of course, the field theory (5.14) does not assign anything in codimension 2, so we must *extend* our notion of field theory to carry out this factorization.

REMARK 5.19. There is no canonical choice for the codomain symmetric monoidal n-category $\mathcal{C}$ of an extended n-dimensional field theory. There are physics-inspired desiderata. At the "top two levels" $\mathcal{C}$ should reduce to $\mathrm{Vect}_{\mathbb{C}}$, or better $s\mathrm{Vect}_{\mathbb{C}}$. To wit, there is a tensor unit object $\mathbf{1}_0$, a tensor unit 1-morphism $\mathbf{1}_1 \colon \mathbf{1}_0 \to \mathbf{1}_0$, a tensor unit 2-morphism $\mathbf{1}_2 \colon \mathbf{1}_1 \to \mathbf{1}_1$, etc., and the "top two levels" of a symmetric monoidal n-category form the 1-category of endomorphisms of $\mathbf{1}_{n-2}$. Also, the complex linearity—which derives from interference and superposition in quantum mechanics—should continue to all levels. Finally, we have a highly evidenced candidate $\Sigma^n I\mathbb{C}^\times$ for the underlying Picard groupoid of $\mathcal{C}$; see §6.8.

REMARK 5.20. The expressions in (5.16) and (5.17) only make sense if the linear category $F(Z)$ is semisimple. This holds in elementary examples, but there are many derived situations of interest in which this reasoning is far too naive. We offer it here only as motivation for Axiom 5.21.

By contrast, the domain of an extended field theory is unambiguous: it is the bordism n-category described in Example 5.4. Since we restrict to topological theories, we can encode the background fields as a flabby tangential structure $\mathcal{X}_n \to B\mathrm{GL}_n\,\mathbb{R}$; see Definition 1.40.

AXIOM SYSTEM 5.21. Fix a symmetric monoidal n-category $\mathcal{C}$. An *extended n-dimensional topological field theory on $\mathcal{X}_n$-manifolds with values in* $\mathcal{C}$ is a symmetric monoidal functor

$$F\colon \mathrm{Bord}_n(\mathcal{X}_n) \longrightarrow \mathcal{C}. \tag{5.22}$$

See [**F3**, **La**, **F4**] for early versions of extended topological field theories and [**L1**, **AF**, **CS**] for modern treatments.

REMARK 5.23. We should incorporate into the axioms evaluation of the theory on smooth *families* of manifolds, as we saw in a non-topological theory (Example 2.46). In the topological case we can and should promote Axiom 5.21 from (n,n)-categories to (∞,n)-categories, where the higher morphisms in $\mathrm{Bord}_n(\mathcal{X}_n)$ are diffeomorphisms, isotopies, isotopies of isotopies, etc.; see [**L1**, §1.4]. Some discrete examples factor down to a map of (n,n)-categories, but many interesting examples, such as those coming from geometric representation theory, for example [**BGN**], do not.

EXAMPLE 5.24. Fix a finite group G and consider the $n = 2$ finite gauge theory $F_G\colon \mathrm{Bord}_{\langle 1,2\rangle} \to \mathrm{Vect}_{\mathbb{C}}$ discussed in Example 1.23. Extend to

$$\widehat{F}_G\colon \mathrm{Bord}_2 \longrightarrow \mathrm{Cat}_{\mathbb{C}.} \tag{5.25}$$

The 1-category $\widehat{F}_G(\mathrm{pt})$ can be computed by a "finite path integral" which linearizes the groupoid $\mathrm{Bun}_G(\mathrm{pt}) \simeq */\!/G$; see [**FHLT**, §3]. We obtain the linear category[50] $\widehat{F}_G(\mathrm{pt}) = \mathrm{Rep}_{\mathbb{C}}(G)$ of finite dimensional complex representations of G. Alternatively, we can consider the codomain 2-category $\mathcal{C} = \mathrm{Alg}_{\mathbb{C}}$, in which case the finite path integral computes the group algebra $\widehat{F}_G(\mathrm{pt}) = \mathbb{C}[G]$ with its convolution product.

EXERCISE 5.26. For $\widehat{F}_G\colon \mathrm{Bord}_2 \to \mathrm{Alg}_{\mathbb{C}}$ compute $\widehat{F}_G(S^1)$ from $\widehat{F}_G(\mathrm{pt})$. Hint: Express S^1 as the composition $e\circ\sigma\circ c$ of the duality data evaluation and coevaluation associated with pt and the symmetry σ.

EXERCISE 5.27. The 2-dimensional Arf theory of spin manifolds (Example 1.59) has an extension $\widehat{F}$ with codomain $\mathcal{C} = s\mathrm{Alg}_{\mathbb{C}}$; in fact, it factors through the underlying Picard subgroupoid of central simple algebras. Suppose the value $\widehat{F}(\mathrm{pt}_+)$ of $\widehat{F}$ on the positively oriented point is the complex Clifford algebra $\mathrm{Cliff}_1^{\mathbb{C}}$ on a single generator. Compute as in Exercise 5.26 that $\widehat{F}(S^1)$ is an odd line. There is a beautiful application of the extended Arf theory to spin Hurwitz numbers [**Gu**].

5.4. Extended operators

Extended locality has been around in quantum field theory for a long time in the form of extended operators. The line operators of Wilson [**Wi**] and 't Hooft [**tH**] in gauge theories are typical examples. Kapustin [**Ka1**] explained the relationship between extended operators and Axiom System 5.21. We give a brief account here. The ideas pertain to all field theories, but we focus on the topological case.

First, in an n-dimensional topological field theory F the vector space of *observables*, or *point operators*, at $x \in X$ is $F\bigl(S_\epsilon(x)\bigr)$, the value of the theory on the linking sphere of $x \in X$ for small radius ϵ. The precise value of $\epsilon > 0$ sufficiently small is irrelevant in a topological theory. Note $F\bigl(S_\epsilon(x)\bigr) \cong F(S^{n-1})$, but not canonically. To view elements of $F(S^{n-1}) \cong F\bigl(S_\epsilon(x)\bigr)$ as point operators we imagine ϵ small, as in (3.3). At the other extreme, for large spheres elements of $F(S^{n-1}) \cong F\bigl(S_\epsilon(x)\bigr)$ may be viewed as *states* in the theory on S^{n-1}. In a topological—or more generally a conformal—field theory the *state-operator correspondence* expresses the isomorphism between the vector space of states and the

[50] Although $\mathrm{Rep}_{\mathbb{C}}(G)$ has a (symmetric) monoidal structure, it is not used here.

vector space of operators. Correlation functions on X^n closed are computed at $x_1, \dots, x_k \in X$ using the bordism

$$\widehat{X}_\epsilon = X \setminus \bigcup_{n=1}^{k} B_\epsilon(x_i) \colon \coprod_{i=1}^{k} S_\epsilon(x_i) \longrightarrow \varnothing^{n-1} \tag{5.28}$$

for sufficiently small ϵ.

Remark 5.29 (Vector spaces of point operators). If the theory has Wick-rotated symmetry type (H_n, ρ_n), then the sphere has an H_n-structure and the vector space of point operators depends on it. If $H_n = \mathrm{SO}_n \times K$ or $H_n = \mathrm{O}_n \times K$, the extra data is a principal K-bundle $Q \to S^{n-1}$ (with connection). So there is a vector space V_Q of point operators for each Q. The group $\operatorname{Aut} Q$ of global gauge transformations acts on V_Q. For the trivial K-bundle this is the representation of the global symmetry group K on local operators. If K is finite, then the "twist operators" in an $n = 2$ dimensional theory evaluated on nontrivial principal K-bundles $Q \to S^1$ are familiar, for example in string theory. They are also familiar when $H_2 = \mathrm{Spin}_2$, in which case the operators associated to the nonbounding spin circle create a defect at the excised point which changes the spin structure on the punctured surface. In 3 dimensions, if H_3 is a Cartesian product as above and $K = \mathbb{T}$, then the twist operators in some sense create a magnetically charged instanton for the global symmetry group $\mathbb{T}$; the $\mathbb{Z}$-grading from the action of $\mathbb{T}$ on the point operators measures the electric charge.

Now instead of a point $x \in X$ we consider a connected 1-dimensional submanifold $L \subset X$. We could again cut out a tubular neighborhood of L, and the vector space attached to the boundary—the total space of the sphere bundle $S_\epsilon(\nu_L) \to L$ of the normal bundle to L—is the space of operators one can insert in a neighborhood of L. But extended locality tells that $F\bigl(S_\epsilon(\nu_L)\bigr)$ depends locally on points of L. Namely, for each $x \in L$ there is an associated invariant $F\bigl(S_\epsilon(\nu_L)_x\bigr)$ of the S^{n-1} link to L at x: the unit sphere in the normal space. For typical codomains $\mathcal{C}$ this invariant is a 1-category or algebra. In a non-topological theory we would take $\epsilon \to 0$, but in a topological theory this is unnecessary. All $F\bigl(S_\epsilon(\nu_L)_x\bigr)$ are isomorphic to $F(S^{n-2})$, but not canonically. In any case $F(S^{n-2})$ is often called the category of *line operators* in the theory.

We can also interpret a 1-dimensional submanifold $L \subset X$ as the Wick rotated version of the *worldline* of a *particle*. In that interpretation $F(S^{n-2})$ is the linear 1-category of particles in the theory. If a, b, c are objects in that category, then the vector space of morphisms $a \otimes b \to c$ parametrizes fusion processes. (The parallel statement for 0-dimensional submanifolds is that a point $x \in X$ is the Wick-rotated version of an *instanton*.)

The story continues to higher dimensional submanifolds in X, with higher categorical objects attached to their links. An $(n-1)$-dimensional submanifold $W \subset X$ is called a *domain wall*; the link is diffeomorphic to S^0 and $F(S^0)$ parametrizes endomorphisms of the theory F. More generally, we can have two theories F_0, F_1 on the two sides of the domain wall and a map[51] $F_0 \to F_1$ on the domain wall.

[51] More precisely, a map of the truncations $\tau_{\leqslant n-1}F_0 \to \tau_{\leqslant n-1}F_1$ to $(n-1)$-dimensional theories. Recall (Lemma 4.44) that a map as n-dimensional theories is necessarily invertible, and we want to allow non-invertible domain walls.

If X is a manifold with boundary, then the link of $\partial X \subset X$ is a single point and $F(\mathrm{pt})$ is interpreted as the collection of *boundary conditions* for F.

5.5. Algebra structures on spheres

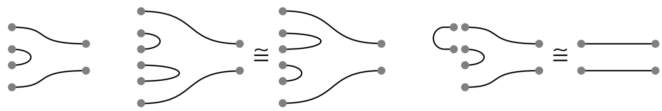

FIGURE 5.7. The algebra structure on S^0

To begin consider S^0 as an object in Bord_1. The bordism $X\colon S^0 \amalg S^0 \to S^0$ depicted on the left of Figure 5.7 defines an algebra structure on S^0. The associativity and unit axioms are illustrated as well. Similarly, the object S^1 in $\mathrm{Bord}_{\langle 1,2\rangle}$ is a *commutative* algebra object; commutativity of multiplication is depicted as the braiding β in Figure 5.8. The images of S^0 and S^1 under a symmetric monoidal functor are also (commutative) algebra objects. We used the commutative algebra structure on S^1 in §4.2; see Figure 4.5.

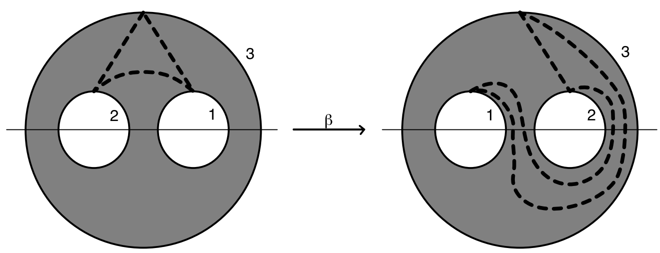

FIGURE 5.8. The algebra structure on S^1

There is an algebra structure on S^{n-1} in $\mathrm{Bord}_{\langle n-1,n\rangle}$; multiplication is the bordism $D^n \backslash (D^n \amalg D^n)$, a closed ball with two smaller balls removed. It is commutative if $n \geqslant 2$.

PROPOSITION 5.30. *Let* $F\colon \mathrm{Bord}_{\langle n-1,n\rangle} \to \mathrm{Vect}_{\mathbb{C}}$ *be a topological field theory. Then* $F(S^{n-1})$ *is an algebra, commutative if* $n \geqslant 2$.

REMARK 5.31. In theories with $\mathcal{X}_n$-structure, let the classifying map of $TD^n \to D^n$ be the constant map $D^n \to B\mathrm{O}_n$ to the basepoint, and the $\mathcal{X}_n$-structure the constant map $D^n \to \mathcal{X}_n$. This induces an $\mathcal{X}_n$-structure on the boundary S^{n-1} and, with that $\mathcal{X}_n$-structure, S^{n-1} is again an algebra.

REMARK 5.32. This algebra structure on the vector space of observables of a topological field theory has an analog in a non-topological theory: the *operator product expansion* [**Wei1**, Chapter 20]. It is a basic piece of structure in quantum field theory.

In an extended n-dimensional field theory $\widehat{F}$ there are algebra structures on spheres of all dimensions $\leqslant n-1$, and so too on their images under $\widehat{F}$. In this higher categorical context there are finer gradations of commutativity. For example, suppose C is a monoidal 1-category with multiplication functor $\mu\colon C \times C \to C$. A *braiding* is a natural transformation $\beta\colon \mu \circ \tau \to \mu$, where $\tau\colon C \times C \to C \times C$ is transposition. The braiding data satisfies some conditions we do not spell out here; see [**JS**]. The existence of a braiding is an expression of commutativity, but there is a stronger form, namely the requirement that the square of the braiding be the identity. Hence the distinction between *braided monoidal categories* and *symmetric monoidal categories.* In an n-dimensional extended topological field theory $F(S^{n-2})$ is braided commutative if $n = 3$ and symmetric commutative if $n \geqslant 4$. (See [**F4**] for additional features of the $n = 3$ case.)

There is a general framework—operads—in which to measure the level of commutativity [**MSS**]. In that framework S^{k-1} is an E_k*-algebra*, hence so is its image under a symmetric monoidal functor.

REMARK 5.33. Versions of these algebra structures show up in non-topological field theories, as in Remark 5.32, but now on line operators, surface operators, etc., and there are sometimes more subtle operations as well; see [**Ka1, BBBDN**].

5.6. Cobordism hypothesis

Baez-Dolan [**BD**] recognized the importance of duality and dualizability in topological field theories. Recall (Definition A.28) that a dualizable object $x \in C$ in a symmetric monoidal category behaves as a finite dimensional vector space: dualizability is a finiteness condition. Every object in a bordism category is dualizable, so too is its image under a field theory; see Lemma 1.20 and Corollary 1.21.

There is a higher notion of dualizability in a higher category [**L1**, §2.3]. Objects in $\mathrm{Bord}_n(\mathcal{X}_n)$ are *fully dualizable* in this sense. The *cobordism hypothesis* asserts that a fully extended topological field theory is determined by its value on a point, and any fully dualizable object in the codomain n-category $\mathcal{C}$ can occur. We proved the discrete 1-dimensional version, Theorem 4.46, in §4.4. The cobordism hypothesis does more; it identifies the homotopy type of the space of theories on manifolds with $\mathcal{X}_n$-structure and a fixed codomain $\mathcal{C}$. The theorem works for theories with values in (∞, n)-categories. Use the same notation as in Theorem 4.46, but now $(\mathcal{C}^{\mathrm{fd}})^\sim$ is the ∞-groupoid underlying the subcategory of fully dualizable objects in $\mathcal{C}$. The simplest case has domain n-framed manifolds (Example 1.42), and that is the version we state.

THEOREM 5.34 (Cobordism hypothesis). *Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. Then the map*

$$\begin{aligned} \Phi\colon\ \mathrm{TFT}_n^{(n\text{-framed})}(\mathcal{C}) &\longrightarrow (\mathcal{C}^{\mathrm{fd}})^\sim \\ F &\longmapsto F(\mathrm{pt}_+) \end{aligned} \tag{5.35}$$

is an equivalence of (∞, n)-groupoids.

The cobordism hypothesis was conjectured by Baez-Dolan [**BD**], proved by Hopkins-Lurie in the 2-dimensional case, and then by Lurie [**L1**] in general; see [**Te1, F5**] for expositions. Ayala-Francis [**AF**] are developing a proof using factorization homology.

There are many powerful applications of the cobordism hypothesis; for some applications in geometric representation theory see [**BN, BGN**]. In a different direction, the cobordism hypothesis promises to be a powerful tool for classification problems (of *non-invertible* theories) in condensed matter physics. In these lectures we eventually focus on *invertible* theories, and as we explain in Lecture 6 we work with infinite loop spaces in place of symmetric monoidal higher categories. In the invertible case the cobordism hypothesis reduces [**L1**, §2.5] to Theorem 6.67, a key ingredient in our story which can be proved using techniques of stable homotopy theory.

5.7. Extended example

Let G be a finite group. Recall from Example 1.23 the associated 3-dimensional gauge theory

$$\mathscr{G}_G\colon \mathrm{Bord}_{\langle 2,3\rangle} \longrightarrow \mathrm{Vect}_{\mathbb{C}} \tag{5.36}$$

which sums the tensor unit theory over the groupoid Bun_G of principal G-bundles. There is an extension [**F3**, **FHLT**] of this finite path integral construction to obtain an extended topological field theory. We tell some of its features here; see [**FT2**, §3] and the references therein for a recent account with more details. This theory is the simplest example of 3-dimensional quantum Chern-Simons field theory [**W6**, **DW**].

The first decision we make is a choice of codomain symmetric monoidal 3-category $\mathcal{C}$. A convenient choice is $\mathcal{C} = \mathrm{TensCat}_{\mathbb{C}}$, the 3-category whose objects are complex linear tensor categories $\mathcal{T}$. A 1-morphism $\mathcal{T} \to \mathcal{T}'$ is an $(\mathcal{T}', \mathcal{T})$-bimodule category, a 2-morphism is a linear functor which commutes with the tensor category actions, and a 3-morphism is a natural transformation of functors. See [**EGNO**, **DSPS**] for detailed developments of $\mathrm{TensCat}_{\mathbb{C}}$ as a 3-category.

Extended finite gauge theory is a symmetric monoidal functor

$$\widehat{\mathscr{G}}_G\colon \mathrm{Bord}_3 \longrightarrow \mathrm{TensCat}_{\mathbb{C}}\,. \tag{5.37}$$

Its value on a point, computed as the (co)limit[52] of the constant functor $\mathrm{Bun}_G(\mathrm{pt}) = */\!/G \to \mathrm{TensCat}_{\mathbb{C}}$ with value the tensor unit $\mathrm{Vect}_{\mathbb{C}}$, is

$$\widehat{\mathscr{G}}_G(\mathrm{pt}) = \mathrm{Vect}_{\mathbb{C}}[G], \tag{5.38}$$

the tensor category of finite rank complex vector bundles over G with convolution product: if $W, W' \to G$ are vector bundles, then

$$(W * W')_h = \bigoplus_{gg'=h} W_g \otimes W'_{g'}, \qquad h \in G; \tag{5.39}$$

the convolution product of morphisms is defined similarly. One may regard $\mathrm{Vect}_{\mathbb{C}}[G]$ as the "group ring" of G with coefficients in $\mathrm{Vect}_{\mathbb{C}}$. See [**FHLT**, Example 3.6] for the analogous 2-categorical (co)limit. Since the circle $S^1\colon \varnothing^0 \to \varnothing^0$ is a 1-morphism from the tensor unit (empty 0-manifold) to itself, $\widehat{\mathscr{G}}_G(S^1)$ is a complex linear category. It is computed by summing over $\mathrm{Bun}_G(S^1) \approx G/\!/G$:

$$\widehat{\mathscr{G}}_G(S^1) = \mathrm{Vect}_G(G) \tag{5.40}$$

is the linear category of complex vector bundles over the groupoid $\mathrm{Bun}_G(S^1) \approx G/\!/G$, i.e., the category of G-equivariant vector bundles over G. The values of $\widehat{\mathscr{G}}_G$ on 2- and 3-manifolds agree with those of $\mathscr{G}_G$, as computed in (1.26) and (1.29).

[52] In the finite path integral construction colimits and limits are equivalent, a property called *ambidexterity* [**HL**].

EXERCISE 5.41. Construct an extension of Dijkgraaf-Witten theory (Example 1.44) to a theory with domain the 3-category $\mathrm{Bord}_3(GL_3^+\mathbb{R})$.

The category of line operators in 3-dimensional finite gauge theory is $\widehat{\mathscr{G}_G}(S^1) \approx \mathrm{Vect}_G(G)$; see §5.4. There are subcategories of *Wilson* and *'t Hooft* operators, and these have a classical gauge-theoretic description. The Wilson operator is defined for $S \subset X$ an oriented connected 1-dimensional submanifold of a closed 3-manifold X together with $\chi\colon G \to \mathbb{T}$ a character of G. Then the function

$$h_{S,\chi}\colon \mathrm{Bun}_G(X) \longrightarrow \mathbb{C} \tag{5.42}$$

maps a principal G-bundle $P \to X$ to χ applied to the holonomy.[53] The finite path integral (1.29) with Wilson operator $(S,\chi)_W$ inserted is

$$\mathscr{G}_G\big(X;(S,\chi)_W\big) = \sum_{[P]\in\pi_0\,\mathrm{Bun}_G(X)} \frac{h_{S,\chi}(P)}{\#\,\mathrm{Aut}\,P}. \tag{5.43}$$

REMARK 5.44. Since $\widehat{\mathscr{G}_G}$ is a theory of unoriented manifolds, we should not need an orientation on S to define the loop operator. Indeed, we can drop the orientation and replace χ by a function from orientations of S to characters of G which inverts the character when the orientation is reversed.

The 't Hooft operator is defined for $S \subset X$ a co-oriented connected 1-dimensional submanifold and $\gamma \subset G$ a conjugacy class. Define the groupoid $\mathrm{Bun}_G\big(X;(S,\gamma)\big)$ whose objects are principal G-bundles $P \to X\backslash S$ with holonomy γ around an oriented linking curve to S. The finite path integral (1.29) with 't Hooft operator $(S,\gamma)_H$ inserted is

$$\mathscr{G}_G\big(X;(S,\gamma)_H\big) = \sum_{[P]\in\pi_0\,\mathrm{Bun}_G(X;(S,\gamma))} \frac{1}{\#\,\mathrm{Aut}\,P}. \tag{5.45}$$

As in Remark 5.44 we can drop the co-orientation and replace γ by a function from co-orientations to conjugacy classes which inverts under co-orientation reversal. The Wilson loop operators form the full subcategory of equivariant vector bundles supported at the identity $e \in G$, which is equivalent to the category $\mathrm{Rep}_{\mathbb{C}}(G)$ of finite dimensional complex representations of G. The 't Hooft operators form the full subcategory of equivariant vector bundles on which the centralizer $Z_x(G)$ of each $x \in G$ acts trivially on the fiber at x. The general loop operator is an amalgam of these two extremes and does not have a classical description.

REMARK 5.46. Let $\widehat{F}$ be an n-dimensional extended topological field theory and $S \subset X$ a connected 1-dimensional submanifold of an n-manifold X. The link of S at each point is diffeomorphic to S^{n-2}, but there is no preferred diffeomorphism. Furthermore, the group of diffeomorphisms of S^{n-2} may act nontrivially on $\widehat{F}(S^{n-2})$. Therefore, to specify a loop operator on S it is not sufficient[54] to give an object of $\widehat{F}(S^{n-2})$. For finite gauge theory $\widehat{\mathscr{G}_G}$ the objects in $\widehat{\mathscr{G}_G}(S^1)$ corresponding to Wilson and 't Hooft operators are SO_2-invariant, so no normal framing is required.

[53] The holonomy is determined up to conjugacy and χ is a class function.

[54] For example, typically in 3-dimensional Chern-Simons theory [**W6**] one imposes a normal framing of S to rigidify the SO_2-action (Dehn twist, ribbon structure) on $\widehat{F}(S^1)$.

Now suppose $G = A$ is a finite abelian group. Then the conjugation action of A on A is trivial, and we identify

$$\mathrm{Vect}_A(A) \simeq \mathrm{Vect}(A \times A^\vee) \tag{5.47}$$

by decomposing the representation of A on each fiber. Wilson operators are labeled by vector bundles pulled back under the projection $A \times A^\vee \to A^\vee$; 't Hooft operators by vector bundles pulled back under the projection $A \times A^\vee \to A$.

There is another extended 3-dimensional topological field theory

$$\widehat{\mathscr{R}}_G\colon \mathrm{Bord}_3 \longrightarrow \mathrm{TensCat}_{\mathbb{C}} \tag{5.48}$$

defined using the cobordism hypothesis by declaring that the value

$$\widehat{\mathscr{R}}_G(\mathrm{pt}) = \mathrm{Rep}_{\mathbb{C}}(G) \tag{5.49}$$

on a point is the tensor category of finite dimensional complex representations of G.

REMARK 5.50. If G is nonabelian we do not know of a "classical model" for $\widehat{\mathscr{R}}_G$. Also, to obtain a theory on general manifolds, as opposed to framed manifolds, we must specify "$GL_3\mathbb{R}$-equivariance data"; see [**FT2**, §3.2]. The tensor categories $t\mathrm{Vect}[G]$ and $\mathrm{Rep}_{\mathbb{C}}(G)$ are isomorphic objects of $\mathrm{TensCat}_{\mathbb{C}}$, i.e., are Morita equivalent. This leads to an equivalence of extended field theories $\widehat{\mathscr{G}}_G \approx \widehat{\mathscr{R}}_G$ by another application of the cobordism hypothesis. (The Morita equivalence only preserves $GL_3^+\mathbb{R}$-equivariance data, not GL_3-equivariance data, so the equivalence of field theories only holds after pullback via $\mathrm{Bord}_3(GL_3^+\mathbb{R}) \to \mathrm{Bord}_3$.) If $G = A$ is *abelian*, then there is a Pontrjagin dual group $A^\vee = \mathrm{Hom}(A, \mathbb{C}^\times)$ and $\widehat{\mathscr{R}}_A \approx \widehat{\mathscr{G}}_{A^\vee}$ is also a finite gauge theory. In that case the equivalence $\widehat{\mathscr{G}}_A \approx \widehat{\mathscr{G}}_{A^\vee}$ is a finite group version of *electromagnetic duality*. For example, if Y is a closed oriented surface, then the isomorphism

$$\widehat{\mathscr{G}}_{A^\vee}(Y) = \mathrm{Fun}\big(H^1(Y;A)\big) \xrightarrow{\;\cong\;} \mathrm{Fun}\big(H^1(Y;A^\vee)\big) = \widehat{\mathscr{G}}_{A^\vee}(Y) \tag{5.51}$$

is the Fourier transform between functions on Pontrjagin dual groups. Electromagnetic duality exchanges A and $A^\vee$ in (5.47), so exchanges Wilson and 't Hooft operators.

LECTURE 6

Invertibility and Stable Homotopy Theory

The link between invertible field theories and stable homotopy theory was introduced in [**FHT**]. The basic idea is simple: if $M \to N$ is a homomorphism of commutative monoids which factors through the abelian group of units $N^\times \subset N$, then there is an induced homomorphism of abelian groups $|M| \to N^\times$, where $|M|$ is the *group completion* of M. For field theory we use a categorified variant: an invertible field theory factors through a map of (higher) Picard groupoids, and the geometric realization of a Picard groupoid is an infinite loop space, the 0-space of a spectrum. Therefore, *we model invertible field theories as maps of spectra.* The issue is to identify the spectra, a task we carry out for *topological* field theories. The domain spectrum is the geometric realization of a higher bordism category (Example 5.4), and this is known to be a Madsen-Tillmann spectrum. For the codomain we need more input, because in the non-invertible case there is no canonical codomain to use as a starting point. In the invertible case there *is* a universal choice—the *character dual* $I\mathbb{C}^\times$ to the sphere spectrum—which we motivate by asking that the partition function determine the entire theory. But then there is magic, revealed at the end of §6.8, which makes that choice work even better than we may have expected. We define invertible topological field theories as maps of spectra in Ansatz 6.89, though in principle it should be a theorem derived from Axiom System 5.21 and some input about the codomain. Once we formulate invertible topological field theories as maps of spectra, we can identify the abelian group of deformation classes, which we do in §6.9, again formally as a definition. Passing to deformation classes naturally brings in new invertible theories, as we explain in §6.10. There we model the deformation class as a "continuous" invertible theory, a notion which does not generalize to the non-invertible case. The framework developed in this lecture incorporates strong locality of extended field theory. However, it does not include unitarity, which is essential for applications to physics and which we take up in the next two lectures.

We begin in §6.1 with the groupoid completion of a category, which we compute explicitly in §6.3 for the 1-dimensional bordism category. We define invertible field theories and give examples in §6.2. In the non-extended case there are simple Morse theory arguments to illustrate the connection between invertible topological field theories and bordism, as we explain in §6.4. Then, after some comments about spectra in §6.5, we give a detailed description of Madsen-Tillmann and Thom spectra in §6.6. The character and Anderson duals to the sphere spectrum, with motivation from algebra, are introduced in §6.7. Extended invertible field theories and deformation classes are discussed in §6.8–§6.10.

6.1. Categorical preliminaries

DEFINITION 6.1. Let C be a category. A *groupoid completion* $(|C|, i)$ of C is a groupoid $|C|$ and a functor $i\colon C \to |C|$ which satisfies the following universal property: If $\mathcal{G}$ is a groupoid and $f\colon C \to \mathcal{G}$ a functor, then there exists a unique groupoid map $\tilde{f}\colon |C| \to \mathcal{G}$ which makes the diagram

$$\begin{tikzcd} C \arrow[rr, "i"] \arrow[dr, "f"'] & & {|C|} \arrow[dl, dashed, "\tilde{f}"] \\ & \mathcal{G} & \end{tikzcd} \tag{6.2}$$

commute.

Intuitively, $|C|$ is obtained from C by "inverting all of the arrows", much in the same way that the group completion of a monoid is constructed. There is a choice whether to require that $\tilde{f}$ in (6.2) be unique; we make that choice. Then $(|C|, i)$ is unique up to unique isomorphism, and the map i is an isomorphism $i_0\colon C_0 \to |C|_0$ on objects. As for existence, one approach is to use the geometric realization of the category C and let $|C|$ be its fundamental groupoid.

EXERCISE 6.3.

(1) Let M be a monoid. Write the universal property for a group completion $i\colon M \to |M|$ and prove existence and uniqueness.
(2) Let C be a (small) category with C_0, C_1 the sets of objects and morphisms, respectively. Generate a free groupoid by formally introducing inverse arrows—another copy of C, with source and target maps swapped. Then construct a groupoid completion as the quotient by the relation that compositions in C equal formal compositions in the free groupoid.
(3) If C is symmetric monoidal, induce a symmetric monoidal structure on $|C|$. If every object in C is dualizable, show that $|C|$ is a Picard groupoid.

REMARK 6.4. One can also define the groupoid completion of an (∞, n)-category; see [**L1**, Remark 2.5.5].

6.2. Invertible field theories

We work with extended topological field theories, but emphasize that the notion of an invertible field theory works in the *non-topological* case as well. Invertible field theories were introduced in [**FM2**] in the context of M-theory for non-extended topological field theories.

DEFINITION 6.5. Fix a nonnegative integer n, a flabby tangential structure $\mathcal{X}_n$, and a symmetric monoidal (∞, n)-category $\mathcal{C}$. Then a topological quantum field theory $\alpha\colon \mathrm{Bord}_n(\mathcal{X}_n) \to \mathcal{C}$ is *invertible* if it factors through the underlying Picard groupoid[55] of $\mathcal{C}$:

$$\begin{tikzcd} \mathrm{Bord}_n(\mathcal{X}_n) \arrow[rr, "\alpha"] \arrow[dr, dashed] & & \mathcal{C} \\ & \mathcal{C}^\times \arrow[ur] & \end{tikzcd} \tag{6.6}$$

If α is invertible, it follows from the universal property of the groupoid completion (Definition 6.1, but now extended to (∞, n)-categories) that there is a factorization

$$\begin{tikzcd} \mathrm{Bord}_n(\mathcal{X}_n) \arrow[r, "\alpha"] \arrow[d] & \mathcal{C} \\ |\,\mathrm{Bord}_n(\mathcal{X}_n)| \arrow[r, "\tilde{\alpha}"] & \mathcal{C}^\times \arrow[u] \end{tikzcd} \tag{6.7}$$

We identify the invertible theory with the map $\tilde{\alpha}$ (and usually omit the tilde.)

REMARK 6.8. The groupoid completion $|\,\mathrm{Bord}_n(\mathcal{X}_n)|$ of a bordism category is a Picard groupoid, since every object of $\mathrm{Bord}_n(\mathcal{X}_n)$ is dualizable (Lemma 1.20).

EXERCISE 6.9. A topological field theory α is invertible if and only if it is an invertible object in the symmetric monoidal category of topological field theories.

EXAMPLE 6.10. We invite the reader to check that several of the theories we have already introduced are invertible, though not always extended. These include "classical" theories such as the classical Dijkgraaf-Witten theory (Example 1.50); theories of characteristic numbers (Example 1.52), including the Euler theory; the 2-dimensional Arf theory (Example 1.59); and the (non-topological) 1-dimensional theory of parallel transport (Example 2.46).

EXAMPLE 6.11. According to Theorem 4.46 to define an oriented 1-dimensional TQFT

$$\alpha\colon \mathrm{Bord}_{\langle 0,1\rangle}(\mathrm{SO}_1) \to s\mathrm{Vect}_k \tag{6.12}$$

we need only specify $\alpha(\mathrm{pt}_+)$, which we take to be the *odd k-line.* It follows easily that $\alpha(S^1) = -1$.

EXAMPLE 6.13 (Euler theory). It is easy to fully extend the n-dimensional Euler theory ϵ_μ in (1.53). Since the vector space in codimension 1 is always the tensor unit, we can extend the theory to higher codimensions by mapping every k-morphism, $k < n$, to the tensor unit. Hence the truncation

$$\tau_{\leqslant n-1}\epsilon_\mu\colon \mathrm{Bord}_{n-1} \longrightarrow \tau_{\leqslant n-1}\mathcal{C} \tag{6.14}$$

is the trivial theory—the tensor unit theory—with codomain the truncation of $\mathcal{C}$ that drops non-invertible n-morphisms.

A more down-to-earth explanation: any compact n-manifold, with or without boundary, has an Euler *number*, so the line attached to the boundary $(n-1)$-manifold is the complex line $\mathbb{C}$. (The exponentiated Euler number takes values in

[55] See §A.4 for the underlying groupoid in the 1-categorical case. For a non-extended field theory the codomain 1-category is typically a category of topological complex vector spaces; the underlying groupoid is the groupoid $\mathrm{Line}_{\mathbb{C}}$ of complex lines and invertible linear maps. We use the extension [**L1**, Remark 2.4.5] to (∞, n)-categories.

this line.) The truncated theory which assigns the trivial line to every closed $(n-1)$-manifold fully extends to the tensor unit theory no matter what the codomain category. By contrast, for other invariants, such as the classical Dijkgraaf-Witten invariant with nonzero cocycle (Example 1.50), the value on a compact n-manifold with boundary is not a number, but rather an element of a nontrivial complex line.

EXAMPLE 6.15 (Kervaire theory). Assume that $n = 4\ell+1$. The *Kervaire semi-characteristic* [**Ke**, **LMP**] of a closed orientable n-manifold X is the mod 2 sum of the even Betti numbers. It is a KO-characteristic number [**AS1**, §3]. Analogous to Example 1.52 there is an invertible field theory of oriented bordisms

$$\kappa\colon \operatorname{Bord}_{\langle n-1,n\rangle}(\mathrm{SO}_n) \longrightarrow s\mathrm{Vect}_{\mathbb{C}} \tag{6.16}$$

whose partition function is -1 to the power of the Kervaire semi-characteristic. Note $\kappa(S^n) = -1$. The case $n = 1$ is the theory in Example 6.11.

EXAMPLE 6.17. There is a nontrivial invertible theory [**FKS**, §5] of spin 3-manifolds X equipped with a map $\phi\colon X \to S^2$. It is a theory of order 2 which restricted to $X = S^3$ gives the mod 2 Hopf invariant of ϕ. The partition function has a geometric description. Fix a regular value $p \in S^2$ of $\phi\colon X \to S^2$, and fix a basis e_1, e_2 of T_pS^2. This produces a normal framing of the 1-manifold $S := \phi^{-1}(p)$, and so a stable normal framing. The partition function is -1 to the power the class of the stably normally framed manifold S in the framed bordism group $\Omega_1^{\text{framed}} \cong \mathbb{Z}/2\mathbb{Z}$.

EXERCISE 6.18.
(1) For $X = S^3$ prove that this gives the mod 2 Hopf invariant.
(2) Construct the non-extended theory explicitly and geometrically.

REMARK 6.19. The input to the Feynman path integral on a closed n-manifold is the exponentiated classical action e^{iS} of a classical field theory. The locality properties are summarized by the requirement that e^{iS} be the partition function of an extended invertible field theory.

6.3. Geometric realization of 1-dimensional bordism

It is illuminating to compute the groupoid completion (Definition 6.1) explicitly for the discrete 1-category $\operatorname{Bord}_{\langle 0,1\rangle}(\mathrm{SO}_1)$. Note that an orientation of a 1-manifold is equivalent to a 1-framing.

THEOREM 6.20. *The groupoid completion of the oriented bordism category has*

$$\pi_0|\operatorname{Bord}_1(\mathrm{SO}_1)| \cong \mathbb{Z}, \qquad \pi_1|\operatorname{Bord}_1(\mathrm{SO}_1)| \cong \mathbb{Z}/2\mathbb{Z}, \tag{6.21}$$

with nontrivial k-invariant.

PROOF. The argument for π_0 is straightforward and amounts to the assertion $\Omega_0^{\mathrm{SO}} \cong \mathbb{Z}$.

Set $B = \operatorname{Bord}_1(\mathrm{SO}_1)$. To compute $\pi_1 B$ we argue as follows. First, $1_B = \varnothing^0$ is the empty 0-manifold, so $\operatorname{End}(1_B) = B(1_B, 1_B)$ consists of diffeomorphism classes of closed oriented 1-manifolds. By the classification of 1-manifolds there is an isomorphism of commutative monoids $\operatorname{End}(1_B) \cong \mathbb{Z}^{\geqslant 0}$ which counts the number of (circle) components of a bordism X. Let X_n denote the disjoint union of n oriented

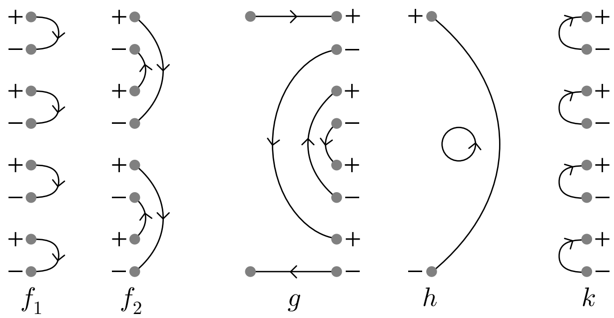

FIGURE 6.1. Some oriented 1-dimensional bordisms

circles. Note that the circle has a unique orientation up to orientation-preserving diffeomorphism. Then using the bordisms defined in Figure 6.1 we have

$$f_1 \circ g = f_2 \circ g = h \tag{6.22}$$

as morphisms in B. In the groupoid completion $|B|$ we can compose on the right with the inverse to $i(g)$ to conclude that $i(f_1) = i(f_2)$, where $i\colon B \to |B|$. That implies that in $|B|$ we have

$$i(f_1) \circ i(k) = i(f_2) \circ i(k), \tag{6.23}$$

hence $i(X_4) = i(X_2)$. Cancel to conclude $i(X_2) = i(\varnothing^0) = 1_{|B|}$. To rule out the possibility that $i(X_1)$ is also the tensor unit, use the invertible topological field theory in Example 6.11. It maps the oriented circle X_1 to a non-tensor unit (which necessarily has order two).

It remains to show that every morphism $\varnothing^0 \to \varnothing^0$ in $|B|$ is equivalent to a union of circles and their formal inverses. Observe first that the inverse of the "right elbow" is the "left elbow" union a circle, since their composition in one order is $i(X_2)$, which is equivalent to the identity map. Next, any morphism $\varnothing^0 \to \varnothing^0$ in $|B|$ is the composition of a finite number of morphisms $Y_{2k} \to Y_{2\ell}$ and inverses of such morphisms, where Y_n is the 0-manifold consisting of n points. Furthermore, each such morphism is the disjoint union of circles, identities, right elbows, left elbows, and their inverses. Identities are self-inverse and the elbows are each others inverse, up to a circle, hence carrying out the compositions of elbows and identities we obtain a union of circles and their inverses, as desired.

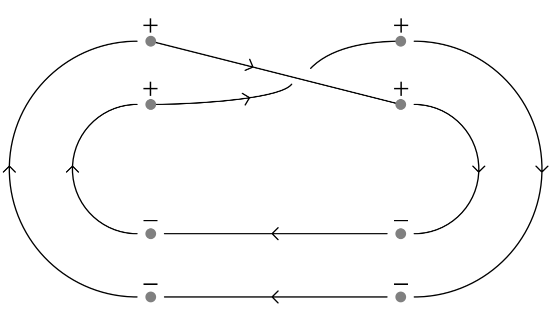

FIGURE 6.2. The k-invariant of $|\operatorname{Bord}_{\langle 0,1\rangle}(\mathrm{SO}_1)|$

Figure 6.2 illustrates the computation of the k-invariant (A.50) of $|B|$. The nontrivial element of $\pi_0|B^{\mathrm{SO}}|$ is represented by pt_+, and the top part of the diagram is the symmetry $\sigma\colon \mathrm{pt}_+ \amalg \mathrm{pt}_+ \to \mathrm{pt}_+ \amalg \mathrm{pt}_+$ in B^{SO}. The inverse of $\mathrm{pt}_+ \amalg \mathrm{pt}_+$ is $\mathrm{pt}_- \amalg \mathrm{pt}_-$, and we tensor with the identity map on the inverse. That tensor product is the disjoint union of the top and bottom four strands. The left and right ends of Figure 6.2 implement the isomorphism $1_{|B|} \cong \mathrm{pt}_+ \amalg \mathrm{pt}_+ \amalg \mathrm{pt}_- \amalg \mathrm{pt}_-$. The result is the oriented circle X_1, which is the generator of $\pi_1|B|$. $\square$

6.4. Non-extended invertible field theories and Reinhart bordism

Theorem 6.27 below is a first indication of the relationship between invertible topological field theories and bordism invariants. We first introduce an unstable notion of bordism.

Definition 6.24. Let X be a closed n-manifold. Then X is (*Reinhart*) *null bordant* if there exists a compact $(n+1)$-manifold W with $\partial W = X$ and a rank n vector bundle $E \to W$ together with an isomorphism $TW \xrightarrow{\cong} \underline{\mathbb{R}} \oplus E$ whose restriction to ∂W is the direct sum $\nu \oplus TX \xrightarrow{\cong} \underline{\mathbb{R}} \oplus E\big|_X$ of an outward trivialization of the normal bundle $\nu \to X$ to the boundary and an isomorphism of TX with $E\big|_X$.

Remark 6.25. The original definition of Reinhart [**Re**] for oriented and unoriented manifolds is equivalent; our formulation works for arbitrary tangential structures. See [**E**, Appendix] for the relationship of Definition 6.24 to Madsen-Tillmann spectra.

If X is Reinhart null bordant, then X is Thom null bordant (Definition 1.1), but the converse is not true. Use Definition 6.24 to define[56] an equivalence relation and Reinhart bordism groups Ω_n^R. We can incorporate an n-dimensional flabby tangential structure $\mathcal{X}_n \to B\mathrm{GL}_n\,\mathbb{R}$, and so define $\Omega_n^R(\mathcal{X}_n)$.

Exercise 6.26.

(1) If n is even prove that S^n is Thom null bordant but not Reinhart null bordant.
(2) Fix an even positive integer n. Prove that for any $\mu \in \mathbb{C}^\times$, the map $X \mapsto \mu^{\mathrm{Euler}(X)}$ is an unoriented Reinhart bordism invariant.
(3) If $n \equiv 1 \pmod 4$ prove that S^n has order 2 in $\Omega_n^R(\mathrm{SO}_n)$. If $n \equiv 3 \pmod 4$ prove that S^n is oriented Reinhart null bordant.
(4) If $n \equiv 1 \pmod 4$ prove that the Kervaire semicharacteristic (Example 6.15) is an oriented Reinhart bordism invariant.

The first part of the following theorem is [**FM2**, Proposition 5.8]. Recall the Euler theory ϵ_μ (1.53) and the Kervaire theory κ (Example 6.15).

Theorem 6.27. *Let* $\alpha\colon \mathrm{Bord}_{\langle n-1,n\rangle}(\mathrm{SO}_n) \to s\mathrm{Vect}_{\mathbb{C}}$ *be an invertible (non-extended) field theory.*

(1) *If* $\alpha(S^n) = 1$, *then the partition function is a Thom bordism invariant*

$$\begin{aligned} \Omega_n(\mathrm{SO}_n) &\longrightarrow \mathbb{C}^\times \\ [X] &\longmapsto \alpha(X) \end{aligned} \tag{6.28}$$

[56] As explained in Remark 6.66 below, the equivalence classes form a commutative monoid whose group completion is the Reinhart bordism group.

(2) *For all α the partition function is a Reinhart bordism invariant*
$$\Omega_n^R(\mathrm{SO}_n) \longrightarrow \mathbb{C}^\times.$$

PROOF. First, we study the behavior of α under Morse surgery. Fix $r \in \mathbb{Z}$, $0 \leqslant r \leqslant n$, and set

$$\begin{aligned} H'_r &= D^r \times S^{n-r} \\ H''_r &= S^{r-1} \times D^{n-r+1}. \end{aligned} \tag{6.29}$$

Orient spheres using the boundary orientation induced from the closed ball, orient H'_r, H''_r (r odd) with the product orientation, and orient H''_r (r even) oppositely to the product orientation; then H'_r, H''_r induce the same orientation on $S^{r-1} \times S^{n-r}$. View H'_r, H''_r as bordisms with outgoing boundary and write[57]

$$\alpha(H''_r) = \lambda_r\, \alpha(H'_r) \quad \in \alpha(S^{r-1} \times S^{n-r}) \tag{6.30}$$

for $\lambda_r \in \mathbb{C}^\times$. Set $r = n+1$ to deduce

$$a := \alpha(S^n) = \lambda_{n+1}. \tag{6.31}$$

Let $\widehat{H}'_r, \widehat{H}''_r$ be the manifolds (6.29) as bordisms with incoming boundary and opposite orientation. Then there are diffeomorphisms (isomorphisms in the bordism category)

$$\begin{aligned} \widehat{H}'_r \circ H''_r \cong \widehat{H}''_r \circ H'_r &\cong S^n \\ \widehat{H}'_r \circ H'_r &\cong S^r \times S^{n-r} \\ \widehat{H}''_r \circ H''_r &\cong S^{r-1} \times S^{n-r+1} \end{aligned} \tag{6.32}$$

Let $\nu_r = \alpha(S^r \times S^{n-r})$. Deduce from (6.30), (6.31), and (6.32) that

$$a = \lambda_r \nu_r = \lambda_r^{-1} \nu_{r-1}, \tag{6.33}$$

Now $\nu_1 = 1$, as follows from footnote [57], from which we conclude

$$\lambda_r = \begin{cases} a^{-1}, & r \text{ even}, \\ a, & r \text{ odd}; \end{cases} \qquad \text{and} \qquad \nu_r = \begin{cases} a^2, & r \text{ even}, \\ 1, & r \text{ odd}. \end{cases} \tag{6.34}$$

Recall that Morse surgery is an elementary bordism of the form

$$W_r(X') \colon H'_r \cup_{S^{r-1} \times S^{n-r}} X' \longrightarrow H''_r \cup_{S^{r-1} \times S^{n-r}} X' \tag{6.35}$$

for some compact n-manifold X' with boundary; $W_r(X')$ is the trace of the surgery with incoming patient the domain manifold and outgoing patient the codomain manifold.

Now to the assertions in the theorem. For (1) we have $a = 1$ and must show that if X is a closed oriented n-manifold, then $\alpha(X)$ is unchanged by Morse surgery (6.35), i.e., $\lambda_r = 1$ for all r. That follows immediately from (6.34). For (2) if n is even set $\alpha' = \alpha \cdot \epsilon_\mu$, where μ is chosen so that $\mu^{-2} = \alpha(S^n)$ and ϵ_μ is the Euler theory (1.53). Then α' satisfies the hypothesis of (1), so its partition function is a Thom bordism invariant. It now follows from Exercise 6.26(2) that α is a Reinhart bordism invariant. If n is odd, then a closed n-manifold X is oriented Reinhart null bordant if and only if it bounds a compact $(n+1)$-manifold W of even Euler

[57] If a closed oriented $(n-1)$-manifold Y is a boundary, i.e., if there exists a bordism $X \colon \varnothing^{n-1} \to Y$, then since $\alpha(X) \colon \mathbb{C} \to \alpha(Y)$ is an isomorphism we conclude that $\alpha(Y)$ is an *even* line.

number. (This is a consequence of Definition 6.24, or see [**Re**, §2]. In fact, the parity of $\mathrm{Euler}(W)$ is determined by X.) Any Morse function expresses W as a composition of elementary bordisms (6.35) and computes $\alpha(X)$ as the product of the corresponding λ_r. The Euler number of the elementary bordism $W_r(X')$ rel its incoming boundary is $(-1)^r$, and the Euler number of $W_r(X')$ is the sum over elementary bordisms. Hence if $\mathrm{Euler}(W)$ is even then an equal number of even and odd r occur. It follows from (6.34) that $\alpha(X) = 1$, which was to be proved. □

REMARK 6.36. If $n \equiv -1 \pmod 4$, then every closed n-manifold bounds a compact manifold with even Euler number, hence one with Euler number zero. Example: $S^3 = \partial\big(\mathbb{CP}^2 \# (S^1 \times S^3) \setminus D^4\big)$. If $n \equiv 1 \pmod 4$ then it follows from (6.34) that $a^2 = 1$, and if $a = -1$ tensor α with the Kervaire theory (Example 6.15) to obtain a theory with unit partition function on the sphere. This gives another proof of Theorem 6.27 for $n \equiv 1 \pmod 4$.

EXERCISE 6.37. Generalize Theorem 6.27 to arbitrary symmetry types.

6.5. Picard groupoids and spectra

Recall (Remark 5.1) that according to Grothendieck's homotopy hypothesis we can pass freely between topological spaces and ∞-groupoids. Now refine that statement by introducing an extra abelian group structure on each side.

DEFINITION 6.38.

(1) Let T_0 be a pointed topological space. Then T_0 is a *loop space* if there exists a pointed topological space T_1 and a homotopy equivalence $T_0 \simeq \Omega T_1$. It is an *infinite loop space* if there exists[58] a sequence $T_0, T_1, \dots$ of pointed topological spaces such that $T_q \simeq \Omega T_{q+1}$ for all $q \in \mathbb{Z}^{\geqslant 0}$.
(2) A *spectrum* T is a sequence $\{T_q\}_{q\in\mathbb{Z}}$ of pointed spaces and maps $s_q\colon \Sigma T_q \to T_{q+1}$.
(3) An Ω-*spectrum* is a spectrum T such that the adjoints $t_q\colon T_q \to \Omega T_{q+1}$ of the structure maps are weak homotopy equivalences. In particular, T_0 is an infinite loop space.

A spectrum can be completed to an Ω-spectrum.

If T_0 is a loop space, then $\pi_0 T_0 \cong \pi_0 \Omega T_1 \cong \pi_1 T_1$ is a group. If T_0 is a double loop space, then $\pi_0 T_0 \cong \pi_0 \Omega^2 T_2 \cong \pi_2 T_2$ is an abelian group. The higher groupoids $\pi_{\leqslant q} T_0$ also inherit group-like structures: if T_0 is an infinite loop space, then $\pi_{\leqslant q} T_0$ is a Picard q-groupoid for all q. In particular, the homotopy groups of a spectrum are abelian groups, and taken together they form a $\mathbb{Z}$-graded abelian group $\pi_\bullet T = \bigoplus_{q\in\mathbb{Z}} \pi_q T$. Thus a spectrum is a refinement of a $\mathbb{Z}$-graded abelian group.

EXERCISE 6.39. Construct the Picard groupoid structure on the fundamental groupoid $\pi_{\leqslant 1} T_0$ of an infinite loop space. How much of the infinite loop structure do you use?

[58] One may want to make the deloopings data rather than a condition, in which case we arrive at (3). We conflate 'infinite loop space' and 'Ω-spectrum' in the sequel.

Conversely, a Picard ∞-groupoid is the fundamental ∞-groupoid of an infinite loop space, or equivalently the 0-space of an Ω-spectrum. (See [**HS**, Proposition B.12] for the 1-groupoid version and [**L1**, Example 2.4.15] for the general case.)

Using this identification, an invertible field theory is a map of spectra; the fact that α in (6.7) is a symmetric monoidal functor translates to the geometric realization $\tilde{\alpha}$ being a spectrum map. We arrive at this conclusion starting from maps of higher categories—which Axiom System 3.1 tells is what a field theory is—but we can take all that as motivation and *define* an invertible field theory to be a map of appropriate spectra. Such a motivated definition we call an *Ansatz*. We will take that approach shortly, but first we need to know which spectra to use. The discussion is quite different for the domain and codomain. The domain is the geometric realization of a bordism spectrum, which we can describe explicitly. The codomain is not determined *a priori*—see Remark 5.19—but we will see that in the invertible case there is a natural choice.

6.6. Madsen-Tillmann and Thom spectra

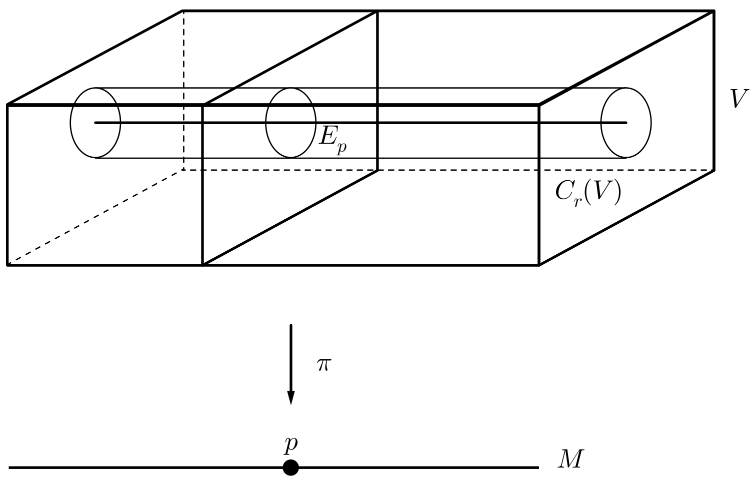

FIGURE 6.3. The pair $\big(V, C_r(V)\big)$

Let $V \to M$ be a real vector bundle over a topological space. Choose an inner product on the fibers, and let $C_r(V) \subset V$ be the set of all vectors of norm at least r. The quotient $V/C_r(V)$ is called the *Thom complex* of $V \to M$ and is denoted $\mathrm{Thom}(M; V)$. Figure 6.3 provides a convenient illustration: imagine the red region collapsed to a point. Note there is no projection from $\mathrm{Thom}(M; V)$ to M; for example, there is no basepoint in M and no distinguished image of the basepoint in $\mathrm{Thom}(M; V)$. Also, note that the zero section induces an inclusion

$$i\colon M \longrightarrow \mathrm{Thom}(M; V). \tag{6.40}$$

EXERCISE 6.41. There is a nontrivial real line bundle $V \to S^1$, often called the *Möbius bundle*. Identify its Thom complex as a familiar pointed space.

PROPOSITION 6.42. *The Thom complex of* $\underline{\mathbb{R}} \oplus V \to Y$ *is homeomorphic to the suspension of the Thom complex of* $V \to Y$.

The Thom complex of the 0-vector bundle—the identity map $Y \to Y$—is the disjoint union of Y and a single point, which is then the basepoint of the disjoint union. That disjoint union is denoted Y_+. Then Proposition 6.42 implies that the Thom complex of $\underline{\mathbb{R}} \to Y$ is ΣY_+, the suspension of Y_+. Iterating, we have $\mathrm{Thom}(Y; \underline{\mathbb{R}^\ell}) \simeq \Sigma^\ell Y_+$. So the Thom complex is a "twisted suspension" of the base space.

Proof. Up to homeomorphism the disk bundle of $\underline{\mathbb{R}} \oplus V \to Y$ is the Cartesian product of the unit disk in $\mathbb{R}$ and the disk bundle of $V \to Y$. Crushing the complement in $\mathbb{R} \times V$ to a point is the same crush which forms the suspension of $\mathrm{Thom}(Y; V)$, as in Figure 6.4. $\square$

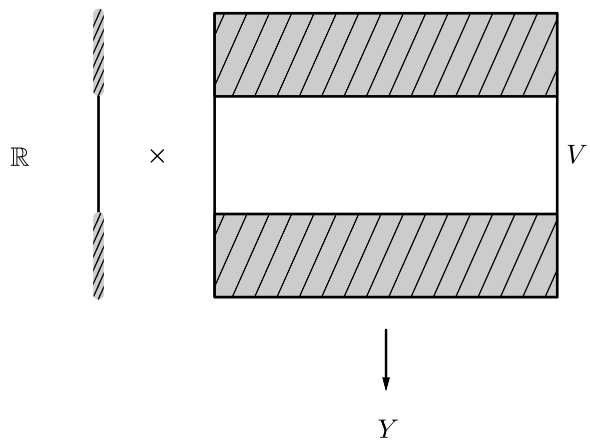

Figure 6.4. The Thom complex of $\underline{\mathbb{R}} \oplus V \to Y$

Let $\pi\colon \mathcal{X}_n \to B\mathrm{GL}_n\,\mathbb{R}$ be an n-dimensional flabby tangential structure (Definition 1.40). Let $S_n \to B\mathrm{GL}_n\,\mathbb{R}$ be the standard rank n real vector bundle. Let $Gr_n(V)$ denote the Grassmannian of n-dimensional subspaces of a finite dimensional real vector space V. There is a short exact sequence

$$0 \longrightarrow S \longrightarrow \underline{V} \longrightarrow Q \longrightarrow 0 \tag{6.43}$$

of real vector bundles over $Gr_n(V)$ in which $S \to Gr_n(V)$ is the canonical subbundle of the trivial bundle and $Q \to Gr_n(V)$ is their quotient. When $V = \mathbb{R}^{n+q}$ we use the notations $S(n)$ and $Q(q)$ for the total spaces of these vector bundles. Define $\mathcal{X}_n(n+q)$ as the pullback

$$\begin{tikzcd} \mathcal{X}_n(n+q) \arrow[r, dashed] \arrow[d, dashed] & \mathcal{X}_n \arrow[d] \\ Gr_n(\mathbb{R}^{n+q}) \arrow[r] & B\mathrm{GL}_n\,\mathbb{R} \end{tikzcd} \tag{6.44}$$

Use the standard metric on $\mathbb{R}^{n+q}$ to split (6.43) as $\underline{\mathbb{R}^{n+q}} = Q(q) \oplus S(n)$ and, by pullback, obtain a direct sum of vector bundles over $\mathcal{X}_n(n+q)$.

Definition 6.45. The *Madsen-Tillmann spectrum* $MT\mathcal{X}_n$ is the spectrum whose $(n+q)^{\mathrm{th}}$ space is the Thom space of $Q(q) \to \mathcal{X}_n(n+q)$. The structure maps of the spectrum are obtained by applying the Thom space construction to the map

$$\begin{tikzcd} \underline{\mathbb{R}} \oplus Q(q) \arrow[r] \arrow[d] & Q(q+1) \arrow[d] \\ \mathcal{X}_n(n+q) \arrow[r] & \mathcal{X}_n(n+q+1) \end{tikzcd} \tag{6.46}$$

of vector bundles. In short, $MT\mathcal{X}_n$ is the Thom spectrum of the virtual bundle $-\pi^* S_n \to \mathcal{X}_n$

The intuition for the last assertion is that, as virtual bundles, $Q(q) = -S(n) + \underline{\mathbb{R}^{n+q}}$, so the Thom space of the vector bundle $Q(q) \to \mathcal{X}_n(n+q)$ is the 0-space of the $(n+q)^{\text{th}}$ suspension of $\operatorname{Thom}(\mathcal{X}_n; -\pi^* S_n)$. The latter is equally the $(n+q)^{\text{th}}$ space of the unsuspended $MT\mathcal{X}_n$, or equivalently the q^{th} space of the suspension $\Sigma^n MT\mathcal{X}_n$.

Remark 6.47. The 'MT' notation is due to Mike Hopkins and is a play on the 'M'-notation for Thom spectra (Definition 6.52 below). 'MT' not only stands for 'Madsen-Tillmann', but also for a Tangential variant of the thoM spectrum. MT spectra are tangential and unstable; Thom spectra are normal and stable. We will see a precise relationship between MT and Thom spectra in (6.61) below. For Madsen-Tillmann spectra constructed from a symmetry type (H_n, ρ_n) we use the notation MTH_n. For example, the Madsen-Tillmann spectrum for oriented bundles is $MT\mathrm{SO}_n$.

Exercise 6.48. There is a homotopy equivalence $MT\mathrm{SO}(1) \simeq S^{-1} = \Sigma^{-1} S^0$.

Let $\mathcal{Y}$ be a stable tangential structure (Definition 1.40). Define $\mathcal{Y}(q)$ as the pullback

$$\begin{tikzcd} \mathcal{Y}(q) \arrow[r, dashed] \arrow[d, dashed] & \mathcal{Y} \arrow[d] \\ BGL_q\mathbb{R} \arrow[r] & BGL \end{tikzcd} \tag{6.49}$$

where $BGL = \operatorname{colim}_{q\to\infty} BGL_q\mathbb{R}$. Consider the diagram

$$\begin{tikzcd} \underline{\mathbb{R}} \oplus S(q) \arrow[r] \arrow[d] & S(q+1) \arrow[d] \\ \mathcal{Y}(q) \arrow[r, "i"] & \mathcal{Y}(q+1) \end{tikzcd} \tag{6.50}$$

There is an induced map on Thom complexes, and by Proposition 6.42 a map

$$s_q \colon \Sigma\big(\operatorname{Thom}(\mathcal{Y}(q);\, S(q))\big) \longrightarrow \operatorname{Thom}(\mathcal{Y}(q+1);\, S(q+1)). \tag{6.51}$$

Definition 6.52. The *Thom spectrum* $M\mathcal{Y}$ of a stable tangential structure $\mathcal{Y}$ is the sequence of pointed spaces

$$M\mathcal{Y}_q = \operatorname{Thom}(\mathcal{Y}(q);\, S(q)) \tag{6.53}$$

together with structure maps (6.51).

The classifying space $B\mathrm{GL}_n\,\mathbb{R}$ is a colimit of Grassmannians:

$$Gr_k(\mathbb{R}^q) \longrightarrow Gr_k(\mathbb{R}^{q+1}) \longrightarrow Gr_k(\mathbb{R}^{q+2}) \longrightarrow \cdots . \tag{6.54}$$

Endow $\mathbb{R}^q$ with the standard inner product. Then the map $W \mapsto W^\perp$ to the orthogonal subspace induces inverse diffeomorphisms

$$Gr_n(\mathbb{R}^m) \longleftrightarrow Gr_{m-n}(\mathbb{R}^m) \tag{6.55}$$

which exchange the tautological subbundles S with the tautological quotient bundles Q. The double colimit of (6.55) as $n, m \to \infty$ yields an involution

$$\iota\colon BGL \longrightarrow BGL \tag{6.56}$$

If $\mathcal{X} \to BGL$ is a stable tangential structure, we define its pullback by ι to be a new stable tangential structure

$$\begin{tikzcd} \mathcal{X}^\perp \arrow[r, dashed] \arrow[d, dashed] & \mathcal{X} \arrow[d] \\ BGL \arrow[r, "\iota"] & BGL \end{tikzcd} \tag{6.57}$$

Our next task is to construct a map

$$\Sigma^n MT\mathcal{X}_n \longrightarrow M\mathcal{X}^\perp \tag{6.58}$$

from the Madsen-Tillmann spectrum to the Thom spectrum. Namely, the perp map followed by stabilization yields the diagram

$$\begin{tikzcd} Q(q) \arrow[r, "\cong"] \arrow[d] & S(q) \arrow[r] \arrow[d] & S(q) \arrow[d] \\ \mathcal{X}_n(n+q) \arrow[r, "\cong"] \arrow[d] & \mathcal{X}_q^\perp(n+q) \arrow[r, dashed] \arrow[d, dashed] & \mathcal{X}_q^\perp \arrow[d] \\ Gr_n(\mathbb{R}^{n+q}) \arrow[r, "\perp", "\cong"'] & Gr_q(\mathbb{R}^{n+q}) \arrow[r] & Gr_q(\mathbb{R}^\infty) \end{tikzcd} \tag{6.59}$$

The induced map on the Thom space of the upper left vertical arrow to the Thom space of the upper right vertical arrow is a map $(\Sigma^n MT\mathcal{X}_n)_q \to (M\mathcal{X}^\perp)_q$ on the q-spaces of the spectra. The maps are compatible with the structure maps of the spectra as q varies, and so we obtain the map (6.58) of spectra.

The stabilization map

$$\begin{tikzcd} Q(q) \arrow[r] \arrow[d] & Q(q) \arrow[d] \\ \mathcal{X}_n(n+q) \arrow[r] & \mathcal{X}_{n+1}(n+q+1) \end{tikzcd} \tag{6.60}$$

induces a map $(\Sigma^n MT\mathcal{X}_n)_q \to (\Sigma^{n+1} MT\mathcal{X}_{n+1})_q$ on Thom spaces, and iterating with n we obtain a sequence of maps

$$MT\mathcal{X}_0 \longrightarrow \Sigma^1 MT\mathcal{X}_1 \longrightarrow \Sigma^2 MT\mathcal{X}_2 \longrightarrow \cdots \tag{6.61}$$

of spectra. Define the colimit to be the *stable Madsen-Tillmann spectrum* $MT\mathcal{X}$. The perp maps (6.58) induce

$$MT\mathcal{X} \longrightarrow M\mathcal{X}^\perp \tag{6.62}$$

on the colimit $n \to \infty$. It is clear from the construction (6.59) that (6.62) is an equivalence. So the (suitably suspended) Madsen-Tillmann spectra (6.61) define a filtration of the Thom spectrum $M\mathcal{X}^{\perp}$.

EXERCISE 6.63. Prove the following equivalences:

$$\begin{aligned} MT\,\mathrm{O} &\simeq M\,\mathrm{O} \\ MT\mathrm{Spin} &\simeq M\mathrm{Spin} \\ MT\mathrm{Pin}^+ &\simeq M\mathrm{Pin}^- \\ MT\mathrm{Pin}^- &\simeq M\mathrm{Pin}^+ \end{aligned} \tag{6.64}$$

REMARK 6.65. Fix a homomorphism of Lie groups $\rho_n\colon H_n \to \mathrm{GL}_n\mathbb{R}$ as in (1.37), and set $\mathcal{X}_n = BH_n$. The Pontrjagin-Thom construction provides the basic relationship between $\mathcal{X}_n$ and H_n-manifolds. Namely, if a map $S^{k+q} \to \mathrm{Thom}(\mathcal{X}_n(n+q); Q(q))$ is transverse to the 0-section of $Q(q) \to X_n(n+q)$, then the inverse image of the 0-section is a k-dimensional submanifold $M \subset S^{k+q}$ whose stable tangent bundle is equipped with an isomorphism to the pullback of the tautological bundle $S(n) \to X_n(n+q)$; the latter is equipped with an H_n-structure. Theorem 6.67 below implies that the abelian group $\pi_k\Sigma^n MTH_n$ is generated by closed k-dimensional H_n-manifolds under disjoint union. The class of a closed manifold M^k is zero if and only if $M = \partial W$ where W is a compact $(k+1)$-manifold whose stable tangent bundle is isomorphic to a rank n bundle with an H_n-structure extending that of M. (Definition 6.24 has a precise statement.) In other words, $\pi_k\Sigma^n MTH_n$ is the Reinhart bordism group $\Omega_k^R(H_n)$; see [**E**, Appendix].

REMARK 6.66. Not every element of the homotopy group $\pi_k\Sigma^n MTH_n$ is represented by a manifold; group completion of the semigroup of manifold classes is needed to obtain the homotopy group. For example, $\pi_0 MT\,\mathrm{O}_0 \cong \mathbb{Z}$ but closed 0-dimensional manifolds only realize the submonoid of nonnegative elements. Note that the sphere S^{2m} represents a nonzero element in $\pi_{2m}\Sigma^{2m}MT\mathrm{SO}_{2m}$, but is zero in the next group $\pi_{2m+1}\Sigma^{2m+1}MT\mathrm{SO}_{2m+1}$: the closed ball D^{2m+1} has nonzero Euler characteristic so no SO_{2m}-structure. As another illustration, the 2-sphere and the genus 2 surface represent opposite elements of $\pi_2\Sigma^2 MT\mathrm{SO}_2$: a genus 2 handlebody with a 3-ball excised admits an SO_2-structure.

The relevance of Madsen-Tillmann spectra for invertible field theories is due to the following identification of the groupoid completion of a bordism category. Recall that this groupoid completion is the domain of the map $\tilde{\alpha}$ in (6.7). We continue to identify Picard ∞-groupoids with infinite loop spaces (stable homotopy hypothesis).

THEOREM 6.67. $|\mathrm{Bord}_n(\mathcal{X}_n)|$ *is weakly equivalent to the 0-space of the Madsen-Tillmann spectrum* $\Sigma^n MT\mathcal{X}_n$.

One version of this theorem is proved in [**BM**], though only for unoriented manifolds and is carried out for "n-uple categories" rather than (∞, n)-categories. The theorem is stated in [**L1**, §2.5] as a corollary of the cobordism hypothesis. Proofs of Theorem 6.67 in the context of (∞, n)-categories have appeared in preprint form. A preprint of Ayala-Francis [**AF**] proves[59] the cobordism hypothesis and Theorem 6.67 for framed manifolds. A preprint by Schommer-Pries [**S-P2**] contains a

[59]modulo a conjecture at this writing

complete proof of Theorem 6.67 independent of the cobordism hypothesis. Section 6.3 illustrates a truncation of Theorem 6.67 in the special case of 1-framed 1-manifolds.

6.7. Duals to the sphere spectrum

An abelian group[60] A has two associated dual abelian groups. The *character dual*[61] $A^\vee = \operatorname{Hom}(A, \mathbb{C}^\times)$ is its group of characters. The naive $\mathbb{Z}$-*dual* is $\operatorname{Hom}(A, \mathbb{Z})$. The naiveté is evident if A is finite, for then $\operatorname{Hom}(A, \mathbb{Z}) = 0$ does not capture any information about A. Enter the *derived* $\mathbb{Z}$-*dual*, which is the collection of Ext groups $\operatorname{Ext}^q(A, \mathbb{Z})$, $q \in \mathbb{Z}$. In fact, only $\operatorname{Ext}^0(A, \mathbb{Z})$ and $\operatorname{Ext}^1(A, \mathbb{Z})$ are possibly nonzero. Recall that $\operatorname{Ext}^1(A, \mathbb{Z})$ is the group of isomorphism classes of abelian group extensions

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{A} \longrightarrow A \longrightarrow 0. \tag{6.68}$$

It is natural to extend the dualities to $\mathbb{Z}$-graded abelian groups

$$A = \bigoplus_{q \in \mathbb{Z}} A_q. \tag{6.69}$$

The character dual is

$$(A^\vee)_q = \operatorname{Hom}(A_{-q}, \mathbb{C}^\times) = (A_{-q})^\vee. \tag{6.70}$$

The character dual and $\mathbb{Z}$-dual are related by an application of the exponential short exact sequence

$$0 \longrightarrow \mathbb{Z}(1) \longrightarrow \mathbb{C} \xrightarrow{\ \exp\ } \mathbb{C}^\times \longrightarrow 1 \tag{6.71}$$

in which[62] $\mathbb{Z}(1) = 2\pi\sqrt{-1}\mathbb{Z}$. Hereafter we replace $\mathbb{Z}$-dual with $\mathbb{Z}(1)$-dual. The derived $\mathbb{Z}(1)$-dual A^* fits into the long exact sequence

$$\cdots \longrightarrow \operatorname{Hom}(A_{-q}, \mathbb{C}) \longrightarrow A^\vee_q \longrightarrow A^*_{q-1} \longrightarrow \operatorname{Hom}(A_{-(q-1)}, \mathbb{C}) \longrightarrow \cdots \tag{6.72}$$

induced from the exponential sequence (6.71).

REMARK 6.73. If $A_q = 0$ for $q \neq 0$, so A is ungraded, then (6.72) reduces to

$$0 \longrightarrow A^*_0 \longrightarrow \operatorname{Hom}(A_0, \mathbb{C}) \longrightarrow A^\vee_0 \longrightarrow A^*_{-1} \longrightarrow 0 \tag{6.74}$$

in which $A^*_0 = \operatorname{Hom}(A_0, \mathbb{Z}(1))$ and $A^*_{-1} = \operatorname{Ext}^1(A, \mathbb{Z}(1))$. If $\phi\colon A_0 \to \mathbb{C}^\times$ is an element of $A^\vee_0$, then its image in A^*_{-1} is represented by the group extension constructed as the pullback of the exponential sequence (6.71) by ϕ; we state this more formally below in Proposition 6.84. If we use the standard topology on $\mathbb{C}^\times$ we can topologize $A^\vee_0 = \operatorname{Hom}(A_0, \mathbb{C}^\times)$ as a complex abelian Lie group $\mathcal{A}$. Then the exact sequence (6.74) is the standard exact sequence

$$0 \longrightarrow \pi_1\mathcal{A} \longrightarrow \operatorname{Lie}(\mathcal{A}) \longrightarrow \mathcal{A} \longrightarrow \pi_0\mathcal{A} \longrightarrow 0 \tag{6.75}$$

in which $\operatorname{Lie}(\mathcal{A})$ is the Lie algebra of $\mathcal{A}$. The identification of $\operatorname{Ext}^1(-, \mathbb{Z})$ as the group of components of the continuously topologized character dual will be generalized in §6.9 to an identification of the deformation classes of a continuously topologized space of field theories.

---

[60] For our present illustrative purposes we take A to be discrete and finitely generated, but without much change it could be a Lie group with finitely generated π_0, π_1.

[61] The closely related *Pontrjagin dual* is $\operatorname{Hom}(A, \mathbb{T})$, where $\mathbb{T} \subset \mathbb{C}^\times$ is the circle group.

[62] The group $\mathbb{Z}(1)$ is defined without specifying a choice for $\sqrt{-1}$.

REMARK 6.76. If we pose as a problem to compute the group of deformation classes of characters of an abelian group A_0, then by following the line of reasoning in Remark 6.73 we are led to the (shifted) derived $\mathbb{Z}(1)$-dual of A_0. Inevitably we discover not only the group A^*_{-1} we were after, but also an extra "piece" $A^*_0 = \operatorname{Hom}(A_0, \mathbb{Z}(1))$. If A_0 is finite, then this extra piece vanishes and the derived $\mathbb{Z}(1)$-dual gives the answer on the nose. But, as is typical with derived geometry, what we compute may not be the answer to the original problem. Here our answer is all the homotopy groups of the abelian *Lie group* $\mathcal{A}$, not just π_0. We encounter a similar phenomenon in our problem about invertible topological field theories; see §6.10 and Lecture 9.

EXERCISE 6.77.

(1) Write the group law on Ext^1 in terms of extensions.
(2) Show that the derived $\mathbb{Z}(1)$-dual of the derived $\mathbb{Z}(1)$-dual of A is isomorphic to A.

A spectrum refines a $\mathbb{Z}$-graded abelian group: its homotopy groups. The character and $\mathbb{Z}$-dual constructions lift to spectra, and reduce to the constructions on $\mathbb{Z}$-graded abelian groups upon passing to homotopy groups; see [**HeSt**] for an exposition. Since the sphere spectrum S^0 plays the role of $\mathbb{Z}(1)$, its character and $\mathbb{Z}(1)$-duals are the universal codomains for these duals. The character dual to the sphere spectrum, denoted $I\mathbb{C}^\times$, is closely related to the *Brown-Comenetz dual* [**BC**] which is the $\mathbb{Q}/\mathbb{Z}$-dual to the sphere spectrum. Since $\mathbb{C}^\times$ is an injective abelian group, the character dual can be defined via Brown representability: if $\mathcal{B}$ is any spectrum then

$$\pi_q \operatorname{Map}(\mathcal{B}, I\mathbb{C}^\times) = (\pi_{-q}\mathcal{B})^\vee. \tag{6.78}$$

There is also a $\mathbb{C}$-dual $H\mathbb{C}$ to the sphere spectrum; it is an Eilenberg-MacLane spectrum with the only nonzero homotopy group $\pi_0 H\mathbb{C} \cong \mathbb{C}$. The $\mathbb{Z}(1)$-dual $I\mathbb{Z}(1)$ to the sphere spectrum, known as the *Anderson dual* [**An**], [**HS**, Appendix B], is defined as the homotopy fiber of $H\mathbb{C} \to I\mathbb{C}^\times$; c.f., the exponential exact sequence (6.71). The spectra $I\mathbb{C}^\times, I\mathbb{Z}(1)$ are *co-connective*: homotopy groups in positive degrees vanish. The low-lying homotopy groups are

(6.79)

q	$\pi_q S^0$	$\pi_q I\mathbb{C}^\times$	$\pi_q I\mathbb{Z}(1)$
4	0	0	0
3	$\mathbb{Z}/24\mathbb{Z}$	0	0
2	$\mathbb{Z}/2\mathbb{Z}$	0	0
1	$\mathbb{Z}/2\mathbb{Z}$	0	0
0	$\mathbb{Z}$	$\mathbb{C}^\times$	$\mathbb{Z}(1)$
-1	0	μ_2	0
-2	0	μ_2	μ_2
-3	0	μ_{24}	μ_2
-4	0	0	μ_{24}

where $\mu_k = (\mathbb{Z}/k\mathbb{Z})^\vee = \operatorname{Hom}(\mathbb{Z}/k\mathbb{Z}, \mathbb{C}^\times)$ is the group of k^{th} roots of unity.

From their definitions, the character and Anderson duals enjoy universal properties. For spectra $\mathcal{B}, \mathcal{I}$ let $[\mathcal{B}, \mathcal{I}]$ denote the abelian group of homotopy classes of spectrum maps $\mathcal{B} \to \mathcal{I}$.

PROPOSITION 6.80. *For any spectrum $\mathcal{B}$ with $\pi_{-1}\mathcal{B}, \pi_0\mathcal{B}$ finitely generated there is an isomorphism*

$$[\mathcal{B}, I\mathbb{C}^\times] \cong \operatorname{Hom}(\pi_0\mathcal{B}, \mathbb{C}^\times) \tag{6.81}$$

and a short exact sequence

$$0 \longrightarrow \operatorname{Ext}^1(\pi_{-1}\mathcal{B}, \mathbb{Z}(1)) \longrightarrow [\mathcal{B}, I\mathbb{Z}(1)] \longrightarrow \operatorname{Hom}(\pi_0\mathcal{B}, \mathbb{Z}(1)) \longrightarrow 0 \tag{6.82}$$

Furthermore, $\operatorname{Ext}^1(\pi_{-1}\mathcal{B}, \mathbb{Z}(1))$ *is the torsion subgroup of* $[\mathcal{B}, I\mathbb{Z}(1)]$, *and its free quotient is* $\operatorname{Hom}(\pi_0\mathcal{B}, \mathbb{Z}(1))$.

Here (6.81) is a special case of (6.78), and the short exact sequence (6.82) echoes the situation for discrete abelian groups (6.72).

From the exponential sequence (6.71) and the definition of $I\mathbb{Z}(1)$ there is a map

$$I\mathbb{C}^\times \longrightarrow \Sigma I\mathbb{Z}(1). \tag{6.83}$$

PROPOSITION 6.84. *For any spectrum $\mathcal{B}$ with $\pi_0\mathcal{B}$ finitely generated, the map*

$$\operatorname{Hom}(\pi_0\mathcal{B}, \mathbb{C}^\times) \longrightarrow \operatorname{Ext}^1(\pi_0\mathcal{B}, \mathbb{Z}(1)) \tag{6.85}$$

induced by mapping $\mathcal{B}$ into (6.83) *maps $\phi\colon \pi_0\mathcal{B} \to \mathbb{C}^\times$ to the isomorphism class of the pullback of the exponential sequence by ϕ.*

EXERCISE 6.86. Prove Proposition 6.84.

REMARK 6.87. We can endow $\operatorname{Hom}(\pi_0\mathcal{B}, \mathbb{C}^\times)$ with the structure of a complex abelian Lie group with Lie algebra $\operatorname{Hom}(\pi_0\mathcal{B}, \mathbb{C})$. Then (6.85) is the map π_0 onto the group of components, as in (6.75); see §6.9.

6.8. Invertible field theories as maps of spectra

While we have no canonical choice for the codomain $\mathcal{C}$ of an *extended* topological field theory (5.22), an extended *invertible* topological field theory only requires a Picard ∞-groupoid, not a full (∞, n)-category. This is equivalent to a choice of infinite loop space, or a spectrum.

ANSATZ 6.88. The codomain of an n-dimensional extended invertible topological field theory is $\Sigma^n I\mathbb{C}^\times$.

For a theory with Wick-rotated symmetry type (H_n, ρ_n) we use Theorem 6.67 to deduce that the domain should be the Madsen-Tillmann spectrum $\Sigma^n MTH_n$. Therefore, we arrive at the following.

ANSATZ 6.89. A *discrete invertible n-dimensional extended topological field theory with symmetry type* (H_n, ρ_n) is a spectrum map

$$\alpha\colon \Sigma^n MTH_n \longrightarrow \Sigma^n I\mathbb{C}^\times. \tag{6.90}$$

The space of theories of this type is $\operatorname{Map}(\Sigma^n MTH_n, \Sigma^n I\mathbb{C}^\times)$.

Here 'Map' indicates the space of maps between the indicated spectra. The restriction to the mapping *space*, as opposed to mapping spectrum, is to avoid negative homotopy groups which in any case are not encountered in parameter *spaces* of theories. The word 'discrete' evokes the discrete topology on $\mathbb{C}^\times$, in contrast to the "continuous" theories we describe in §6.10.

Our choice of $\Sigma^n I\mathbb{C}^\times$ in Ansatz 6.88 can be motivated by the requirement that the partition function of an extended field theory determine the theory. This may be something one expects, or would desire, of physical field theories, but it is not true in general, even in the topological case. Ansatz 6.88 enforces this condition for invertible topological theories, as follows from the first statement in Proposition 6.80, since $\operatorname{Hom}(\pi_n\Sigma^n MTH_n, \mathbb{C}^\times)$ is the group of $\mathbb{C}^\times$-valued Reinhart bordism invariants, or equivalently the group of partition functions. That same statement, read backwards, implies that any $\mathbb{C}^\times$-valued Reinhart bordism invariant is fully local in the sense that it is the partition function of a fully extended invertible topological field theory.

REMARK 6.91. 'Partition functions determine the theory' is one motivation for Ansatz 6.88, and having made that choice we observe some magic. First, if $\alpha\colon \Sigma^n MTH_n \to \Sigma^n I\mathbb{C}^\times$ is an invertible theory, then the induced map on π_n is the partition function of n-manifolds. The values of α on $(n-1)$-manifolds are computed by the induced map on Picard groupoids $\pi_{\langle n-1,n\rangle}$. The codomain $\pi_{\langle n-1,n\rangle}\Sigma^n I\mathbb{C}^\times \cong \pi_{\langle -1,0\rangle} I\mathbb{C}^\times$ has homotopy groups $\pi_0 \cong \mu_2$ and $\pi_1 \cong \mathbb{C}^\times$, after shifting up, and there is a nonzero k-invariant. This is equivalent to the Picard groupoid $s\mathrm{Line}_\mathbb{C}$ (Exercise 5.9) of $\mathbb{Z}/2\mathbb{Z}$-graded complex lines. This is a physically sensible home for the state space of an invertible theory: there is a single state which is either bosonic or fermionic. Magically, the Bose-Fermi statistics are encoded in the homotopy groups of $I\mathbb{C}^\times$, so ultimately in the homotopy groups of the sphere spectrum! Turning to $(n-2)$-manifolds, after shifting we have the target Picard 2-groupoid $\pi_{\langle 0,1,2\rangle}\Sigma^2 I\mathbb{C}^\times$ with $\pi_0 \cong \mu_2$, $\pi_1 \cong \mu_2$, $\pi_2 \cong \mathbb{C}^\times$, and nontrivial k-invariants connecting neighboring groups. It is equivalent to the Morita-Picard 2-groupoid of central simple complex superalgebras (Example 5.8); see [**DeGu**, §4.3.5] for a proof of the equivalence. The $\pi_0 \cong \mu_2$ in this nonconnected delooping of $s\mathrm{Line}_\mathbb{C}$ also fits the physics; see [**GK**, **ALW**] where objects with nontrivial μ_2-grading are termed 'Majorana'.

6.9. Deformation classes of invertible field theories

As a warmup, let S be a compact topological space and consider $A = H^q(S;\mathbb{C}^\times)$ for some $q \in \mathbb{Z}^{\geqslant 0}$. As written A is a discrete abelian group, but it can be topologized as a complex abelian Lie group with Lie algebra $H^q(S;\mathbb{C})$, analogous to the discussion in Remark 6.73. Namely, the exponential sequence (6.71) induces a long exact sequence

$$\cdots \longrightarrow H^q(S;\mathbb{C}) \xrightarrow{\exp} H^q(S;\mathbb{C}^\times) \xrightarrow{\beta_{\mathbb{Z}(1)}} H^{q+1}(S;\mathbb{Z}(1)) \longrightarrow \cdots \tag{6.92}$$

and the image of exp includes a neighborhood of the identity of A. The continuous topology on the complex vector space $H^q(S;\mathbb{C})$ induces a topology on the neighborhood of the identity element of $A = H^q(S;\mathbb{C}^\times)$, and the group law translates it around to the Lie group topology on A.

LEMMA 6.93. *Elements $a_0, a_1 \in A$ lie in the same path component if and only if $a_1 - a_0$ is in the image of exp. The group $\pi_0 A$ is the torsion subgroup of $H^{q+1}(S;\mathbb{Z}(1))$ and $\pi_0\colon A \to \pi_0 A$ is identified with the integer Bockstein map $\beta_{\mathbb{Z}(1)}$ (with codomain the torsion subgroup).*

Ansatz 6.89 identifies a space $\operatorname{Map}(\Sigma^n MTH_n, \Sigma^n I\mathbb{C}^\times)$ of invertible topological field theories, or more precisely a homotopy type. It is topologized analogous to the discrete topology on A: its group of path components $[\Sigma^n MTH_n, \Sigma^n I\mathbb{C}^\times]$ consists of *isomorphism classes* of field theories. In principle, we would like to define a second topology on $\mathrm{Map}(\Sigma^n MTH_n, \Sigma^n I\mathbb{C}^\times)$ whose path components are *deformation classes*, but instead we use Lemma 6.93 and the fibration

$$H\mathbb{C} \xrightarrow{\exp} I\mathbb{C}^\times \longrightarrow \Sigma I\mathbb{Z}(1) \tag{6.94}$$

induced from the exponential sequence to motivate the following.

DEFINITION 6.95. Theories $\alpha_0, \alpha_1 \in \mathrm{Map}(\Sigma^n MTH_n, \Sigma^n I\mathbb{C}^\times)$ are *deformation equivalent* if there exists $\xi \in H^n(\Sigma^n MTH_n; \mathbb{C})$ whose image under exp is the difference $[\alpha_1] - [\alpha_0]$ of the isomorphism classes $[\alpha_0], [\alpha_1] \in [\Sigma^n MTH_n, \Sigma^n I\mathbb{C}^\times]$.

The fibration (6.94) then immediately implies the following.

THEOREM 6.96. *There is a 1:1 correspondence*

$$\left\{\begin{matrix}\textit{deformation classes of invertible}\\ n\textit{-dimensional extended topological field}\\ \textit{theories with symmetry type } (H_n, \rho_n)\end{matrix}\right\} \cong [\Sigma^n MTH_n, \Sigma^{n+1} I\mathbb{Z}(1)]_{\mathrm{tor}}. \tag{6.97}$$

The subscript 'tor' denotes the torsion subgroup of the indicated abelian group. This appears, at least implicitly, in a joint paper [**FHT**] of the author, Mike Hopkins, and Constantin Teleman; Theorem 6.96 has been the basis of many investigations since.

6.10. Continuous invertible topological field theories

Theorem 6.96 raises an obvious question: What is the field theoretic interpretation of the abelian group

$$[\Sigma^n MTH_n, \Sigma^{n+1} I\mathbb{Z}(1)]\ ? \tag{6.98}$$

As we saw in §6.7, *derived* hom into $\mathbb{Z}(1)$ often includes extra characters not in our original cast (Remark 6.76). Torsion elements in (6.98) are deformation classes of discrete invertible topological field theories, and now we inquire about elements of infinite order. It turns out the infinite order elements are also included in the answer to our physics classification Problem 2.33, as we will see in Lecture 9, but they are not represented by discrete invertible topological field theories. Here we give an "integral" interpretation of the full abelian group (6.98).

As a warmup, let M be a smooth manifold and consider the following three geometric objects on M:

(a) principal $\mathbb{C}^\times$-bundles $P \to M$ with connection
(b) principal $\mathbb{C}^\times$-bundles $P \to M$ with flat connection
(c) principal $\mathbb{C}^\times$-bundles $P \to M$

The abelian groups of equivalence classes are isomorphic to cohomology groups:[63] (a) $\check{H}^2(M)$, (b) $H^1(M; \mathbb{C}^\times)$, and (c) $H^2(M; \mathbb{Z}(1))$. Of more direct relevance is the structure induced on the free loop space $LM = \mathrm{Map}(S^1, M)$:

(a) a smooth function $LM \to \mathbb{C}^\times$
(b) a locally constant function $LM \to \mathbb{C}^\times$
(c) a principal $\mathbb{Z}(1)$-bundle $E \to LM$

[63] We discuss *differential cohomology groups* $\check{H}^\bullet(M)$ in §9.4.

In (a) and (b) the function is the holonomy. The *isomorphism class* of the bundle $E \to LM$ in (c) is determined by the *homotopy class* of the holonomy $LM \to \mathbb{C}^\times$ of *any* connection. Better, we construct the bundle $E \to LM$ by "integrating" over the affine space $\mathcal{A}_P$ of connections on $P \to M$. Namely, an element $\lambda \in \mathbb{C}^\times$ determines a $\mathbb{Z}(1)$-torsor $E_\lambda \subset \mathbb{C}$ of all $x \in \mathbb{C}$ such that $\exp(x) = \lambda$, and so the holonomy function $LM \to \mathbb{C}^\times$ of a connection $\Theta \in \mathcal{A}_P$ on $P \to M$ determines a principal $\mathbb{Z}(1)$-bundle $E_\Theta \to LM$. Altogether, we obtain a principal $\mathbb{Z}(1)$-bundle $\mathcal{E} \to \mathcal{A}_P \times LM$. Since the affine space $\mathcal{A}_P$ of connections is contractible, $\mathcal{E} \to \mathcal{A}_P \times LM$ descends to a principal $\mathbb{Z}(1)$-bundle $E \to LM$.

We have the following types of invertible field theories:

(a) a non-topological theory (such as Example 2.46)
(b) a discrete invertible theory (Ansatz 6.89)
(c) a topological field theory with partition functions $\mathbb{Z}(1)$-torsors

An n-dimensional theory of type (c) is often the truncation of an $(n+1)$-dimensional theory whose partition functions lie in $\mathbb{Z}$. We make (c) precise using stable homotopy theory.

ANSATZ 6.99. A *continuous invertible n-dimensional extended topological field theory with symmetry type* (H_n, ρ_n) is a spectrum map

$$\varphi\colon \Sigma^n MTH_n \longrightarrow \Sigma^{n+1} I\mathbb{Z}(1). \tag{6.100}$$

The space of theories of this type is $\mathrm{Map}(\Sigma^n MTH_n, \Sigma^{n+1} I\mathbb{Z}(1))$.

A discrete invertible topological field theory F gives rise to a continuous invertible topological field theory φ_F: compose $F\colon \Sigma^n MTH_n \to \Sigma^n I\mathbb{C}^\times$ with the map $\Sigma^n I\mathbb{C}^\times \to \Sigma^{n+1} I\mathbb{Z}(1)$. The continuous theory φ_F retains the homotopical information in F, in particular its deformation class.

LECTURE 7

Wick-Rotated Unitarity

In this lecture and the next we take up unitarity in quantum field theory. Now that we have an enhanced notion of locality for Wick-rotated field theories (Lecture 5), we face the problem of incorporating unitarity into the enhancement. No definition of extended unitarity is known in general, and it is a very interesting open problem to produce one. We propose a solution for invertible topological theories in Lecture 8. Axiom System 3.1 is for *Wick-rotated* quantum field theory, so our first task is to understand the manifestation of ordinary non-extended unitarity under Wick rotation. This is *reflection positivity* in Euclidean field theory, which we explain in §7.2 and §7.3. Our treatment separates reflection from positivity: reflection is data whereas non-extended positivity is a condition. One novelty is our implementation of reflection for general symmetry types (H_n, ρ_n), which we achieve via a co-extension $\widehat{H}_n$ of the group H_n. We explain the positivity condition in §7.4 as it appears in the theory on curved manifolds, and in §7.5 give a necessary condition for positivity (Proposition 7.44). We conclude in §7.6 with some inconclusive musings about positivity in codimension 2. This lecture begins with a categorical interpretation of positivity (§7.1) that motivates Lecture 8.

7.1. Positive definite Hermitian vector spaces

Let V be a finite dimensional complex vector space. A Hermitian form on V is a nondegenerate pairing $h\colon \overline{V} \times V \to \mathbb{C}$ between V and its complex conjugate $\overline{V}$, the same underlying real vector space with $\sqrt{-1}$ acting oppositely. The space of nondegenerate pairings has components distinguished by pairs (p, q) of nonnegative integers with $p+q = \dim V$. For each form h there exist maximal subspaces $P \subset V$ on which h is positive definite and maximal subspaces $Q \subset V$ on which h is negative definite; for any choices P, Q we have $V = P \oplus Q$. The dimensions p, q of P, Q are invariants of h, and are also invariant under deformations of h. Positivity of h is equivalent to $P = V$, and is a *condition* separate from the *data* of the Hermitian form.

We now reformulate positivity in categorical terms. The material in §A.5 on categorical involutions is a prerequisite. Let $f\mathrm{Vect}_{\mathbb{C}}$ denote the topological groupoid of finite dimensional vector spaces and invertible linear maps. Use the standard topology on invertible linear maps. Also, $f\mathrm{Vect}_{\mathbb{C}}$ is symmetric monoidal under the usual tensor product. The *bar involution* $\beta\colon f\mathrm{Vect}_{\mathbb{C}} \longrightarrow f\mathrm{Vect}_{\mathbb{C}}$ maps complex vector spaces and linear maps to their complex conjugates. *Duality* is a twisted involution (Definition A.59(1)) $\delta\colon f\mathrm{Vect}_{\mathbb{C}} \longrightarrow f\mathrm{Vect}_{\mathbb{C}}^{\mathrm{op}}$ which takes a vector space to its dual and reverses the direction of arrows. The composite $\beta\delta$ is a twisted involution which maps a complex vector space V to $\overline{V}^*$. A fixed point of $\beta\delta$ in the sense of Definition A.59(2) is a vector space with a Hermitian form.

Let $f\mathrm{Vect}_{\mathbb{C}}^{\mathrm{pos}}$ be the topological groupoid of finite dimensional complex vector spaces equipped with a positive definite Hermitian inner product, and unitary transformations. Since the inclusion $\mathrm{U}_n \hookrightarrow GL_n(\mathbb{C})$ is a homotopy equivalence, the functor $f\mathrm{Vect}_{\mathbb{C}}^{\mathrm{pos}} \to f\mathrm{Vect}_{\mathbb{C}}$ which forgets the inner products is a weak equivalence of topological categories. The combined bar star twisted involution $\beta\delta$ is trivialized on $f\mathrm{Vect}_{\mathbb{C}}^{\mathrm{pos}}$ by the Hermitian inner products. The trivialization is noncanonical, however. We can choose negative definite vector spaces in place of positive definite ones. Or, for each prime p, we can make a choice of positive or negative definite Hermitian inner products on vector spaces of dimension p and then extend to all finite dimensional vector spaces by tensoring.

Remark 7.1. This categorical interpretation of positivity suggests that whatever complex conjugation is in the codomain of an extended field theory, on the higher categories in which $\mathbb{C}$ is regarded as having a topology, the combined action of complex conjugation and duality should be trivializable.

Remark 7.2. On the subgroupoid $\mathrm{Line}_{\mathbb{C}}$ of complex lines, the homotopy theory picks out positive definite Hermitian lines (as opposed to negative definite Hermitian lines). This is a consequence of requiring the trivialization of bar star to be compatible with the monoidal structure: the tensor product of positive definite lines is positive definite, whereas the same is not true with 'positive' replaced by 'negative'.

7.2. Wick-rotated unitarity in quantum mechanics

In quantum mechanics unitarity is manifest: the Hilbert space $\mathcal{H}$ of states has a positive definite Hermitian structure and time evolution U_t of states is by unitary operators. Since Axiom System 3.1 models Wick-rotated systems, to incorporate unitarity into the Axiom System we must first understand its transformation under Wick rotation. Here, for quantum mechanics, we use the material in §2.5 about Wick rotation and §2.6 about the interpretation of quantum mechanics in the Axiom System.

Let

$$F\colon\ \mathrm{Bord}_1(\mathrm{SO}_1^{\nabla}) \longrightarrow t\mathrm{Vect}_{\mathbb{C}} \tag{7.3}$$

be a Wick-rotated quantum mechanical model. Then $F(\mathrm{pt}_+)$ is the state space $\mathcal{H}$ as a topological vector space, but there is no inner product.[64] Observe that (a completion of) the topological vector space $F(\mathrm{pt}_-)$ is in duality with $\mathcal{H}$, but there is no Hermitian form. Thus we need new *data* which identifies $F(\mathrm{pt}_-)$ as the complex conjugate of $F(\mathrm{pt}_+)$. In other words, the Hermitian structure can be encoded by implementing the slogan

$$\text{orientation-reversal} \longmapsto \text{complex conjugation} \tag{7.4}$$

For then $F(\mathrm{pt}_-)$ is isomorphic to both $\overline{\mathcal{H}}$ and also isomorphic to $\mathcal{H}^*$; the composition of these isomorphisms is a Hermitian form. We have not made this argument precise as we have not spelled out which topological vector spaces to use. The analog in the topological case is for finite dimensional vector spaces and is elementary linear algebra, and that is all we use in these lectures.

[64] We actually obtain a family of topological vector spaces; see the text preceding Proposition 4.36.

This motivates encoding Wick-rotated unitarity as additional data for the functor (7.3). The domain has the involution of orientation-reversal, the codomain the involution of complex conjugation, and what is needed is equivariance *data* (Definition A.57) together with a positivity *condition*.

EXAMPLE 7.5. A particle on S^1 with θ-angle illustrates (7.4); see §2.2 for the case $\theta = 0$. In this case we construct a family of theories F_θ parametrized by $\theta \in \mathbb{R}$, and orientation-reversal maps F_θ to $F_{-\theta}$. There is a lagrangian description of F_θ: if $\lambda(s) = e^{i\phi(s)}$ describes the motion of a particle, $\phi\colon X^1 \to \mathbb{R}$, then the lagrangian density is

$$L = \frac{1}{2}\dot{\phi}^2|ds| - \theta\lambda^*(\omega), \tag{7.6}$$

where $\omega \in \Omega^1_{S^1}$ satisfies $\int_{S^1}\omega = 1$ for some fixed orientation of S^1. Let $\mathcal{L}_\theta \to S^1$ be the complex line bundle with covariant derivative of holonomy $e^{i\theta}$. Then the Hilbert space of states of the theory at θ is $\mathcal{H}_\theta = L^2(S^1;\mathcal{L}_\theta)$. The Hamiltonian $H_\theta = \Delta_\theta$ is the Laplace operator on $\mathcal{H}_\theta$, and the Wick-rotated operator attached to an interval of length τ is $e^{-\tau\Delta_\theta}$. The implementation of orientation-reversal is complex conjugation:

$$\mathcal{H} \longmapsto \overline{\mathcal{H}} \tag{7.7}$$

$$e^{-\tau\Delta} \longmapsto \overline{e^{-\tau\Delta}} = e^{-\Delta_{-\theta}} \tag{7.8}$$

It maps the theory at θ to the theory at $-\theta$.

EXERCISE 7.9. Construct a lift of the $\mathbb{Z}$-action by translation $\theta \mapsto \theta + 2\pi$ to this family of theories. What is the group generated by this action and the involution (7.7), (7.8)?

7.3. Wick-rotated unitarity in Euclidean quantum field theory

We turn to field theories in higher dimensions. Consider the basic symmetry type $H_n = \mathrm{SO}_n$ with $\rho_n\colon \mathrm{SO}_n \hookrightarrow \mathrm{O}_n$ the inclusion. In Euclidean field theory unitarity is expressed as *reflection positivity*, an essential ingredient in the reconstruction of a unitary theory in Minkowski spacetime [**OS**]. Reflection positivity is another manifestation of the slogan (7.4).

Let n be the spacetime dimension and $\mathbb{E}^n$ standard Euclidean n-space. Fix an affine hyperplane $\Pi \subset \mathbb{E}^n$ and let σ denote (affine) reflection about Π. Let $\mathcal{O}$ be an operator, or product of operators, in the quantum theory which is supported in the open half-space $\mathbb{E}^n_+$ on one side of Π; the reflected operator $\sigma(\mathcal{O})$ has support in the complementary half-space $\mathbb{E}^n_-$. Let $\langle\mathcal{O}\rangle_{\mathbb{E}^n_+} \in \mathcal{H}$ denote the half-space correlation function, which is a vector in the Hilbert space of the theory. In a lagrangian field theory it is the functional integral over the half-space $\mathbb{E}^n_+$. Then the *reflection* part of 'reflection positivity' is

$$\langle\sigma(\mathcal{O})\rangle_{\mathbb{E}^n_-} = \overline{\langle\mathcal{O}\rangle_{\mathbb{E}^n_+}}; \tag{7.10}$$

see (7.8) for the analog in quantum mechanics. The Hilbert space $\mathcal{H}$ is associated to $(\Pi, \mathfrak{o})$, where $\mathfrak{o}$ is an orientation of the normal line to Π, the arrow of time. The reflection σ reverses $\mathfrak{o}$, and the Hilbert space associated to $(\Pi, -\mathfrak{o})$ is the complex

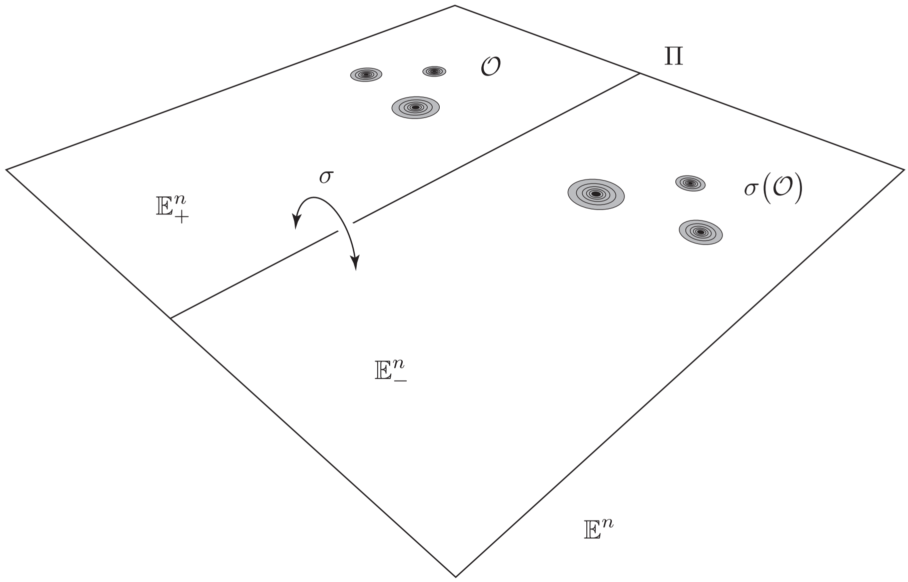

FIGURE 7.1. Reflection positivity in Euclidean space

conjugate

$$\mathcal{H}_{(\Pi,-\mathfrak{o})} \xrightarrow{\cong} \overline{\mathcal{H}_{(\Pi,\mathfrak{o})}}; \tag{7.11}$$

see (7.7) for the analog in quantum mechanics. Therefore, $\langle\sigma(\mathcal{O})\rangle_{\mathbb{E}^n_-} \in \overline{\mathcal{H}}$ and (7.10) is an equation in the complex conjugate Hilbert space $\overline{\mathcal{H}}$. The *positivity* part of 'reflection positivity' is the positive definiteness of $\mathcal{H}$, which implies in particular that the norm square of the vector $\langle\mathcal{O}\rangle_{\mathbb{E}^n_+}$ is nonnegative:

$$\langle\sigma(\mathcal{O})\,\mathcal{O}\rangle_{\mathbb{E}^n} \geqslant 0 \tag{7.12}$$

Our next task is to extend this formulation to a general symmetry type (H_n, ρ_n). Essentially, we must extend the SO_n story to include an internal symmetry group K. As motivation we make the following heuristic comments. For simplicity consider the Cartesian product symmetry type $H_n = \mathrm{SO}_n \times K$. Let X be Euclidean space with an open neighborhood of the support of the operators $\mathcal{O}$, $\sigma(\mathcal{O})$ removed. Let $Y = \partial X \cap \mathbb{H}_+$ and assume $\sigma(Y) = \partial X \cap \mathbb{H}_-$. In general there are twist operators which are defined by a principal K-bundle $P \to X$, as in Remark 5.29. The reflection σ must account for the K-bundle, and it might seem at first that σ should "reverse" the bundle by an involution on K. But that does not happen; rather σ lifts to $P \to X$. We give three arguments to justify the existence of a lift of σ.

- If $\mathcal{O}$ is a point operator, then Y is a sphere. Identify $\sigma(Y)$ with Y via translation. Then σ acts on Y as reflection in the equatorial plane parallel to Π. If we one-point compactify X to S^n minus the two balls and assume P extends over the compactification, then the restrictions of P to Y and $\sigma(Y)$ are isomorphic, since the compactification is diffeomorphic to $[0,1] \times S^{n-1}$.

- Continuing, suppose $P \to X$ is the trivial bundle and V is the vector space of local operators attached to Y. (In a geometric theory we take a limit as the radius of the removed ball shrinks to zero.) The automorphism group K of the trivial bundle over Y acts on V, producing K-multiplets of point operators. Postulate a lift of σ to $P \to X$. The hyperplane reflection σ induces an isomorphism $V \to \overline{V}$ which commutes with the K-action, since geometrically the lift of reflection to the trivial bundle $P \to X$ commutes with the global gauge transformations. So a K-multiplet in V is mapped to a K-multiplet in $\overline{V}$ which transforms in the complex conjugate representation. This is the expected behavior.
- Let $n = 1$ and $H_1 = \mathrm{SO}_1 \times \mathbb{Z}/3\mathbb{Z}$. Let $\alpha\colon \mathrm{Bord}_{\langle 0,1\rangle}(H_1) \to \mathrm{Vect}_{\mathbb{C}}$ be the invertible theory which attaches a nontrivial character $\chi\colon \mathbb{Z}/3\mathbb{Z} \to \mathbb{T}$ to the positively oriented point with its trivial $\mathbb{Z}/3\mathbb{Z}$-bundle. (That object Y of the bordism category has automorphism group $\mathbb{Z}/3\mathbb{Z}$, which then acts on the vector space $\alpha(Y)$.) This theory is unitary, a finite analog of Example 2.46. Now $\alpha(P \to S^1)$ is χ applied to the holonomy of the principal $\mathbb{Z}/3\mathbb{Z}$-bundle $P \to S^1$. Reflection reverses the orientation of S^1, and if the bundle stays the same under reflection, then the holonomy complex conjugates, which is precisely what it should do in a reflection positive theory.

We use reflection symmetry (7.10) to construct a larger symmetry group $\widehat{H}_n$ from H_n by adjoining an involution. In the special case $H_n = \mathrm{Spin}_n$, we define $\widehat{H}_n = \mathrm{Pin}_n^+$; the general case is a bootstrap from this. We choose Pin_n^+ rather than Pin_n^- so that the lift of a hyperplane reflection squares to the identity. The arguments above motivate the triviality of the hyperplane reflection automorphism of the internal symmetry group K. Regard $\widehat{H}_n$ as a symmetry group of the Euclidean quantum field theory; the action of an element in $\widehat{H}_n \backslash H_n$ on the Hilbert space $\mathcal{H}$ is by an anti-unitary transformation. The context for the next result is §3.4; the proof is in [**FH1**, §3.3].

THEOREM 7.13.

(1) *There exists a canonical group extension*

$$1 \longrightarrow H_n \xrightarrow{j_n} \widehat{H}_n \longrightarrow \{\pm 1\} \longrightarrow 1, \tag{7.14}$$

split (noncanonically) by a choice of hyperplane reflection $\sigma \in \mathrm{O}_n$, such that the splitting induces an automorphism of $\widetilde{SH}_n \cong \mathrm{Spin}_n \times K$ which is the product of conjugation by σ on Spin_n and is the identity automorphism of K.

(2) *There is a homomorphism $\hat{\rho}_n$ which fits into the pullback diagram*

$$\begin{tikzcd} H_n \arrow[r, "j_n"] \arrow[d, "\rho_n"'] & \widehat{H}_n \arrow[d, "\hat{\rho}_n"] \\ \mathrm{O}_n \arrow[r] & \{\pm 1\} \times \mathrm{O}_n \end{tikzcd} \tag{7.15}$$

(3) *There are inclusions $\hat{\imath}_n\colon \widehat{H}_n \to \widehat{H}_{n+1}$ which, together with the inclusions $i_n\colon H_n \to H_{n+1}$, induce a commutative diagram linking* (7.15) *for varying n.*

(4) *For each $n \geqslant 1$ there is an inclusion of group extensions*

$$\begin{tikzcd} 1 \arrow[r] & H_n \arrow[r] \arrow[d, hook, "i_n"] & \{\pm 1\} \times H_n \arrow[r] \arrow[d, hook, "s_{n+1} * j_{n+1} i_n"] & \{\pm 1\} \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & H_{n+1} \arrow[r, "j_{n+1}"] & \widehat{H}_{n+1} \arrow[r] & \{\pm 1\} \arrow[r] & 1 \end{tikzcd} \tag{7.16}$$

in which i_n is the inclusion in (3.25)*, j_n the inclusion in* (7.14)*, and $s_n\colon \{\pm 1\} \to \widehat{H}_n$ the splitting of* (7.14) *induced by the hyperplane reflection which reverses the first coordinate of $\mathbb{R}^n$ and fixes the others. Furthermore, the inclusions i_n and $\hat{\imath}_n$ induce a commutative diagram linking* (7.16) *for varying n.*

For the basic symmetry groups in (3.23) the extended symmetry groups are listed here:

(7.17)

states/symmetry	H_n	$\widehat{H}_n$
bosons only	SO_n	O_n
fermions allowed	Spin_n	Pin_n^+
bosons, time-reversal (T)	O_n	$\{\pm 1\} \times \mathrm{O}_n$
fermions, $T^2 = (-1)^F$	Pin_n^+	$\widehat{\mathrm{Pin}}_n^+$
fermions, $T^2 = \mathrm{id}$	Pin_n^-	$\widehat{\mathrm{Pin}}_n^-$

The splitting of $\widehat{\mathrm{O}}_n$ is a consequence of the fact that hyperplane reflections are inner in O_n. A similar argument proves that the 4-component group $\widehat{\mathrm{Pin}}_n^\pm$ can be constructed from $\mathrm{Pin}_n^\pm$ by adjoining the automorphism which is the identity on $\mathrm{Spin}_n \subset \mathrm{Pin}_n^\pm$ and multiplication by the central element $-1 \in \mathrm{Spin}_n$ on the off-component of $\mathrm{Pin}_n^\pm$.

REMARK 7.18. We deploy the co-extension (7.14) to formulate reflection positivity on $\mathbb{E}^n$ for a theory with arbitrary symmetry type (H_n, ρ_n). Adjoining translations via the pullback

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, equal] & \mathcal{H}_n \arrow[r, dashed] \arrow[d, dashed] & \mathrm{Euc}_n \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & K \arrow[r] & H_n \arrow[r, "\rho_n"] & \mathrm{O}_n \arrow[r] & 1 \end{tikzcd} \tag{7.19}$$

we obtain a larger group $\mathcal{H}_n$ and a homomorphism $\mathcal{H}_n \to \mathrm{Euc}_n$ to the Euclidean group. The complex point observables form a vector bundle $\mathcal{O}_\mathbb{C} \to \mathbb{E}^n$, and $\mathcal{O}_\mathbb{C}$ carries an action of $\mathcal{H}_n$ lifting that of Euc_n on $\mathbb{E}^n$. Theorem 7.13 gives a co-extension $\widehat{\mathcal{H}}_n$ of $\mathcal{H}_n$ and a homomorphism $\widehat{\mathcal{H}}_n \to \{\pm 1\} \times \mathrm{Euc}_n$. As in (7.10)–(7.12) fix a hyperplane reflection σ and now fix a lift $\hat{\sigma} \in \widehat{\mathcal{H}}_n$ of $(-1, \sigma) \in \{\pm 1\} \times \mathrm{Euc}_n$. Part of the data of the reflection structure is a lift of $\hat{\sigma}$ to an antilinear map of the complex vector bundle $\mathcal{O}_\mathbb{C} \to \mathbb{E}^n$. Now (7.10)–(7.12) apply with $\hat{\sigma}$ replacing σ.

7.4. Reflection structures and naive positivity

Now we turn to curved manifolds and the implementation of reflection positivity in Axiom System 3.1, though we only treat the topological case in these lectures.

Recall (Definition 3.33) that an H_n-manifold is a Riemannian n-manifold equipped with a reduction (P,θ) of its orthonormal frame bundle $\mathcal{B}_O(X) \to X$ to H_n. Extend the principal H_n-bundle $P \to X$ to a principal $\widehat{H}_n$-bundle $j_n(P) \to X$, where j_n is the inclusion in (7.14). If $P \to X$ has a connection, then the connection extends canonically to $j_n(P) \to X$. Use (7.15) to extend the isomorphism $\theta\colon \mathcal{B}_O(X) \to \rho_n(P)$ to an isomorphism $\hat{\theta}\colon \{\pm 1\} \times \mathcal{B}_O(X) \to \hat{\rho}_n\big(j_n(P)\big)$.

Definition 7.20. The *opposite* H_n*-structure* (P',θ') is the principal H_n-bundle $P' := j_n(P)\backslash P \to X$ and the restriction θ' of $\hat{\theta}$ to $\{-1\} \times \mathcal{B}_O(X)$.

The same definition works for differential H_n-structures: simply carry along the connection. Taking opposites is involutive: there is a canonical isomorphism $(P,\theta) \xrightarrow{\cong} (P'',\theta'')$.

Remark 7.21. Let $\sigma \in \mathrm{O}_n$ be a hyperplane reflection and ϕ_σ the automorphism of H_n resulting from the splitting of (7.14). Then we can identify the principal H_n-bundle $P' \to X$ as the projection $P \to X$ of manifolds with the original H_n-action on P modified by precomposition with the automorphism ϕ_σ. For if $\tilde{\sigma} \in \widehat{H}_n$ is the splitting element, then we have an isomorphism

$$\begin{aligned} P &\longrightarrow j_n(P)\backslash P \\ p &\longmapsto p \cdot \tilde{\sigma} \end{aligned} \tag{7.22}$$

of principal H_n-bundles.

Example 7.23. An SO_n-structure is an orientation, and the opposite SO_n-structure is the reverse orientation: $P \to X$ is the principal SO_n-bundle of oriented orthonormal frames, $j_n(P) \to X$ the principal O_n-bundle $\mathcal{B}_O(X) \to X$ of all orthonormal frames, and $j_n(P)\backslash P \to X$ the principal SO_n-bundle of oppositely oriented orthonormal frames.

Example 7.24. For simplicity, we sometimes abbreviate '$\mathrm{Pin}_n^{\pm}$-structure' to 'pin structure', just as 'Spin_n-structure' is abbreviated to 'spin structure'. The opposite functor acts trivially on unoriented manifolds (O_n-structures). The opposite of a pin structure is obtained by tensoring with the orientation double cover, as is easily derived from the text following (7.17).

Use the involution in Definition 7.20 to construct an involution of geometric bordism categories

$$\beta_{\mathcal{B}} = \beta\colon \mathrm{Bord}_{\langle n-1,n\rangle}(H_n^{\nabla}) \to \mathrm{Bord}_{\langle n-1,n\rangle}(H_n^{\nabla}). \tag{7.25}$$

In Definition A.52 we explain that an involution on a category $\mathcal{B}$ is a functor $\beta\colon \mathcal{B} \to \mathcal{B}$ and a natural transformation of functors $\eta\colon \mathrm{id}_{\mathcal{B}} \to \beta^2$. The objects and morphisms in $\mathrm{Bord}_{\langle n-1,n\rangle}(H_n^{\nabla})$ are Riemannian manifolds with differential H_n-structure: the functor β fixes the underlying Riemannian manifold and flips the differential H_n-structure to its opposite. The equivalence η implements the canonical isomorphism indicated after Definition 7.20. We emphasize that the "bar involution" β is covariant: a morphism $X\colon Y_0 \to Y_1$ maps to a morphism $\beta X\colon \beta Y_0 \to \beta Y_1$. Put differently, the arrows of time on objects are unchanged under β.

Now we specialize to the topological case. First, recall that every object in the unoriented topological bordism category has a dual (Lemma 1.20). This assertion extends to an arbitrary symmetry type (H_n,ρ_n) as follows. Recall that an object Y in $\mathrm{Bord}_{\langle n-1,n\rangle}(H_n)$ is a compact $(n-1)$-manifold without boundary, an

H_n-structure on $\underline{\mathbb{R}} \oplus TY \to Y$, and an orientation of the subbundle $\underline{\mathbb{R}} \to Y$, called the *arrow of time*.

LEMMA 7.26. *Y has a dual $Y^\vee$ which equals Y with the reversed arrow of time.*

PROOF. As manifolds evaluation e_Y and coevaluation c_Y are $[0,1] \times Y$, but the arrows of time are different; see Figure 1.4. $\square$

Every object Y in a topological bordism category has a canonical Hermitian structure in the sense of Definition A.62.

PROPOSITION 7.27. *For any object Y in $\mathrm{Bord}_{\langle n-1,n\rangle}(H_n)$ there is a canonical isomorphism*

$$\beta Y \xrightarrow{\cong} Y^\vee \tag{7.28}$$

In other words, reversing the H_n-structure (βY) is equivalent to reversing the arrow of time ($Y^\vee$). We defer to [**FH1**, Proposition 4.8] for the proof.

REMARK 7.29. As an illustration of these involutions, we point out that in the proof of Theorem 6.27 the bordisms $\widehat{H}'_r, \widehat{H}''_r$ are obtained from h'_r, h''_r by applying the commuting involutions β and δ, where δ is the duality involution. Because of Proposition 7.27 the boundary—with orientation—is unchanged by $\beta\delta$.

Let

$$\beta_{\mathcal{C}} = \beta\colon \mathrm{Vect}_{\mathbb{C}} \longrightarrow \mathrm{Vect}_{\mathbb{C}} \tag{7.30}$$

be the involution of complex conjugation. Recall the involution $\beta_{\mathcal{B}}$ on bordism in (7.25). Let

$$F\colon \mathrm{Bord}_{\langle n-1,n\rangle}(H_n) \to \mathrm{Vect}_{\mathbb{C}} \tag{7.31}$$

be a non-extended topological field theory.

DEFINITION 7.32. A *reflection structure* on F is equivariance data for the involutions $\beta_{\mathcal{B}}, \beta_{\mathcal{C}}$.

Equivariance data is spelled out in Definition A.57. For every closed $(n-1)$-manifold Y with H_n-structure we have an isomorphism of vector spaces

$$F(\beta Y) \xrightarrow{\cong} \overline{F(Y)}, \tag{7.33}$$

the curved space analog of (7.11). Combining with the isomorphism (7.28), we see that $F(e_Y)$ becomes the Hermitian form

$$h_Y\colon F(Y^\vee) \otimes F(Y) \cong F(\beta Y) \otimes F(Y) \cong \overline{F(Y)} \otimes F(Y) \longrightarrow \mathbb{C}, \tag{7.34}$$

which by the "S-diagram" argument in Figure 1.5 is nondegenerate. Sesquilinearity is a consequence of the isomorphism

$$\begin{aligned} e_Y &\longrightarrow \beta(e_Y) \\ (t,y) &\longmapsto (1-t,y) \end{aligned} \tag{7.35}$$

The reflection structure includes a curved space analog of (7.10): if X is a closed H_n-manifold, then

$$F(\beta X) = \overline{F(X)}. \tag{7.36}$$

We remark that (7.36) is a strong constraint. For example, in a theory of unoriented manifolds ($H_n = \mathrm{O}_n$) β is the identity functor and (7.36) implies that all partition functions are real.

Definition 7.32 is the 'reflection' half of 'reflection positivity'. The 'positivity' half ensures quantum state spaces are Hilbert spaces.

DEFINITION 7.37. A reflection structure is *positive* if the induced Hermitian form h_Y is positive definite for all $Y \in \mathrm{Bord}_{\langle n-1,n\rangle}(H_n)$.

REMARK 7.38. In a non-extended field theory reflection is *data* and positivity is a *condition*. In the extended case taken up in 7.6 and Lecture 8, both reflection and positivity are data.

EXAMPLE 7.39. Let n be even and $\mu \in \mathbb{C}^\times$. The invertible Euler theory $\epsilon_\mu \colon \mathrm{Bord}_{\langle n-1,n\rangle}(\mathrm{O}_n) \to \mathrm{Line}_{\mathbb{C}}$ in (1.53) has $\epsilon_\mu(S^n) = \mu^2$. If ϵ_μ admits a reflection structure, then we must have $\mu^2 \in \mathbb{R}$. Positivity imposes a stronger constraint. Let D be the manifold D^n with outgoing boundary $S = S^{n-1}$. Write S^n as the composition $\varnothing^{n-1} \xrightarrow{D} S^{n-1} \xrightarrow{\beta D^\vee} \varnothing^{n-1}$ of two closed balls. This composition may be rewritten as

$$\varnothing^{n-1} \xrightarrow{\beta D \amalg D} S \amalg S \xrightarrow{e_S} \varnothing^{n-1}, \tag{7.40}$$

which ϵ_μ evaluates to $h_S\big(\beta\epsilon_\mu(D), \epsilon_\mu(D)\big)$. Therefore, if ϵ_μ is positive, then $\mu^2 > 0$ and so $\mu \in \mathbb{R}$. This is a special case of an argument about doubles we give in the next section. We can also conclude $\mu \in \mathbb{R}$ from $\mu = \epsilon_\mu(\mathbb{RP}^n)$, since the partition function of a closed manifold must be real in an unoriented theory with reflection structure. However, the argument with doubles also applies to the oriented Euler theory $\mathrm{Bord}_{\langle n-1,n\rangle}(\mathrm{SO}_n) \to \mathrm{Line}_{\mathbb{C}}$, whereas the argument with $\mathbb{RP}^n$ does not.

7.5. Positivity and doubles

Continuing with the symmetry type (H_n, ρ_n), let

$$F \colon \mathrm{Bord}_{\langle n-1,n\rangle}(H_n) \longrightarrow \mathrm{Vect}_{\mathbb{C}} \tag{7.41}$$

be a non-extended topological field theory with reflection structure (Definition 7.32). In §7.4 we discussed the analogs (7.33) and (7.36) of the data (7.11) and the equation (7.10) in Euclidean field theory, which are encoded by the reflection structure. Now we take up the Axiom System analog of the positivity condition (7.12) in Euclidean field theory. We begin with the definition of the double of a manifold with boundary. Recall the bar involution β on bordisms (7.25).

DEFINITION 7.42. Let X be a compact H_n-manifold with boundary, viewed as a bordism $\varnothing^{n-1} \to \partial X$. The *double* of X is the closed H_n-manifold

$$\Delta X = e_{\partial X}(\beta X, X), \tag{7.43}$$

where $e_{\partial X}$ is the evaluation bordism.

The double is illustrated in Figure 7.2. In that picture $Y = \partial X$. The double construction is standard in differential geometry for unoriented and oriented manifolds, but less so for general tangential structures such as spin and pin. Our definition uses the group co-extension (7.14), through the involution β, so is ultimately based on hyperplane reflections in Euclidean space, as conforms with intuition for the double and the model case: $\mathbb{E}^n$ as the double of a half space.

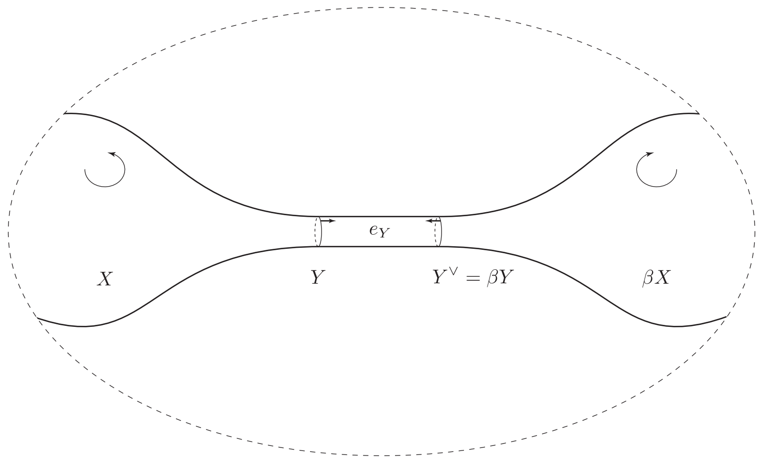

FIGURE 7.2. The double of X

PROPOSITION 7.44. *If a topological field theory* $F\colon \mathrm{Bord}_{\langle n-1,n\rangle}(H_n) \to \mathrm{Vect}_{\mathbb{C}}$ *admits a positive reflection structure, then* $F(\Delta X) \geqslant 0$ *for all compact* H_n*-manifolds* X *with boundary.*

Note that the value of a theory on a closed n-manifold does not depend on the reflection structure, so this is a necessary condition on non-extended theories as defined by Axiom System 1.17.

PROOF. Assume F has a positive reflection structure. From (7.43) and (7.36) we deduce

$$F(\Delta X) = F(e_{\partial X})\bigl(F(\beta X), F(X)\bigr) = h_{\partial X}\bigl(\overline{F(X)}, F(X)\bigr) = \|F(X)\|^2_{F(\partial X)}, \tag{7.45}$$

which is nonnegative by positivity (Definition 7.37). □

EXAMPLE 7.46. Consider the 2-dimensional theory $F\colon \mathrm{Bord}_{\langle 1,2\rangle}(\mathrm{SO}_2) \to \mathrm{Vect}_{\mathbb{C}}$ constructed from the Frobenius algebra $A = H^\bullet(S^2;\mathbb{C})$; see Theorem 4.22 and Example 4.21. Write S^2 as the composition (7.40) of two disks to deduce that $F(S^2) = \tau(1) = 0$, where $\tau\colon A \to \mathbb{C}$ is the trace and $1 \in A$ the unit. It follows that F does not admit a positive reflection structure, since the norm square of 1 is positive for any Hilbert space structure on A.

The homogeneous space H_{n+1}/H_n is diffeomorphic to the sphere S^n. This follows from the fact that the squares in (3.25) are pullbacks. The homogeneous principal H_n-bundle $H_{n+1} \to H_{n+1}/H_n$ is part of an H_n-structure on S^n; the associated bundle to $\rho_n\colon H_n \to \mathrm{O}_n$ is isomorphic to the orthonormal frame bundle.

PROPOSITION 7.47. *The sphere* S^n *with* H_n*-structure* $H_{n+1} \to H_{n+1}/H_n$ *is a double.*

PROOF. Reflection σ in the hyperplane perpendicular to e_1 is an involution of S^n with fixed point set the equatorial S^{n-1} perpendicular to e_1. The reflection lifts to an isomorphism of the principal H_n-bundle $H_{n+1} \to H_{n+1}/H_n$ with the pullback of its opposite; see [**FH1**, Proposition 3.13]. □

EXERCISE 7.48. Which spin circle is a double? (There are two spin structures on S^1.)

The universal family of H_n-spheres is the fiber bundle

$$H_{n+1}/H_n \longrightarrow BH_n \longrightarrow BH_{n+1}. \tag{7.49}$$

It plays an important role in the obstruction theory analysis in §8.1.

We now prove that any two H_n-doubles are bordant. They are not Reinhart bordant, which would be bordant through an H_n-manifold, but rather they are Thom bordant.

PROPOSITION 7.50. *Let Y_0, Y_1 be closed $(n-1)$-dimensional H_n-manifolds and $X\colon Y_0 \to Y_1$ an H_n-bordism. Then*

$$\beta X \amalg e_{Y_1} \amalg X\colon \beta Y_0 \amalg Y_0 \longrightarrow \varnothing^{n-1} \tag{7.51}$$

is H_{n+1}-bordant to e_{Y_0}.

PROOF. The bordism[65] is $[0,1] \times X$. ☐

COROLLARY 7.52. *The double ΔX of a compact H_n-manifold with boundary is null bordant through an H_{n+1}-manifold.*

By Corollary 7.47 this applies to S^n with its canonical H_n-structure, and so every double is H_{n+1}-bordant to S^n.

PROOF. Apply Proposition 7.50 to $X\colon \varnothing^{n-1} \to \partial X$ (and smooth the corners of $[0,1] \times X$). Figure 7.3 illustrates the construction. ☐

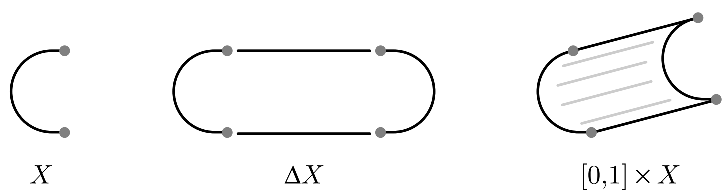

FIGURE 7.3. $\partial\big([0,1] \times X\big) = \{0\} \times X \,\cup\, [0,1] \times \partial X \,\cup\, \{1\} \times X$

REMARK 7.53. If X is the 2-dimensional disk, viewed as a bordism from the empty 1-manifold to the circle, then ΔX is the 2-dimensional sphere S^2 and the null bordism $[0,1] \times X$ is the 3-dimensional ball D^3. The nonzero Euler number of S^2 obstructs the existence of an H_2-structure on D^3 which restricts to the given H_2-structure on S^2 (for any stable tangential structure H).

[65]It is a bordism of manifolds with boundary, or better a higher morphism in a multi-bordism category. We only use $Y_0 = \varnothing^{n-1}$, as in Corollary 7.52, in which case $[0,1] \times X$ is a null bordism of a closed manifold.

7.6. Introduction to extended reflection structures and positivity

Once one has in hand a theory of involutions on higher categories, then there should be no difficulty defining a reflection structure on an extended topological field theory (5.22) with symmetry type (H_n, ρ_n). One need only define the bar involutions on the extended bordism n-category $\mathrm{Bord}_n(H_n)$ and on the codomain symmetric monoidal n-category $\mathcal{C}$. On $\mathrm{Bord}_n(H_n)$ the objects and morphisms at all levels have underlying H_n-manifolds with corners; the involution reverses the H_n-structure[66] as in Definition 7.20. The situation is different for the codomain $\mathcal{C}$, since we do not make any formal hypotheses on the nature of $\mathcal{C}$. Abstractly we simply require an n-category $\mathcal{C}$ with involution $\beta_{\mathcal{C}}$, but we may also require $(\mathcal{C}, \beta_{\mathcal{C}})$ to agree "near the top" with $\mathrm{Vect}_{\mathbb{C}}$ and complex conjugation. We focus on $n = 2$ and $\mathcal{C} = \mathrm{Alg}_{\mathbb{C}}$, as in Example 5.8, with involution $\beta_{\mathcal{C}}$ complex conjugation of algebras, bimodules, and intertwiners.

We want to "impose" positivity on an extended reflection structure. General principles of category number suggest that whereas in the non-extended case positivity is a *condition*, in the extended case it should be *data*. That is indeed the case for invertible topological theories, which is the only situation in which we have a notion of extended positivity, which we develop in Lecture 8 using homotopy theory.

QUESTION 7.54. What is extended positivity in an extended field theory with reflection structure?

This is an open problem, even for topological field theories. The two pillars of quantum field theory are locality and unitarity, so it is a fundamental issue to marry unitarity with extended locality. To illustrate Question 7.54, we conclude this lecture by exploring positivity for complex conjugation on the 2-category $\mathrm{Alg}_{\mathbb{C}}$. (We use ordinary, not super, algebras to avoid signs.) It would be interesting to have a categorical approach analogous to that in §7.1 for the 1-category $\mathrm{Vect}_{\mathbb{C}}$.

To begin we identify duals in $\mathrm{Alg}_{\mathbb{C}}$.

LEMMA 7.55. *The dual to an algebra* $A \in \mathrm{Alg}_{\mathbb{C}}$ *is the opposite algebra* A^{op}.

Note that every object has a dual: 1-dualizability is automatic. Higher dualizability imposes stringent finiteness conditions (for a few examples, see [**Dav**, §3.2] as well as [**DSPS**, **BJS**].

PROOF. Let evaluation $e_A\colon A^{\mathrm{op}} \otimes A \to \mathbb{C}$ be A as a right $(A^{\mathrm{op}} \otimes A)$-module, and let coevaluation $c_A\colon \mathbb{C} \to A \otimes A^{\mathrm{op}}$ be A as a left $(A \otimes A^{\mathrm{op}})$-module. □

EXERCISE 7.56. Check the S-diagram relations.

Hence the duality involution $\delta\colon \mathrm{Alg}_{\mathbb{C}} \to \mathrm{Alg}_{\mathbb{C}}^{\mathrm{op}}$ takes an algebra A to A^{op}; a 1-morphism $B\colon A_0 \to A_1$, which is an (A_1, A_0)-bimodule, to the $(A_0^{\mathrm{op}}, A_1^{\mathrm{op}})$-bimodule B; and a 2-morphism $f\colon B \to B'$ between 1-morphisms $B, B'\colon A_0 \to A_1$ to itself. Note δ reverses the direction of 1-morphisms and preserves the direction of 2-morphisms.[67] The 2-category $\mathrm{Alg}_{\mathbb{C}}$ has an (untwisted) bar involution β which complex conjugates at all levels: $A \mapsto \overline{A}$, $B \mapsto \overline{B}$, $f \mapsto \overline{f}$.

[66] Recall that we either stabilize the tangent bundle to have rank n or embed in a germ of an n-manifold. Thus H_n-structures are defined on manifolds of dimension $< n$ in terms of the frame bundle of a rank n vector bundle.

[67] An alternative twisted duality involution: map B to $B^*\colon A_0^{\mathrm{op}} \to A_1^{\mathrm{op}}$ and f to f^*, thereby preserving the direction of 1-morphisms and reversing the direction of 2-morphisms.

Now suppose $A \in \mathrm{Alg}_{\mathbb{C}}$ is a Hermitian object, a complex algebra equipped with an isomorphism $\overline{A} \xrightarrow{\cong} A^{\mathrm{op}}$ in $\mathrm{Alg}_{\mathbb{C}}$. In other words, the data is an invertible $(A^{\mathrm{op}}, \overline{A})$-bimodule, or equivalently a right $(A \otimes \overline{A})$-module B. Invertibility is exhibited by a left $(\overline{A} \otimes A)$-module B' and isomorphisms

$$\begin{aligned} B \otimes_A B' &\xrightarrow{\cong} \overline{A} \qquad && \text{of } (\overline{A}, \overline{A})\text{-bimodules,} \\ B \otimes_{\overline{A}} B' &\xrightarrow{\cong} A \qquad && \text{of } (A, A)\text{-bimodules.} \end{aligned} \tag{7.57}$$

A special case is a $*$-structure.

Definition 7.58. Let A be a complex algebra. A *$*$-structure* on A is an algebra isomorphism $*\colon \overline{A} \to A^{\mathrm{op}}$ such that $** = \mathrm{id}_A$.

Alternatively, $*\colon A \to A$ is an anti-linear anti-homomorphism. The Hermitian structure on A is exhibited by the module $B = A$ with right $(A \otimes \overline{A})$-action

$$b \cdot (a' \otimes \bar{a}) = a^* b a', \qquad a, a', b \in A. \tag{7.59}$$

The inverse is $B' = A$ with left $(\overline{A} \otimes A)$-action

$$(\bar{a} \otimes a') \cdot b' = a' b' a^*, \qquad a, a', b' \in A, \tag{7.60}$$

and the isomorphisms (7.57) are

$$\begin{aligned} b \otimes b' &\longmapsto b'^* b^* \\ b \otimes b' &\longmapsto b' b \end{aligned} \tag{7.61}$$

respectively. We are led to the following special case of Question 7.54.

Question 7.62. What is positivity for a $*$-structure or a more general Hermitian structure on A?

Remark 7.63. $*$-algebras and their infinite dimensional topological versions, C^*-algebras and von Neumann algebras, make frequent appearances in mathematical treatments of quantum mechanics and quantum field theory. Indeed, quantum mechanics was a main impetus for the development of operator algebras [**vN2**]. In that context positivity is built in from the beginning since one starts with a positive definite Hilbert space.

Here is a guess about positivity, which we leave as an extended exercise for the reader. Let a Hermitian structure on A be given as an invertible $(A^{\mathrm{op}}, \overline{A})$-module B, as in the paragraph containing (7.57). Then $\overline{B}^*$ is an invertible $(A, \overline{A}^{\mathrm{op}})$-bimodule. Form the complex vector space

$$L = \overline{A} \otimes_{\overline{A} \otimes \overline{A}^{\mathrm{op}}} (B \otimes \overline{B}^*) \otimes_{A^{\mathrm{op}} \otimes A} A. \tag{7.64}$$

Exercise 7.65.

(1) Construct a real structure on L.
(2) If A is invertible, prove that $\dim L = 1$. (A standard theorem about central simple algebras states that A is invertible if and only if the natural map $A \otimes A^{\mathrm{op}} \to \operatorname{End} A$ is an isomorphism.)
(3) What is L in the case of a $*$-structure?
(4) Extend the entire discussion to a $\mathbb{Z}/2\mathbb{Z}$-graded algebra $A \in s\mathrm{Alg}_{\mathbb{C}}$.

Assuming (1) and (2), define *positivity data* on A in the invertible case to be a real trivialization of L, i.e., a real isomorphism $\mathbb{C} \to L$. This fits well with what we find using homotopy theory in Lecture 8. But then what is positivity if A is not invertible?

LECTURE 8

Extended Positivity and Stable Homotopy Theory

This lecture contains our main theorems, which determine the homotopy type of various spaces of invertible field theories. In particular, the abelian group of path components is isomorphic to the abelian group of isomorphism classes or deformation classes of various spaces of invertible unitary extended topological field theories with fixed dimension and symmetry type. For unitary theories the answer is a cohomology group of a Thom spectrum, and there are techniques in stable homotopy theory to compute these groups. We apply the theorems in Lecture 10, where we report on computations for fermion systems and compare to the condensed matter literature. To state these theorems we need to define reflection positivity—the Wick rotation of unitarity—for extended invertible field theories. The framework for reflection structures involves involutions, here on spectra, and so we work in equivariant stable homotopy theory. We must choose involutions on both the domain $\mathcal{B}$ and codomain $\mathcal{I}$ of an invertible field theory $\alpha\colon \mathcal{B} \to \mathcal{I}$; the reflection structure is then equivariance data for α. We define positivity for field theories in homotopy theoretic terms as well. This only involves the codomain $\mathcal{I}$: a reflection structure on an n-dimensional theory allows a factorization of its $(n-1)$-dimensional truncation through a Hermitian version of $\mathcal{I}$ and we ask that it further factor through a positive version of $\mathcal{I}$. At the top level this reduces to naive positivity (7.4). Given these definitions the desired homotopy types are computed using standard techniques in topology.

Here is an outline, then, of the lecture. In §8.1 we explain the relationship between naive positivity—more precisely, the consequence that partition functions of doubles are positive (Proposition 7.44)—and stability, the latter a special notion (Definition 8.8) for invertible theories in which the theory factors through the map from a Madsen-Tillmann spectrum to a Thom spectrum. In §8.2 we give a brief introduction to equivariant spectra, focusing on spectra with involutions. Then in §8.3 we specify the involutions on codomain spectra $\mathcal{I}$ which implement complex conjugation. This leads to several variations, which at the level of super lines correspond to Hermitian structures, positive Hermitian structures, real structures, flat structures, etc. These higher versions, which we call 'super k-lines', are described in homotopy theoretic terms in §8.4. We define extended positivity in §8.5, of course only working in the invertible context. (As stated in Question 7.54, it is an important open problem to define extended positivity in general.) One crucial maneuver is an argument to "split off a reflection". We define the space of extended reflection positive theories as a fiber product (8.54). A reworking leads to Definition 8.62 of an *extended positivity structure* on a particular theory. It is the usual positivity of a Hermitian form in dimension $n-1$, and is data in lower dimensions. Finally, in §8.6 we state the main theorems which determine the homotopy types of relevant spaces of reflection positive unitary theories. We do not include proofs in this lecture; they are all contained in [**FH1**, §8].

8.1. Naive positivity and stability

Recall (Definition 6.45) that to a symmetry type (H_n, ρ_n) we attach the Madsen-Tillmann spectrum $\Sigma^n MTH_n$, the Thom spectrum of the virtual bundle

$$\underline{\mathbb{R}^n} - S_n \longrightarrow BH_n. \tag{8.1}$$

Stabilization (3.25) determines a spectrum map $\Sigma^n MTH_n \to \Sigma^{n+1} MTH_{n+1}$.

PROPOSITION 8.2. *The fiber of the map*

$$\Sigma^n MTH_n \longrightarrow \Sigma^{n+1} MTH_{n+1}$$

is $\Sigma^n(BH_{n+1})_+$. The map $\Sigma^n(BH_{n+1})_+ \to \Sigma^n MTH_n$ is represented by the universal family $BH_n \to BH_{n+1}$ of H_n-spheres.

PROOF. Equivalently, as we prove, there is a fibration sequence

$$\Sigma^{-1} MTH_n \longrightarrow MTH_{n+1} \longrightarrow (BH_{n+1})_+ \tag{8.3}$$

in which the first map is stabilization. Begin with the cofibration built from the sphere and ball bundles of the universal bundle $S_n \to BH_n$:

$$S(S_n)_+ \hookrightarrow B(S_n)_+ \twoheadrightarrow \big(B(S_n), S(S_n)\big)\ , \tag{8.4}$$

Then identify BH_{n-1} as the unit sphere bundle $S(S_n)$ and write (8.4) in terms of Thom spaces:

$$\mathrm{Thom}(BH_{n-1}; \underline{\mathbb{R}^0}) \hookrightarrow \mathrm{Thom}(BH_n; \underline{\mathbb{R}^0}) \twoheadrightarrow \mathrm{Thom}(BH_n; S_n)\ . \tag{8.5}$$

Here $\underline{\mathbb{R}^0}$ is the vector bundle of rank zero. Now add $-S_n$ to each of the vector bundles in (8.5) and note that the restriction of S_n to BH_{n-1} is $S_{n-1} \oplus \underline{\mathbb{R}^1}$. □

EXERCISE 8.6. Use Proposition 8.2 to construct the Euler theory (Example 6.13) via as a map of spectra with an explicit trivialization of its truncation (6.14) in codimension one.

The sequence of MT spectra

$$\Sigma^n MTH_n \xrightarrow{i_n} \Sigma^{n+1} MTH_{n+1} \xrightarrow{i_{n+1}} \cdots \tag{8.7}$$

with colimit the Thom spectrum MTH allows us to make a special definition for invertible topological field theories which is not part of general quantum field theory.

DEFINITION 8.8. An n-dimensional invertible topological field theory with domain $\Sigma^n MTH_n$ is *stable* if it is the restriction of a theory defined on MTH.

Note that the partition function of a stable invertible theory is a Thom bordism invariant. (Compare with the assertion in Theorem 6.27(2) about arbitrary invertible theories).

We apply the relationship (§7.5) between naive positivity and doubles to deduce a criterion for stability (compare with Theorem 6.27(1).)

THEOREM 8.9. *An invertible theory $\alpha\colon \Sigma^n MTH_n \to \Sigma^n I\mathbb{C}^\times$ is stable if and only if $\alpha(S^n) = 1$. The subspace of* $\mathrm{Map}(\Sigma^n MTH_n, \Sigma^n I\mathbb{C}^\times)$ *consisting of theories α with $\alpha(S^n) = 1$ is homotopy equivalent to the mapping space* $\mathrm{Map}(MTH, \Sigma^n I\mathbb{C}^\times)$.

We refer to [**FH1**, Theorem 7.22] for the obstruction theory proof. In one direction the theorem is trivial: if α extends to $\Sigma^{n+1}MTH_{n+1}$, then since S^n is null bordant as an H_{n+1}-manifold it follows that $\alpha(S^n) = 1$. For the converse at the first stage we must factor

$$\begin{tikzcd} \Sigma^n MTH_n \arrow[r, "i_n"] \arrow[dr, "\alpha"'] & \Sigma^{m+1} MTH_{m+1} \arrow[d, dashed] \\ & \Sigma^n I\mathbb{C}^\times \end{tikzcd} \tag{8.10}$$

which we can do if and only if the restriction of α to the homotopy fiber of i_n is null homotopic. But by Proposition 8.2 that fiber is the base of the universal family of H_n-spheres, so the obstruction is the value of the theory on that family of spheres. The triviality is precisely the hypothesis. It turns out there are no further obstructions. Furthermore, the obstruction theory argument can be carried out all at once on the space of invertible theories, hence the second assertion in the theorem.

Remark 8.11. The partition function of a stable invertible topological field theory is a Thom bordism invariant; Theorem 8.9 reduces to Theorem 6.27(1) on the level of partition functions. (In Lecture 6 we gave a different proof for the statement about partition functions using Morse theory.)

8.2. Equivariant spectra

Our next task is to implement reflection structures on invertible field theories. Recall (Definition 7.32) that a theory with reflection structure is a $\mathbb{Z}/2\mathbb{Z}$-equivariant functor between symmetric monoidal categories with a $\mathbb{Z}/2\mathbb{Z}$-action. In the invertible case (Ansatz 6.89) a theory is represented as a spectrum map, so naturally a theory with reflection structure is a $\mathbb{Z}/2\mathbb{Z}$-equivariant spectrum map between spectra with a $\mathbb{Z}/2\mathbb{Z}$-action. In other words, to implement reflection structures on invertible topological theories we work in *equivariant* stable homotopy theory. In this section we give a brief introduction to Borel equivariant spectra; see [**FH1**, §6] for a more generous exposition and [**HHR**, Chapter 2] and the references therein for a comprehensive development of equivariant homotopy theory. In §8.3 we implement complex conjugation on the codomain spectra of invertible field theories. In §8.5 we implement the "β-involution" (7.25) on the domain bordism spectra and then define reflection structures.

Let G be a finite group; in our application $G = \mathbb{Z}/2\mathbb{Z}$. A (naive) *$G$-spectrum* $\mathcal{X}$ is a sequence $\mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2, \dots$ of pointed G-spaces and equivariant maps $s_q\colon S^1 \wedge \mathcal{X}_q \to \mathcal{X}_{q+1}$. Here S^1 has the trivial G-action. If $\mathcal{X}, \mathcal{Y}$ are G-spectra, there is a derived equivariant mapping space $\mathrm{Map}^G(\mathcal{X}, \mathcal{Y})$ of G-equivariant spectrum maps $\mathcal{X} \to \mathcal{Y}$; its group of path components is denoted $[\mathcal{X}, \mathcal{Y}]^{hG}$. The category of G-spectra has a symmetric monoidal structure: smash product. An ordinary spectrum $\mathcal{B}$ determines a G-spectrum with trivial G-action. There are natural adjoints to this functor. First to a pointed G-space Z are associated two natural spaces. The *homotopy fixed point space* Z^{hG} is the space of sections of the fibration

$$EG \times_G Z \longrightarrow BG \tag{8.12}$$

with disjoint basepoint. The *homotopy orbit space* $Z_{hG} = EG\times_G Z \,/\, EG\times\{*\}$ is the Borel construction with the basepoint of Z crushed to a point. We perform these space-level constructions component-wise on a G-spectrum $\mathcal{X}$ to obtain ordinary spectra $\mathcal{X}^{hG}$ and $\mathcal{X}_{hG}$. For $\mathcal{X}$ a G-spectrum and $\mathcal{B}$ a spectrum, there are functorial weak equivalences

$$\begin{aligned}\mathrm{Map}^G(\mathcal{B},\mathcal{X}) &\simeq \mathrm{Map}(\mathcal{B},\mathcal{X}^{hG}),\\ \mathrm{Map}^G(\mathcal{X},\mathcal{B}) &\simeq \mathrm{Map}(EG_+\wedge_G \mathcal{X},\mathcal{B}).\end{aligned} \tag{8.13}$$

These generalize the corresponding statements for G-spaces and express adjoint relations among functors between spectra and G-spectra.

For any real linear representation $\rho\colon G\to \mathrm{Aut}(V)$ the one-point compactification S^V of V is a pointed G-space. The suspension G-spectrum of S^V, also denoted S^V, is invertible under smash product. Write S^{-V} for its inverse. Intuitively, S^{-V} is the suspension G-spectrum associated to the virtual representation $-V$. (See [**FH1**, Example 6.17].)

Specialize to $G=\mathbb{Z}/2\mathbb{Z}$. The fixed point spectrum $\mathcal{X}^{h\mathbb{Z}/2}$ is computed as

$$\begin{aligned}\mathrm{Map}^{\mathbb{Z}/2\mathbb{Z}}(S^0,\mathcal{X}) \;&\simeq\; \mathrm{Map}(B\mathbb{Z}/2\mathbb{Z}_+,\mathcal{X})\\ &\xleftarrow{\;\simeq\;} \mathcal{X}\vee \mathrm{Map}(B\mathbb{Z}/2\mathbb{Z},\mathcal{X}) \xrightarrow{\;\simeq\;} \mathcal{X}\times \mathrm{Map}(B\mathbb{Z}/2\mathbb{Z},\mathcal{X}),\end{aligned} \tag{8.14}$$

in which the left pointing map involves a choice of a basepoint in $B\mathbb{Z}/2\mathbb{Z}$. It is the sum of the map $B\mathbb{Z}/2\mathbb{Z}_+\to S^0$ sending $B\mathbb{Z}/2\mathbb{Z}$ to the non-basepoint and the map $B\mathbb{Z}/2\mathbb{Z}_+\to B\mathbb{Z}/2\mathbb{Z}$ which is the identity map on $B\mathbb{Z}/2\mathbb{Z}$ and sends the disjoint basepoint on the left to the new basepoint on the right.

Let σ be the real sign representation of $\mathbb{Z}/2\mathbb{Z}$. Define reduced virtual representations

$$\begin{aligned}\gamma &= 1-\sigma,\\ \delta &= \sigma-1.\end{aligned} \tag{8.15}$$

If $\mathcal{B}$ is a spectrum, set

$$\begin{aligned}\mathcal{B}^\gamma &= S^\gamma\wedge\mathcal{B},\\ \mathcal{B}^\delta &= S^\delta\wedge\mathcal{B}.\end{aligned} \tag{8.16}$$

Both S^γ and S^δ are $\mathbb{Z}/2\mathbb{Z}$-equivariant refinements of the sphere spectrum S^0, and the induced involution on $\pi_\bullet S^0$ is inversion $a\mapsto -a$ in both cases. In fact, the representation sphere of any odd multiple of $\sigma-1$ has the same property. These representation sphere spectra can be distinguished if we look at the induced actions on Picard q-groupoids extracted from S^0.

EXERCISE 8.17. Identify the Picard 1-groupoid $\pi_{\leqslant 1}S^0$ with ${}_{\mathbb{Z}}\mathrm{Line}_{\mathbb{R}}$, the groupoid of $\mathbb{Z}$-graded real lines ($\pi_0\cong\mathbb{Z}$, $\pi_1\cong\mathbb{Z}/2\mathbb{Z}$, nontrivial k-invariant). What is the induced $\mathbb{Z}/2\mathbb{Z}$-action on ${}_{\mathbb{Z}}\mathrm{Line}_{\mathbb{R}}$ from S^γ? From S^δ? Try this exercise with other models of $\pi_{\leqslant 1}S^0$, such as $|\,\mathrm{Bord}_1(\mathrm{SO}_1)|$; see Theorem 6.20.

For any spectrum $\mathcal{B}$ there is a universal interpretation of the $\mathbb{Z}/2\mathbb{Z}$-equivariant spectrum $\mathcal{B}^\delta$. To put it in perspective, suppose C is a symmetric monoidal category such that every object is dualizable. Implement duality as a functor $C\to C^{\mathrm{op}}$; it is part of a *twisted* $\mathbb{Z}/2\mathbb{Z}$-action on C (Theorem A.59). If C is a Picard groupoid, then we can compose with the twisted involution $C^{\mathrm{op}}\to C$ which inverts morphisms to obtain a $\mathbb{Z}/2\mathbb{Z}$-action on C. The composed duality involution is $c\mapsto c^\vee$ on objects and $f \mapsto (f^\vee)^{-1}$ on morphisms. The same works for Picard q-groupoids, $q \in \mathbb{Z}^{>0} \cup \{\infty\}$. The geometric realization—an infinite loop space, the 0-space of a spectrum—inherits an involution.

PROPOSITION 8.18. *Let $\mathcal{B}$ be a spectrum. Then the $\mathbb{Z}/2\mathbb{Z}$-equivariant spectrum $\mathcal{B}^\delta$ models the duality involution.*

See [**FH1**, §6.3.2] for the complete proof. As motivation, consider a (higher) Picard groupoid C. Let D be the category of pairs $(c_1, c_2) \in C \times C$ equipped with an isomorphism $c_1 \otimes c_2 \to 1$. Then $(c_1, c_2) \mapsto c_1$ is an equivalence $D \xrightarrow{\cong} C$. The duality $\mathbb{Z}/2\mathbb{Z}$-action is $(c_1, c_2) \mapsto (c_2, c_1)$. If $\mathcal{B}$ is a spectrum, then the spectrum which corresponds to D is the homotopy fiber of the product map

$$\mathcal{B} \vee \mathcal{B} \longrightarrow \mathcal{B}. \tag{8.19}$$

To compute the homotopy fiber, smash $\mathcal{B}$ with the cofibration

$$\mathbb{Z}/2\mathbb{Z}_+ \longrightarrow S^0 \longrightarrow S^\sigma \tag{8.20}$$

of pointed $\mathbb{Z}/2\mathbb{Z}$-spaces to construct a fibration

$$\mathcal{B}^\delta \longrightarrow \mathbb{Z}/2\mathbb{Z}_+ \wedge \mathcal{B} \longrightarrow \mathcal{B}. \tag{8.21}$$

This identifies $\mathcal{B}^\delta$ as the equivariant homotopy fiber of (8.19).

8.3. Complex conjugation

Our task is to implement complex conjugation on the codomain $\Sigma^n I\mathbb{C}^\times$ of *discrete* invertible field theories (Ansatz 6.88) and on the codomain $\Sigma^{n+1} I\mathbb{Z}(1)$ of their deformation classes (Theorem 6.96). Equivalently, $\Sigma^{n+1} I\mathbb{Z}(1)$ is the codomain of *continuous* invertible field theories (Ansatz 6.99). There is a long discussion in [**FH1**, §6.3.3] which explains the choices we now make; we do not reproduce it here. In the end we settle on the natural (universal) involution[68] $\gamma = 1 - \sigma$ on $I\mathbb{Z}(1)$: the associated equivariant spectrum is $I\mathbb{Z}(1)^\gamma$. Choices of equivariant enhancements of $I\mathbb{C}^\times$ and $H\mathbb{C}$ are made so that the exponential sequence works, but these involutions are not natural. It is easy to see why not. In terms of the polar decomposition $\mathbb{C}^\times = \mathbb{T} \times \mathbb{R}^{>0}$ complex conjugation acts as inversion on $\mathbb{T}$ and the identity on $\mathbb{R}^{>0}$. Hence define[69]

$$(I\mathbb{C}^\times)^{\nu_0'} = I\mathbb{T}^\gamma \vee H\mathbb{R}^{>0}, \tag{8.22}$$

where the second factor is a nonequivariant spectrum with trivial $\mathbb{Z}/2\mathbb{Z}$-action. The equivariant spectrum $H\mathbb{C}^{\nu_0'}$ is defined similarly using the product decomposition $\mathbb{C} = \mathbb{R}(1) \times \mathbb{R}$. The compatibility of these choices leads to a cofibration sequence

$$I\mathbb{Z}(1)^\gamma \longrightarrow H\mathbb{C}^{\nu_0'} \xrightarrow{\exp} (I\mathbb{C}^\times)^{\nu_0'} \tag{8.23}$$

[68] A *natural* involution is one which is coherently defined for all spectra; see [**FH1**, §6.3.1].

[69] The cumbersome notation ν_0' is kept to be consistent with [**FH1**, §6.3.3].

of $\mathbb{Z}/2\mathbb{Z}$-spectra; the induced map on π_0 is the exponential sequence (6.71) with involutions

$$\sqrt{-1}\,n \longmapsto -\sqrt{-1}\,n \qquad \text{on } \mathbb{Z}(1) \tag{8.24}$$

$$z \longmapsto \bar{z} \qquad \text{on } \mathbb{C} \tag{8.25}$$

$$\lambda \longmapsto \bar{\lambda} \qquad \text{on } \mathbb{C}^\times \tag{8.26}$$

Although (8.26) is the desired complex conjugation involution, we could have achieved it by replacing $\gamma = 1 - \sigma$ with $N\gamma$ for any odd N. The motivation for $N = 1$ is the discussion in §7.1: we want the composition of complex conjugation and duality to have fixed point spectrum equivalent to the original spectrum. This follows from Proposition 8.18, since

$$\left(\mathcal{I}^\delta\right)^\gamma = \mathcal{I} \tag{8.27}$$

for any spectrum $\mathcal{I}$.

Remark 8.28. We give another justification for the choice of involution γ on $I\mathbb{Z}(1)$ and the involutions on $I\mathbb{C}^\times$ and $H\mathbb{C}$ which flow from it. Recall the magic revealed in Remark 6.91: the identification of the low-lying Picard q-groupoids of $I\mathbb{C}^\times$ as $s\mathrm{Line}_{\mathbb{C}}$ ($q = 1$) and $s\mathrm{Alg}_{\mathbb{C}}$ ($q = 2$). The same identifications hold for $\Sigma I\mathbb{Z}(1)$, but with a different topology on morphisms. We claim the involutions ν_0' on $I\mathbb{C}^\times$ and γ on $\Sigma I\mathbb{Z}(1)$ model complex conjugation on these Picard 1- and 2-groupoids. This is justified in [**FH1**, Remark 6.40] where the homotopy fixed point spectra are computed and they model the real counterparts $s\mathrm{Line}_{\mathbb{R}}$ and $s\mathrm{Alg}_{\mathbb{R}}$ to $s\mathrm{Line}_{\mathbb{C}}$ and $s\mathrm{Alg}_{\mathbb{C}}$.

8.4. Higher super lines

We introduce names for the objects assigned to closed manifolds of arbitrary codimension in an invertible field theory. In codimension 0 we have a complex number and in codimension 1 a complex $\mathbb{Z}/2\mathbb{Z}$-graded line, or in common parlance a complex super line. Hence in codimension k we introduce the term 'complex super k-line'.[70] The following definition relies on Proposition 8.18, §8.3, and §7.1.

Definition 8.29.

(1) $I\mathbb{Z}(1)$ is the spectrum of *higher complex super lines*;
(2) $\left(I\mathbb{Z}(1)^\gamma\right)^{h\mathbb{Z}/2}$ is the spectrum of *higher real super lines*;
(3) $I\mathbb{Z}(1)_H := (I\mathbb{Z}(1)^\gamma \wedge S^{\sigma-1})^{h\mathbb{Z}/2}$ is the spectrum of *higher Hermitian super lines*;
(4) $I\mathbb{C}^\times$ is the spectrum of *higher flat complex super lines*;
(5) The k^{th} space in the spectrum $I\mathbb{Z}(1)$ is the space of *complex super k-lines.*

In this nomenclature a complex central simple superalgebra is a complex super 2-line. The discussion in §7.1 is the motivation for (3). There are analogs of (4) and (5) for real and Hermitian super lines. For example, the fixed point spectrum

$$I\mathbb{C}^\times_H := ((I\mathbb{C}^\times)^{\nu_0'} \wedge S^{\sigma-1})^{h\mathbb{Z}/2} \tag{8.30}$$

[70] Kapranov [**Kap**, §3.4] suggests a higher use of 'super' based on the sphere spectrum.

is the spectrum of *higher flat Hermitian super lines*, and the k^{th} space of that spectrum is the space of *flat Hermitian super k-lines.* As for the fixed point spectrum in (3), since $S^{1-\sigma} \wedge S^{\sigma-1}$ is the sphere spectrum with the trivial $\mathbb{Z}/2$-action—the "bar star" involution—we deduce from (8.14) a canonical identification

$$I\mathbb{Z}(1)_H = \operatorname{Map}\bigl(B\mathbb{Z}/2_+, I\mathbb{Z}(1)\bigr). \tag{8.31}$$

Pulling back along $B\mathbb{Z}/2 \to \mathrm{pt}$ we obtain a map

$$I\mathbb{Z}(1) \longrightarrow I\mathbb{Z}(1)_H; \tag{8.32}$$

the image is a summand, split by a choice of point in $B\mathbb{Z}/2$. Again using §7.1 as motivation, we posit the following.

DEFINITION 8.33. The image $I\mathbb{Z}(1)_{\text{pos}}$ of (8.32) is the *spectrum of higher positive definite Hermitian super lines.*

The k^{th} space in $I\mathbb{Z}(1)_{\text{pos}}$ is the space of *positive definite Hermitian super k-lines.* Define the spectrum of *higher* flat *positive definite Hermitian super lines* as the homotopy pullback

$$\begin{tikzcd} I\mathbb{C}^\times_{\text{pos}} \arrow[r] \arrow[d] & \Sigma I\mathbb{Z}(1)_{\text{pos}} \arrow[d] \\ I\mathbb{C}^\times_H \arrow[r] & \Sigma I\mathbb{Z}(1)_H. \end{tikzcd} \tag{8.34}$$

We illustrate this homotopy-theoretic definition of positivity by focusing on the top piece, first in the ungraded case and then in the $\mathbb{Z}/2\mathbb{Z}$-graded case.

EXAMPLE 8.35 (Hermitian lines). Consider the spectrum $\Sigma^2 H\mathbb{Z}$. Its zero-space $\Omega^\infty\Sigma^2 H\mathbb{Z}$ is the geometric realization of the topological groupoid of complex lines; morphisms are invertible linear maps, and morphisms sets have the continuous topology. There is a contractible space of trivializable involutions, and bar star is a point in that space. The analog of (8.31) for $H\mathbb{Z}$ replacing $I\mathbb{Z}(1)$ implies that the set of components of the fixed point spectrum of any such involution is

$$\pi_0 \operatorname{Map}(B\mathbb{Z}/2_+, \Sigma^2 H\mathbb{Z}) = \pi_0\Sigma^2 H\mathbb{Z} \oplus \pi_0 \operatorname{Map}(B\mathbb{Z}/2, \Sigma^2 H\mathbb{Z}) = \{0\} \oplus \mathbb{Z}/2. \tag{8.36}$$

The zero-space of $\operatorname{Map}(B\mathbb{Z}/2_+, \Sigma^2 H\mathbb{Z})$ represents the groupoid of Hermitian lines; the $\mathbb{Z}/2\mathbb{Z}$ tracks the sign of the Hermitian form. The positive subspace, obtained by pulling back along $B\mathbb{Z}/2 \to \mathrm{pt}$, picks out the positive definite forms.

EXAMPLE 8.37 (Super Hermitian lines). The zero-space of the spectrum $\Sigma^2 I\mathbb{Z}(1)$ represents the groupoid of super lines L with continuous topology on morphisms. We compute the set of components of the fixed point spectrum of a trivializable involution:

$$\pi_0 \operatorname{Map}(B\mathbb{Z}/2_+, \Sigma^2 I\mathbb{Z}(1)) = \pi_0\Sigma^2 I\mathbb{Z}(1) \oplus \pi_0 \operatorname{Map}(B\mathbb{Z}/2, \Sigma^2 I\mathbb{Z}(1)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2. \tag{8.38}$$

This is the group of isomorphism classes of super Hermitian lines. The first $\mathbb{Z}/2\mathbb{Z}$ is the grading of the line, the second the "sign" of the form. But the sesquilinearity condition $\langle \bar\ell_1, \ell_2\rangle = (-1)^{|\ell_1||\ell_2|}\overline{\langle \bar\ell_2, \ell_1\rangle}$ implies that if L is odd then $\langle \ell, \ell\rangle \in \sqrt{-1}\mathbb{R}$ for all $\ell \in L$. The notion of positivity in this case chooses a ray in $\sqrt{-1}\mathbb{R}$; there is no canonical choice. In the literature, e.g. [**DM**, (4.4.2)], an arbitrary choice is made. In our homotopy theoretic presentation, this choice lies in the identification of the space of super Hermitian lines with the 0-space of $\Sigma^2 I\mathbb{Z}(1)$. As we descend deeper into extended field theories, there are further choices to be made.

8.5. Spaces of invertible field theories; extended positivity

First, we implement involution (7.25) on bordism spectra. Fix a symmetry type (H_n, ρ_n). Recall the group co-extension (7.14)

$$1 \longrightarrow H_n \longrightarrow \widehat{H}_n \longrightarrow \{\pm 1\} \longrightarrow 1. \tag{8.39}$$

Passing to classifying spaces we obtain the fibration

$$BH_n \longrightarrow B\widehat{H}_n \longrightarrow B\mathbb{Z}/2\mathbb{Z}, \tag{8.40}$$

which induces a $\mathbb{Z}/2\mathbb{Z}$-action on BH_n. Then (7.15) induces a lift to the bundle $S_n \to BH_n$ and so to the Madsen-Tillmann spectrum MTH_n (see Definition 6.45). We refer to [**FH1**, §6.2.2] for more details.

Notation 8.41. Denote the spectrum MTH_n with this involution as MTH_n^β.

Remark 8.42. We warn the reader not to confuse this notation with (8.16), which contains examples of *natural* involutions [**FH1**, §6.3.1]. The involution on the Madsen-Tillmann spectrum is geometric and specific to MTH_n, not universally defined for all spectra. In fact, surely MTH_n^β is the geometric realization of the involution on $\mathrm{Bord}_n(H_n)$ discussed at the beginning of §7.6—an equivariant version of Theorem 6.67—but we do not attempt to prove this.

Ansatz 6.89 defines the space of *discrete* invertible n-dimensional field theories with symmetry type (H_n, ρ_n) as the mapping space

$$\mathcal{I}_n^\delta(H_n) := \mathrm{Map}(\Sigma^n MTH_n\,,\, \Sigma^n I\mathbb{C}^\times), \tag{8.43}$$

and Ansatz 6.99 defines the space of *continuous* n-dimensional theories of the same type as

$$\mathcal{I}_n(H_n) := \mathrm{Map}(\Sigma^n MTH_n\,,\, \Sigma^{n+1} I\mathbb{Z}(1)). \tag{8.44}$$

Invertible theories with reflection structure form the corresponding equivariant mapping spaces:

$$\mathcal{I}_n^\delta(H_n)_{\text{reflection}} := \mathrm{Map}^{\mathbb{Z}/2\mathbb{Z}}\big(\Sigma^n MTH_n^\beta\,,\, (\Sigma^n I\mathbb{C}^\times)^{\nu_0'}\big) \tag{8.45}$$

$$\mathcal{I}_n(H_n)_{\text{reflection}} := \mathrm{Map}^{\mathbb{Z}/2\mathbb{Z}}\big(\Sigma^n MTH_n^\beta\,,\, \Sigma^{n+1} I\mathbb{Z}(1)^\gamma\big) \tag{8.46}$$

Now we implement positivity. First, positivity of an n-dimensional field theory does not involve n-manifolds (numerical partition functions), but rather only depends on the truncation to an $(n-1)$-dimensional theory whose value on an $(n-1)$-manifold is a line.[71] Recall (7.34) that the truncation of a theory with reflection structure assigns a *Hermitian* line to a closed $(n-1)$-manifold, so to keep track of the Hermitian lines we use a truncation map

$$\mathcal{I}_n(H_n)_{\text{reflection}} \longrightarrow \mathcal{I}_n(H_{n-1})_{\text{Hermitian}} \tag{8.47}$$

into the space

$$\mathcal{I}_n(H_{n-1})_{\text{Hermitian}} := \mathrm{Map}\big(\Sigma^{n-1} MTH_{n-1}, \Sigma^{n+1} I\mathbb{Z}(1)_H\big) \tag{8.48}$$

[71] The collection of truncations of continuous theories with reflection structure form the equivariant mapping space $\mathrm{Map}^{\mathbb{Z}/2\mathbb{Z}}(\Sigma^{n-1} MTH_{n-1}^\beta\,,\, \Sigma^{n+1} I\mathbb{Z}(1)^\gamma)$.

of Hermitian $(n-1)$-dimensional theories, which assign a Hermitian super line to a closed $(n-1)$-manifold. (Recall Definition 8.29(3) of $I\mathbb{Z}(1)_H$.) To construct (8.47) we transport to homotopy theory the maneuvers which lead to (7.34), particularly Proposition 7.27.

The relevant move is "splitting off a reflection" [**FH1**, §8.1.1]. The splitting of interest is contained in (7.16) and exists whenever there is an "auxiliary" direction. The middle vertical homomorphism in (7.16) induces

$$BH_{n-1}\times B\mathbb{Z}/2\mathbb{Z}\longrightarrow B\hat{H}_n, \tag{8.49}$$

which factors the projection

$$BH_{n-1}\times B\mathbb{Z}/2\mathbb{Z}\longrightarrow B\hat{H}_n\longrightarrow B\mathbb{Z}/2\mathbb{Z}. \tag{8.50}$$

This, in turn, gives a sequence of equivariant maps

$$\Sigma^{n-1}MTH_{n-1}\wedge S^{1-\sigma}\longrightarrow \Sigma^n MTH_n^{\beta}\longrightarrow MTH\wedge S^{1-\sigma} \tag{8.51}$$

factoring the smash product of the identity map of $S^{1-\sigma}$ with the defining inclusion of $\Sigma^{n-1}MTH_{n-1}$ into MTH. Pullback equivariant maps into $\Sigma^{n+1}I\mathbb{Z}(1)^{\gamma}$ along the first map in (8.51) to construct (8.47).

Now we are in position to define extended positivity for invertible theories. Recall Definition 8.33 of the spectrum $I\mathbb{Z}(1)_{\mathrm{pos}}$ of higher positive definite Hermitian super lines. The space

$$\mathcal{I}_n(H_{n-1})_{\mathrm{positive}} := \mathrm{Map}(\Sigma^{n-1}MTH_{n-1}\,,\,\Sigma^{n+1}I\mathbb{Z}(1)_{\mathrm{pos}}) \tag{8.52}$$

of positive truncated theories maps into $\mathcal{I}_n(H_{n-1})_{\mathrm{Hermitian}}$. We define extended reflection positivity as a reflection structure which "is" positive, first for the entire space of theories; the result is expressed as a fiber product of topological spaces.

Definition 8.53. Fix $n>0$ and a symmetry type (H_n,ρ_n). Define the space $\mathcal{I}_n(H_n)_{\substack{\mathrm{reflection}\\ \mathrm{positive}}}$ of *continuous invertible n-dimensional reflection positive topological field theories with symmetry type* (H_n,ρ_n) as the homotopy pullback

$$\begin{tikzcd} \mathcal{I}_n(H_n)_{\substack{\mathrm{reflection}\\ \mathrm{positive}}} \arrow[r] \arrow[d] & \mathcal{I}_n(H_{n-1})_{\mathrm{positive}} \arrow[d] \\ \mathcal{I}_n(H_n)_{\mathrm{reflection}} \arrow[r, "(8.47)"] & \mathcal{I}_n(H_{n-1})_{\mathrm{Hermitian}} \end{tikzcd} \tag{8.54}$$

The corresponding space of discrete theories is also defined as a pullback, bootstrapping from the continuous case.

Definition 8.55. Fix $n>0$ and a symmetry type (H_n,ρ_n). Define the space $\mathcal{I}^{\delta}_n(H_n)_{\substack{\mathrm{reflection}\\ \mathrm{positive}}}$ of *discrete invertible n-dimensional reflection positive topological field theories with symmetry type* (H_n,ρ_n) as the homotopy pullback

$$\begin{tikzcd} \mathcal{I}^{\delta}_n(H_n)_{\substack{\mathrm{reflection}\\ \mathrm{positive}}} \arrow[r] \arrow[d] & \mathcal{I}_n(H_n)_{\substack{\mathrm{reflection}\\ \mathrm{positive}}} \arrow[d] \\ \mathcal{I}^{\delta}_n(H_n)_{\mathrm{reflection}} \arrow[r] & \mathcal{I}_n(H_n)_{\mathrm{reflection}} \end{tikzcd} \tag{8.56}$$

The theorems in the next section identify the homotopy types of $\mathcal{I}_n(H_n)_{\substack{\text{reflection}\\\text{positive}}}$ and $\mathcal{I}^\delta_n(H_n)_{\substack{\text{reflection}\\\text{positive}}}$. We conclude this section by re-expressing extended positivity as an additional structure on an n-dimensional invertible theory which trivializes an associated invertible $(n-1)$-dimensional field theory. For this we need another space of invertible field theories, based on the target spectrum of higher *real* super lines (Definition 8.29(2)).

DEFINITION 8.57. The space of *continuous invertible* $(n-1)$*-dimensional* real *extended topological field theories with symmetry type* (H_{n-1}, ρ_{n-1}) is

$$\mathcal{I}^{\mathbb{R}}_{n-1}(H_{n-1}) = \operatorname{Map}\bigl(\Sigma^{n-1}MTH_{n-1}, (\Sigma^n I\mathbb{Z}(1)^\gamma)^{h\mathbb{Z}/2}\bigr). \tag{8.58}$$

Recall from Remark 6.91 that the top three truncations of $\Sigma^{n+1}I\mathbb{Z}(1)$ are spectra which represent the abelian group $\mathbb{C}^\times$, the Picard groupoid $\mathrm{Line}_{\mathbb{C}}$, and the Picard 2-groupoid $\mathrm{Alg}_{\mathbb{C}}$, respectively. (In each case $\mathbb{C}^\times$ has the continuous topology.) The homotopy fixed points under complex conjugation represent real versions of these groups and groupoids; see Remark 8.28. Hence for a field theory in $\mathcal{I}^{\mathbb{R}}_{n-1}(H_{n-1})$, the partition function of a closed $(n-1)$-manifold lies in $\{\pm 1\}$, the value on a closed $(n-2)$-manifold is a real super line, etc.

To begin, for any pointed space X there is an equivalence of spectra $X_+ \approx X \vee S^0$, which leads to a cofibration sequence

$$X \longrightarrow X_+ \longrightarrow S^0. \tag{8.59}$$

Set $X = B\mathbb{Z}/2$, smash with $\Sigma^{n-1}MTH_{n-1}$, and apply $\operatorname{Map}\bigl(-, \Sigma^{n+1}I\mathbb{Z}(1)\bigr)$ to obtain the fibration sequence (see [**FH1**, §8.1.3] for details)

$$\mathcal{I}_n(H_{n-1})_{\text{positive}} \longrightarrow \mathcal{I}_n(H_{n-1})_{\text{Hermitian}} \longrightarrow \mathcal{I}^{\mathbb{R}}_{n-1}(H_{n-1}). \tag{8.60}$$

Therefore, the space $\mathcal{I}_n(H_n)_{\substack{\text{reflection}\\\text{positive}}}$ may also be defined as the homotopy fiber of the composition

$$\kappa\colon \mathcal{I}_n(H_n)_{\text{reflection}} \longrightarrow \mathcal{I}_n(H_{n-1})_{\text{Hermitian}} \longrightarrow \mathcal{I}^{\mathbb{R}}_{n-1}(H_{n-1}). \tag{8.61}$$

This leads to the following definition.

DEFINITION 8.62. An (*extended*) *positivity structure* on a continuous n-dimensional field theory $\varphi \in \mathcal{I}_n(H_n)_{\text{reflection}}$ is a trivialization of $\kappa(\varphi)$.

That is, a positivity structure is a path from $\kappa(\varphi)$ to the basepoint in $\mathcal{I}^{\mathbb{R}}_{n-1}(H_{n-1})$. This discussion identifies the space of continuous reflection positive invertible field theories as the space of continuous invertible field theories with both a reflection structure and a positivity structure.

REMARK 8.63. The partition function of the field theory

$$\kappa(\varphi)\colon \Sigma^{n-1}MTH_{n-1} \to \Sigma^n(I\mathbb{Z}(1)^\gamma)^{h\mathbb{Z}/2}$$

is the homomorphism

$$\pi_{n-1}\Sigma^{n-1}MTH_{n-1} \longrightarrow \{\pm 1\} \tag{8.64}$$

induced on π_{n-1} which tracks the sign of the Hermitian lines in the theory φ. The highest piece of the positivity structure is therefore the standard positivity *constraint* in Definition 7.37. The theory $\kappa(\varphi)$ assigns a real super line to a closed $(n-2)$-manifold and more complicated objects in lower dimensions; their trivializations are *data*.

8.6. Main theorems

We determine the homotopy types of the spaces of reflection positive invertible theories, first in the continuous case and then in the discrete case. Proofs of all theorems are in [**FH1**, §8].

Recall that a symmetry type (H_n, ρ_n) stabilizes: there is a homomorphism $\rho\colon H \to \mathrm{O}$ and the symmetry type can be identified as (H, ρ). Define the space of *stable* continuous invertible n-dimensional topological field theories of symmetry type (H, ρ) as the mapping space

$$\mathcal{I}_n(H)_{\mathrm{stable}} = \mathrm{Map}(MTH, \Sigma^{n+1} I\mathbb{Z}(1)). \tag{8.65}$$

The main theorem identifies its underlying homotopy type with that of reflection positive theories. We construct a map

$$\mathcal{I}_n(H)_{\mathrm{stable}} \longrightarrow \mathcal{I}_n(H_n)_{\substack{\text{reflection}\\ \text{positive}}} \tag{8.66}$$

using the "splitting off a reflection" maneuver (8.51) as follows. Map the composition

$$\Sigma^{n-1} MTH_{n-1} \wedge B\mathbb{Z}/2_+ \longrightarrow \Sigma^{n-1} MTH_{n-1} \longrightarrow MTH \tag{8.67}$$

into $\Sigma^{n+1} I\mathbb{Z}(1)$ to obtain a map of $\mathcal{I}_n(H)_{\mathrm{stable}}$ into the northeast corner of (8.54). Use equivariant maps of the sequence (8.51) into $\Sigma^{n+1} I\mathbb{Z}(1)^\gamma$ to map $\mathcal{I}_n(H)_{\mathrm{stable}}$ into the southwest corner of of (8.54). The two compositions into the southeast corner are canonically homotopic, so the fact that the right square in (8.54) is a homotopy pullback yields a map (8.66).

THEOREM 8.68. *The map $\mathcal{I}_n(H)_{\mathrm{stable}} \longrightarrow \mathcal{I}_n(H_n)_{\substack{\text{reflection}\\ \text{positive}}}$ in (8.66) is a homotopy equivalence.*

This theorem is effective in that the space of (nonequivariant) maps (8.65) is computable, as we illustrate in some examples in Lecture 10. Those computations only use the information about path components, which we single out next.

COROLLARY 8.69. *There is an isomorphism*

$$\pi_0\, \mathcal{I}_n(H_n)_{\substack{\text{reflection}\\ \text{positive}}} \cong [MTH, \Sigma^{n+1} I\mathbb{Z}(1)]. \tag{8.70}$$

In the language of field theory, Corollary 8.69 is a 1:1 correspondence

$$\left\{\begin{array}{l}\text{isomorphism classes of continuous invertible}\\ n\text{-dimensional reflection positive extended topological}\\ \text{field theories with symmetry type } (H_n,\rho_n)\end{array}\right\} \cong [MTH, \Sigma^{n+1} I\mathbb{Z}(1)]. \tag{8.71}$$

REMARK 8.72. Since the rational cohomology of BH vanishes in odd degrees, elements of infinite order in (8.70) occur only for n odd.

Recall that the notion of a "continuous" theory (§6.10) is special to the invertible case. For the application to physics, we are interested in *deformation classes* (§6.9) of possibly non-topological invertible theories. We take up non-topological invertible theories in Lecture 9, where we motivate a closely related interpretation of the right hand side of (8.71), formulated as Conjecture 9.34.

Next, we turn to discrete invertible theories. The computation of π_0 is easily stated.

THEOREM 8.73. *The image of the homomorphism*

$$\pi_0\,\mathcal{I}_n^\delta(H_n)_{\substack{\text{reflection}\\ \text{positive}}} \longrightarrow \pi_0\,\mathcal{I}_n(H_n)_{\substack{\text{reflection}\\ \text{positive}}} \tag{8.74}$$

is the torsion subgroup of $\pi_0\,\mathcal{I}_n(H_n)_{\substack{\text{reflection}\\ \text{positive}}}$.

Combining Theorem 8.73 and Corollary 8.69 we obtain the following field theoretic classification.

THEOREM 8.75. *There is a 1:1 correspondence*

$$\left\{\begin{array}{l}\textit{deformation classes of invertible } n\textit{-dimensional}\\ \textit{reflection positive extended topological}\\ \textit{field theories with symmetry type } (H_n,\rho_n)\end{array}\right\} \cong [MTH, \Sigma^{n+1}I\mathbb{Z}(1)]_{\mathrm{tor}}. \tag{8.76}$$

'Tor' denotes the torsion subgroup of the abelian group of spectrum maps. See (8.71) and Conjecture 9.34 for field theoretic interpretations of the entire group.

Finally, we determine the entire homotopy type of $\mathcal{I}_n^\delta(H_n)_{\substack{\text{reflection}\\ \text{positive}}}$. Note that for any n-manifold X the disjoint union $\beta X \amalg X$ is null bordant, and so in a stable theory the partition functions have unit norm, consistent with the appearance of $I\mathbb{T}$ in the following theorem.

THEOREM 8.77. *For n odd there is a homotopy equivalence*

$$\mathrm{Map}(MTH, \Sigma^n I\mathbb{T}) \xrightarrow{\approx} \mathcal{I}_n^\delta(H_n)_{\substack{\text{reflection}\\ \text{positive}}} \tag{8.78}$$

For n even there is a fibration sequence

$$\mathrm{Map}(MTH, \Sigma^n I\mathbb{T}) \longrightarrow \mathcal{I}_n^\delta(H_n)_{\substack{\text{reflection}\\ \text{positive}}} \xrightarrow{s} \mathbb{R}^{>0} \tag{8.79}$$

in which $\mathbb{R}^{>0}$ has the discrete topology and s maps a discrete theory F to $F(S^n)$.

There is a canonical section of s given by Euler theories (Example 1.52): given $x \in \mathbb{R}^{>0}$ define the Euler theory as the composition

$$\Sigma^n MTH_n^\beta \longrightarrow \Sigma^n(BH_n^\beta)_+ \longrightarrow \Sigma^n S^0 \xrightarrow{\sqrt{x}} \Sigma^n H\mathbb{R}^{>0} \longrightarrow \Sigma^n(I\mathbb{C}^\times)^\gamma \tag{8.80}$$

The restriction to $\Sigma^{n-1}MTH_{n-1}^\beta$ is trivialized; using (8.56) we obtain a reflection positive theory.

EXERCISE 8.81. Investigate extended reflection positivity for the invertible theories in §6.2.

LECTURE 9

Non-Topological Invertible Field Theories

Now that we have developed the field theory ideas we need, particularly extended locality and extended unitarity—the latter in the invertible case—we return to Problem 2.33. We consider, at least heuristically, a moduli space of invertible quantum mechanical systems of fixed dimension and symmetry type. Since this moduli space is not a well-defined mathematical object yet, we transform Problem 2.33 to a classification problem in field theory. Recall the basic dichotomy (Remark 2.15) of quantum systems: gapped vs. gapless. The traditional view, before the advent of topological field theory, was that the long range limit of a gapped theory is a trivial field theory. Now it is understood that topological effects survive in this limit, and they determine the deformation class of the theory; see [**W4**, p. 405] for an early articulation. Moreover, these topological effects have strong ramifications for and constraints on the long range effective field theory. However, the expectation that the long range approximation of a gapped theory is a *topological* field theory is not met in the case of 3-dimensional Yang-Mills theory with a Chern-Simons term, as we explain in §9.2. What we meet instead is a theory which locally factors as the tensor product of a purely topological theory with an invertible theory. In many cases the non-topological invertible theory is trivial, but in many important examples it is not. We give more perspective on the appearance of an invertible theory in Lecture 11. In §9.1 we call these possible long range limits 'topological* theories' to emphasize their topological nature. If the entire long range theory is invertible, then the 'topological' in 'topological*' does not have any force and all we can say is that the long range theory is invertible. We give several examples of non-topological invertible theories in §9.3, where we relate them to secondary invariants in differential geometry. The modern point of view on secondary invariants is via differential cohomology, as we describe in §9.4. This discussion forms the basis of Conjecture 9.34, which extends Theorem 8.75.

Some of this lecture is speculative, since we have not provided mathematical foundations. The material in §9.1 and §9.2 about the nature of long range approximations to gapped theories is motivational; the parts about non-topological invertible field theories, including Conjecture 9.34, are mathematically within reach given the existing literature on differential cohomology theories.

9.1. Short-range entangled lattice systems; topological* field theories

We consider lattice systems, such as the toric code (§2.3), but add a crucial hypothesis—sometimes called "short range entanglement"—which is *not* satisfied by the toric code. First, we remind that the Hilbert space of states of a lattice system is presented as a tensor product (2.21) of finite dimensional vector spaces. This displays locality in space explicitly. Some notions, such as the triviality we define shortly, are defined in terms of this tensor product structure. There are a few different meanings for the term "short range entanglement" [**CGW**, **Ki2**] in the condensed matter community. We prefer the term "invertible", which is now in wide use, since the short range entangled systems we consider are invertible under the composition law (Remark 2.14) on quantum mechanical systems, provided that we introduce an equivalence relation on lattice systems. Often the ground state, rather than the Hamiltonian, is said to determine a lattice system, and it is in these terms that we indicate the equivalence relation which defines "triviality": a ground state is trivial if it can be transformed to a decomposable vector in the tensor product via operators (sometimes called "quantum gates") which are local in space and in time. The composition law in this context is sometimes called "stacking".

Fix a dimension d of space and an internal symmetry group I, which we assume to be a compact Lie group. (We see below that we need to remember a bit more data attached to I.) As in §2.4 we throw out the locus of phase transitions and of gapless theories, and so we imagine a moduli space $\mathcal{M}(d, I)$ of gapped invertible lattice systems of dimension d with internal symmetry group I. It is not a mathematically well-defined object. Nonetheless, Problem 2.33 asks to compute its set of path components, which is in fact an abelian group. As explained earlier, two physical principles transport us to a field theory problem:

(1) the deformation class of a quantum system is determined by its low energy behavior;
(2) the low energy physics of a *gapped* system is well-approximated by a topological* field theory.

We explain 'topological*' shortly. Since the lattice systems we envision are (i) invertible, (ii) local, and (iii) unitary, we expect the long range effective field theory to be so as well. Let $\mathcal{M}'(n, H)$ denote the moduli space of invertible fully extended reflection positive field theories of spacetime dimension n and symmetry type (H, ρ). (The developments recounted in Lecture 8 give a rigorous mathematical framework for the subspace of topological theories as well as for continuous invertible theories.) The physical principles lead us to expect a map

$$\mathcal{M}(d, I) \longrightarrow \mathcal{M}'(n, H) \tag{9.1}$$

once we match the discrete parameters (n, H) and (d, I). We first observe the following.

- Although we wrote a map (9.1) our expectation is that only its homotopy class is well-defined, once the moduli space $\mathcal{M}(d, I)$ is constructed: there are choices in constructing a long range effective field theory. The homotopy class of (9.1) determines a map

$$\pi_0\mathcal{M}(d, I) \longrightarrow \pi_0\mathcal{M}'(n, H) \tag{9.2}$$

  on phases, and that is what we need. Should topological questions about phases arise touching on the topology of the moduli spaces beyond π_0, a homotopy class of maps (9.1) will also be sufficient to address them.
- The physical principles above lead to the expectation that (9.2) is injective. Surjectivity is the statement that any deformation class of invertible theories can be realized as the long range effective theory of a lattice system. There are theoretical results along these lines, for example [**WW**, §4], and also the agreement of homotopy theory computations

with results in the condensed matter literature do strongly suggest that (9.2) is an isomorphism.

Next, we match parameters. The dimensions are easy to match: $n = d + 1$. Next, we would like to determine a symmetry type (H, ρ) from the internal symmetry group I, but the natural map is in the other direction. Recall that $\rho\colon H \to \mathrm{O}$ induces a homomorphism $\rho\colon H_n \to \mathrm{O}_n$ for each n, and the kernel K is the internal *spacetime* symmetry group, which is independent of n. The group I is the internal *space* symmetry group. To recover I from (H_n, ρ_n), fix a splitting

$$\mathbb{R}^{1,n-1} = \mathbb{R}^1 \oplus \mathbb{R}^{n-1} \tag{9.3}$$

of spacetime translations into the direct sum of time translations and space translations. The subgroup $\mathrm{O}_1 \times \mathrm{O}_{n-1} \subset \mathrm{O}_{1,n-1}$ preserves that splitting, and the subgroup $\mathrm{O}_1 \times \{\mathrm{id}\} \subset \mathrm{O}_1 \times \mathrm{O}_{n-1}$ is the subgroup which acts trivially on space.[72] So for the symmetry type (H_n, ρ_n), define the nonrelativistic internal subgroup I_n as the pullback

$$\begin{tikzcd} I_n \arrow[rr, hook] \arrow[d, "\phi"'] & & H_n \arrow[d, "\rho_n"] \\ \mathrm{O}_1 \times \{\mathrm{id}\} \arrow[r, hook] & \mathrm{O}_1 \times \mathrm{O}_{n-1} \arrow[r, hook] & \mathrm{O}_n \end{tikzcd} \tag{9.4}$$

The inclusion $H_n \hookrightarrow H_{n+1}$ induces an isomorphism $I_n \xrightarrow{\cong} I_{n+1}$. The colimit of the resulting directed system of groups I_n as $n \to \infty$ is the internal space symmetry group I. As an example, if $(H_n, \rho_n) = (\mathrm{O}_n, \mathrm{id})$ then K is trivial whereas I is cyclic of order two: a reflection in time is internal from a space perspective, but not from a spacetime perspective.

The nonrelativistic internal symmetry group I_n defined in (9.4) comes with two additional pieces of data. First, there is the homomorphism $\phi\colon I_n \to \mathrm{O}_1 = \{\pm 1\}$, a $\mathbb{Z}/2\mathbb{Z}$-grading on I_n which tracks whether a symmetry reverses time. Also, $K = \ker \phi$ contains the distinguished central element k_0 which acts as the grading operator "$(-1)^F$" that distinguishes bosonic and fermionic states; see Proposition 3.16(2). In condensed matter models the triple (I_n, k_0, ϕ) is given and, to match parameters, we must construct the symmetry type (H_n, ρ_n) of the long range effective theory. The construction is explained in [**FH1**, Remark 9.36]. The reader

[72]The splitting (9.3) induces transverse affine foliations of Minkowski spacetime $\mathbb{M}^n$, which may be called a *nonrelativistic structure*. The subgroup of isometries which preserve (9.3) acts on the quotient affine space $\mathbb{M}^n/\mathbb{R}^1$, which is space. The vector groups in the text act on the vector space $\mathbb{R}^{1,n-1}/\mathbb{R}^1$ of spatial translations of $\mathbb{M}^n/\mathbb{R}^1$.

may enjoy working it out from the commutative diagram

$$
\begin{tikzcd}
\mathrm{Spin}_1 \arrow[rr] \arrow[dr] & & \mathrm{Spin}_n \arrow[dr] & & \\
& \widetilde{I}_n \arrow[rr] \arrow[dl] \arrow[dd] & & \widetilde{H}_n \arrow[dl] \arrow[dd] \arrow[dr] & \\
I_n \arrow[rr] \arrow[dd] & & H_n \arrow[dd] & & J \arrow[dd] \\
& \mathrm{Pin}_1^+ \arrow[rr] \arrow[dl] & & \mathrm{Pin}_n^+ \arrow[dr] & \\
\mathrm{O}_1 \arrow[rr] & & \mathrm{O}_n \arrow[rr] & & \{\pm 1\}
\end{tikzcd} \tag{9.5}
$$

in which every parallelogram is a pullback, the kernel of every vertical map is K, and the northeast diagonal composition is exact. Most of these groups appear in §3.4.

Finally, and crucially, it remains to explain the meaning of topological* in physical principle (2) above. The reason one expects a *topological* field theory at long range in a gapped system is that the gap cleanly separates higher oscillating modes from the ground states, and so at long range one only has the ground states. At first glance it is reasonable to postulate that the field theory which describes the ground states is topological. But that turns out not to be quite true. There may be a non-topological invertible field theory as well, in the sense that the long range theory is “locally” the tensor product of a topological theory and an invertible theory. In other terms, the energy-momentum tensor and currents may depend on continuous background fields—metrics and connections—but only mildly: as operators they must be multiples of the identity. (See the discussions in [**GK**, §1.1] and in §11.4 of these lectures.) In the invertible case this implies that we simply cross out ‘topological*’: the low energy theory should be invertible but not necessarily topological since

$$\text{invertible topological}^* = \text{invertible}. \tag{9.6}$$

We illustrate in §9.2 with an example from field theory.

In the invertible case there is an alternative heuristic for the low energy effective field theory. Recall that in §6.10 we introduced the notion of a *continuous* invertible n-dimensional topological field theory φ. It assigns a $\mathbb{Z}(1)$-torsor $\varphi(X)$ to a closed n-manifold X, whereas a usual *discrete* invertible theory F assigns a nonzero complex number $F(X)$. We observed that elements of $\mathbb{C}^\times$ give rise to a $\mathbb{Z}(1)$-torsor via the exponential sequence (6.71), so given a discrete theory F we can define $\varphi_F(X)$ from $F(X)$. In fact, the entire discrete invertible theory F gives rise to a continuous invertible theory φ_F which encodes its deformation class. On the other hand, depending on dimension and symmetry type, there may be continuous invertible theories which do not arise from discrete invertible theories.[73] One can imagine that—in this invertible case—the indeterminacy in defining a low energy

[73] This can only happen if n is odd; see Remark 8.72.

approximation leads to a well-defined isomorphism class of continuous invertible theories. Corollary 8.69, or more accurately (8.71), gives an explicit formula for the abelian group of isomorphism classes of reflection positive continuous invertible theories of fixed dimension and symmetry type. This is the group we compute as the group of invertible gapped phases.

9.2. The long range limit of 3-dimensional Yang-Mills + Chern-Simons

Recall (Example 4.38) that in $n = 2$ dimensions the Yang-Mills action only depends on the area form, but not on the full Riemannian metric. In $n = 3$ dimensions the Yang-Mills action depends on the entire metric, and so the quantum theory does not simplify as it does in $n = 2$ dimensions. It is believed that pure Yang-Mills is a gapped theory in $n = 3$ dimensions and, I believe, that the long range effective theory is trivial. There is a special feature unique to $n = 3$ dimensions, namely the possibility of an explicit mass term for the connection (gauge field). In case the gauge group is the abelian group $G = \mathbb{T}$, the theory on Minkowski spacetime $\mathbb{M}^3$ is *free*—the 3-dimensional version of Maxwell electromagnetism—and we can compute everything explicitly. The mass term is the Chern-Simons [**CS**] term; see [**W5**, Problem FP4]. For a general compact Lie group G there is a Chern-Simons term which depends on a *level* $\lambda \in H^4(BG;\mathbb{Z}(1))$. It too functions as a mass if λ is nondegenerate. So the naive expectation is that the long range behavior of Yang-Mills + Chern-Simons is governed by a topological field theory, and naturally we expect precisely the quantum Chern-Simons theory $F_{G,\lambda}$. The quantum Chern-Simons theory was investigated by Witten [**W6**] in the late 1980s and has been the catalyst for many developments in topological field theory; see [**ABHH**] and the references therein. The expectation that pure Chern-Simons theory describes the long range limit has been formulated by Witten [**W7**], and it is perhaps one of the more mathematically interesting, yet accessible, problems about long range limits. (The corresponding problem in dimension $n = 4$—no Chern-Simons term—is worth one million dollars [**JW**].)

Let us accept that Chern-Simons theory is the long range limit of Yang-Mills + Chern-Simons so that we can examine the formal structure. Yang-Mills + Chern-Simons, if rigorously defined in a Wick-rotated version, would be a homomorphism

$$F_{(G,\lambda)}(g)\colon \mathrm{Bord}_{\langle 2,3\rangle}(\mathrm{SO}_3^{\nabla}) \longrightarrow t\mathrm{Vect}_{\mathbb{C}} \tag{9.7}$$

with domain the bordism category of oriented Riemannian manifolds and codomain a symmetric monoidal category of topological vector spaces. Here $g \in \mathbb{R}$ is naively the coupling constant of the Yang-Mills action, but the 1-parameter family of quantum theories has a parameter with a different description. Barring accidental symmetry breaking or symmetry enhancement, we expect the long range approximation

$$F_{(G,\lambda)}\colon \mathrm{Bord}_{\langle 2,3\rangle}(\mathrm{SO}_3^{\nabla}) \longrightarrow \mathrm{Vect}_{\mathbb{C}} \tag{9.8}$$

to be a theory with the same domain bordism category. Indeed, Witten [**W6**, §2] shows by explicit computation that the quantum Chern-Simons invariant does have a mild metric dependence. He then shows how to get a purely topological invariant by introducing a framing,[74] and it is this purely topological theory on the bordism category $\mathrm{Bord}_{\langle 2,3\rangle}(e)$ of 3-framed manifolds which has been the object of intense

[74]Not a full framing: a 2-framing [**A2**] or a p_1-structure [**BHMV**]. Be warned that the '2' in '2-framing' is different from the '3' in '3-framing'. In the text we use 3-framings, or parallelisms,

mathematical study. But the long range limit of Yang-Mills + Chern-Simons is the metric-dependent version (9.8) of pure Chern-Simons, which is not a topological theory.[75]

REMARK 9.9. Observe that a 3-framing violates relativistic invariance; recall that the Wick-rotated manifestation of relativistic invariance is the statement that the image of $\rho_3\colon H_3 \to \mathrm{O}_3$ contains SO_3. We certainly do not expect the long range approximation of a quantum field theory to break relativistic invariance. (Conversely, a basic premise of our application of field theory to condensed matter systems is that relativistic invariance emerges in the long range approximation.)

The "mildness" of the metric dependence means that the dependence is invertible in the following sense. Let $\mathrm{Bord}_{\langle 2,3\rangle}(e^{\nabla})$ denote the bordism category of 3-framed *Riemannian* manifolds. There is a factorization of the lift of (9.8) under $\mathrm{Bord}_{\langle 2,3\rangle}(e^{\nabla}) \to \mathrm{Bord}_{\langle 2,3\rangle}(\mathrm{SO}_3^{\nabla})$:

$$F_{(G,\lambda)} = T_{(G,\lambda)} \otimes \alpha_c : \mathrm{Bord}_{\langle 2,3\rangle}(e^{\nabla}) \longrightarrow \mathrm{Vect}_{\mathbb{C}}, \tag{9.10}$$

where $T_{(G,\lambda)}$ is the purely topological framing-dependent Chern-Simons theory—it does not depend on the Riemannian metric—and α_c is an invertible field theory which is not topological—it depends on the Riemannian metric and the 3-framing. (We describe α_c in Remark 9.29 below.) It is in this sense that the long range limit $F_{(G,\lambda)}$ factors. We call the factorization 'local' since it depends on an extra structure—a 3-framing compatible with the orientation—which exists and is unique[76] up to a noncanonical isomorphism on an oriented ball.

9.3. Examples of non-topological invertible theories

EXAMPLE 9.11 (Holonomy). In Example 2.46 we use holonomy as the partition function of an invertible theory

$$\alpha_k\colon \mathrm{Bord}_{\langle 0,1\rangle}(\mathrm{SO}_1 \times \mathbb{T}^{\nabla}) \longrightarrow \mathrm{Line}_{\mathbb{C}} \tag{9.12}$$

of oriented 0- and 1-manifolds equipped with a circle bundle with connection. Now we locate its deformation class in homotopy-theoretic terms. We expect its deformation class not to depend on the connection, so the domain is the Thom spectrum[77] $M\mathrm{SO} \wedge B\mathbb{T}_+$ of oriented manifolds with circle bundle. The codomain in general is $\Sigma^2 I\mathbb{Z}(1)$, as explained in §6.9. The Anderson dual spectrum $\mathrm{Map}(M\mathrm{SO}, \Sigma^2 I\mathbb{Z}(1))$ to $M\mathrm{SO}$ has homotopy groups which vanish in degree > 2, and $\pi_2 \cong \mathbb{Z}$, $\pi_1 = \pi_0 = 0$. Hence, since $B\mathbb{T}_+ \simeq \mathbb{CP}^\infty$,

$$[M\mathrm{SO} \wedge B\mathbb{T}_+, \Sigma^2 I\mathbb{Z}(1)] \cong [B\mathbb{T}_+, \mathrm{Map}(M\mathrm{SO}, \Sigma^2 I\mathbb{Z}(1))] \cong H^2(B\mathbb{T}; \mathbb{Z}(1)) \cong \mathbb{Z}(1), \tag{9.13}$$

and the theory α_k maps to $2\pi i k$ under the isomorphism (9.13).

---

for convenience; the theory factors through the bordism category of oriented manifolds with p_1-structure.

[75] In these lectures we have defined a theory to be topological if it factors through the bordism category with no differential structure—no metrics or connections.

[76] but is not a contractible choice

[77] The invertible theory α_k is unitary, so Theorem 8.68 implies that the domain of its continuous version is a Thom spectrum.

REMARK 9.14. These assertions about non-topological invertible theories have the status of conjectures. In [**FH1**] we did not develop the mathematical infrastructure to make them theorems.

EXAMPLE 9.15 (The associated continuous invertible theory). The non-topological invertible field theory α_k gives rise to a continuous invertible topological theory

$$\varphi_k\colon \operatorname{Bord}_{\langle 0,1\rangle}(\mathrm{SO}_1 \times \mathbb{T}) \longrightarrow \operatorname{Gerbe}_{\mathbb{Z}(1)}\,. \tag{9.16}$$

The domain is the bordism category of oriented 0- and 1-manifolds equipped with a principal $\mathbb{T}$-bundle (no connection). The codomain is the Picard 1-groupoid whose objects are $\mathbb{Z}(1)$-gerbes and whose morphisms are isomorphism classes of gerbe isomorphisms. A $\mathbb{Z}(1)$-gerbe is a groupoid on which the groupoid of $\mathbb{Z}(1)$-torsors acts simply transitively. Associated to a $\mathbb{T}$-torsor T is a $\mathbb{Z}(1)$-gerbe whose objects are $\mathbb{R}(1)$-torsors lifting T with respect to the real exponential sequence

$$0 \longrightarrow \mathbb{Z}(1) \longrightarrow \mathbb{R}(1) \xrightarrow{\exp} \mathbb{T} \longrightarrow 1 \tag{9.17}$$

Then $\varphi_k(P \to Y)$ is the $\mathbb{Z}(1)$-gerbe associated to $\alpha_k(P \to Y)$, where $P \to Y$ is a principal $\mathbb{T}$-bundle over an oriented 0-manifold. The isomorphism class of the theory φ_k is an element of the abelian group (9.13); see (8.71). Also, φ_k extends to a theory $\tilde{\varphi}_k$ with domain $\operatorname{Bord}_{\langle 0,1,2\rangle}(\mathrm{SO}_1 \times \mathbb{T})$ and codomain the Picard 2-groupoid of $\mathbb{Z}(1)$-gerbes. The integer-valued partition function is

$$\tilde{\varphi}_k(P \to X) = k\langle c_1(P), [X]\rangle \tag{9.18}$$

for X a closed oriented 2-manifold.

EXAMPLE 9.19 (Classical Chern-Simons). Holonomy, which is the partition function of the theories in Example 9.11, is a *secondary invariant* of curvature. It is an example of a secondary invariant in the theory of connections. In 1973 Chern-Simons [**CS**] introduced a secondary invariant to the primary Chern-Weil invariants; in the simplest case it reduces to holonomy. The primary integer-valued invariants are nonzero only in even dimensions; the secondary invariants can be nonzero in any dimension. In its general form [**F6**, Appendix] the Chern-Simons invariant associated to a compact Lie group G and a *level* $\lambda \in H^{n+1}(BG;\mathbb{Z}(1))$ is the partition function of an invertible field theory (which in general is non-topological, but is topological if λ is finite order)

$$\alpha_{(G,\lambda)}\colon \operatorname{Bord}_{\langle n-1,n\rangle}(\mathrm{SO}_n \times G^{\nabla}) \longrightarrow \operatorname{Line}_{\mathbb{C}}\,. \tag{9.20}$$

It can be fully extended to the domain $\operatorname{Bord}_n(\mathrm{SO}_n \times G^{\nabla})$; see [**F7**]. The deformation class of the fully extended theory is—conjecturally—the image of the level λ under:

$$\begin{aligned} H^{n+1}(BG;\mathbb{Z}(1)) &\cong [BG_+, \Sigma^{n+1}H\mathbb{Z}(1)] \\ &\xrightarrow{U} [M\mathrm{SO}\wedge BG_+, \Sigma^{n+1}H\mathbb{Z}(1)] \longrightarrow [M\mathrm{SO}\wedge BG_+, \Sigma^{n+1}I\mathbb{Z}(1)], \end{aligned} \tag{9.21}$$

where U is multiplication by the universal Thom class (of degree zero).

REMARK 9.22. Usually $\alpha_{(G,\lambda)}$ is called *classical Chern-Simons theory*. It is the input into the Feynman path integral which defines the *quantum Chern-Simons* theory (9.8).

REMARK 9.23. As in Example 9.15 there is an associated continuous n-dimensional invertible theory $\varphi_{G,\lambda}$. It extends to an $(n+1)$-dimensional theory $\tilde{\varphi}_{(G,\lambda)}$ whose $\mathbb{Z}(1)$-valued partition function is the Chern-Simons primary invariant evaluated on the fundamental class. Observe that if the level λ is nonzero and torsion in $H^{n+1}(BG;\mathbb{Z})$, then the theory $\tilde{\varphi}_{(G,\lambda)}$ is nontrivial but all of its partition functions vanish.[78]

EXAMPLE 9.24 (Exponentiated η-invariants). The Chern-Weil and Chern-Simons invariants are primary and secondary invariants for Eilenberg-MacLane integral cohomology. There are also primary and secondary invariants in generalized cohomology theories. For complex and real K-theory the primary invariants were introduced in the very first paper on K-theory by Atiyah-Hirzebruch [**AH**]; they have an analytic interpretation via the Atiyah-Singer index theorem [**AS2**] as Fredholm indices of elliptic operators. Atiyah-Patodi-Singer [**APS**] constructed the corresponding secondary invariants. These *exponentiated η-invariants* are the partition functions of invertible field theories. They exist in both odd and even dimensions; in even dimensions they are necessarily finite order (roots of unity) topological invariants. For example, the signature is a primary invariant of a closed oriented 4-manifold, and there is an associated invertible field theory

$$\alpha\colon \operatorname{Bord}_{\langle 2,3\rangle}(\mathrm{SO}_3^{\nabla}) \longrightarrow \operatorname{Line}_{\mathbb{C}} \tag{9.25}$$

which assigns to a closed oriented Riemannian 3-manifold X its exponentiated η-invariant $\alpha(X)$. The theory α is not topological; its deformation class has infinite order. The value of α on a closed oriented Riemannian 2-manifold is the determinant line of a Dirac-type operator. In [**DaFr**] we define the exponentiated η-invariant of a compact 3-manifold with boundary and prove the gluing law which shows that (9.25) is a symmetric monoidal functor. There is a more refined theory

$$\alpha'\colon \operatorname{Bord}_{\langle 2,3\rangle}(\mathrm{Spin}_3^{\nabla}) \longrightarrow s\operatorname{Line}_{\mathbb{C}} \tag{9.26}$$

whose partition function is the secondary invariant associated to $\frac{1}{2}\hat{A}$ of a closed spin 4-manifold. The value on a closed spin Riemannian 2-manifold is the super pfaffian line of the Dirac operator. See [**FW**, §3] and the references therein for a review of these geometric index theory invariants.

These theories can be fully extended. The deformation class of the extended α' is—conjecturally—the composition

$$M\mathrm{Spin} \xrightarrow{[\mathbf{ABS}]} KO \xrightarrow{\text{Pfaff}} \Sigma^4 I\mathbb{Z}(1) \tag{9.27}$$

of the Atiyah-Bott-Shapiro map from spin bordism to real K-theory and the pfaffian map which exhibits the Anderson self-duality of KO-theory [**FMS**, Proposition B.4], [**HeSt**]; see Conjecture 10.25 and Conjecture 11.23 for related assertions.

REMARK 9.28. These examples exemplify the need to evaluate field theories on *families* of smooth manifolds. For example, the determinant and pfaffian line bundles of families of Dirac operators on surfaces are related to classical ϑ- and Dedekind η-functions [**A3**, **F8**].

REMARK 9.29. The invertible theory α_c which occurs in the factorization (9.10) is the c^{th} power of the theory (9.25), after pullback to the bordism category

[78] That does not occur for $\mathbb{C}^\times$-valued theories (Remark 6.91): the group $\mathbb{C}^\times$ is divisible but $\mathbb{Z}(1)$ is not.

$\mathrm{Bord}_{\langle 2,3\rangle}(e^{\nabla})$ of 3-framed Riemannian manifolds. On a closed 3-framed Riemannian 3-manifold X the η-invariant has a canonical real lift $\eta_X \in \mathbb{R}$, and $\alpha_c(X) = \exp(2\pi i c\eta_X/24)$.

EXAMPLE 9.30 (Holonomy revisited). This is a variation of Example 9.11 and is closely related to the discussion in §6.10. Let M be a smooth manifold and $P \to M$ a principal $\mathbb{C}^\times$-bundle with connection Θ. There is an invertible field theory[79]

$$\alpha_{(P,\Theta)}\colon \mathrm{Bord}_{\langle 0,1\rangle}(\mathrm{SO}_1)[M] \longrightarrow \mathrm{Line}_{\mathbb{C}} \tag{9.31}$$

whose partition function on a map $\phi\colon S^1 \to M$ is the holonomy of $\phi^*\Theta$. If Θ is a *flat* connection, then $\alpha_{(P,\Theta)}$ is a topological field theory—it only depends on the homotopy class of the map ϕ—and the *isomorphism class* of the topological theory is the equivalence class of (P,Θ) in

$$\begin{aligned} H^1(M;\mathbb{C}^\times) &\cong [M_+, \Sigma^1 H\mathbb{C}^\times] \\ &\cong [M_+, \mathrm{Map}(M\mathrm{SO}, \Sigma^1 I\mathbb{C}^\times)] \cong [M\mathrm{SO} \wedge M_+, \Sigma^1 I\mathbb{C}^\times]. \end{aligned} \tag{9.32}$$

If, however, Θ is not flat then the theory is not topological. In every case—conjecturally—the *deformation class* of the theory is the equivalence class of P in

$$\begin{aligned} H^2(M;\mathbb{Z}(1)) &\cong [M_+, \Sigma^2 H\mathbb{Z}(1)] \\ &\cong [M_+, \mathrm{Map}(M\mathrm{SO}, \Sigma^2 I\mathbb{Z}(1))] \cong [M\mathrm{SO} \wedge M_+, \Sigma^2 I\mathbb{Z}(1)]. \end{aligned} \tag{9.33}$$

This is also—rigorously—the isomorphism class of the associated continuous invertible 2-dimensional theory $\tilde{\varphi}_{(P,\Theta)}$.

9.4. Differential cohomology and a conjecture

For a symmetry type (H_n, ρ_n), Theorem 8.75 identifies the torsion subgroup of $[MTH, \Sigma^{n+1} I\mathbb{Z}(1)]$ in field-theoretic terms. We ask: Identify the entire group in field-theoretic terms. One solution is (8.71), but we prefer field-theoretic concepts not particular to the invertible case, which continuous theories are. It is typical in derived geometry, say in enumerative problems, that the natural object one ends up enumerating is larger than originally envisioned. We encountered an algebraic example in Remark 6.76. Motivated by this experience and by the examples in §9.3, we formulate a conjectural answer to our query.

CONJECTURE 9.34. *There is a 1:1 correspondence*

$$\left\{ \begin{array}{l} \textit{deformation classes of invertible } n\textit{-dimensional} \\ \textit{reflection positive extended field theories} \\ \textit{with symmetry type } (H_n, \rho_n) \end{array} \right\} \cong [MTH, \Sigma^{n+1} I\mathbb{Z}(1)]. \tag{9.35}$$

As compared with (8.76) we have dropped ‘topological’ on the left hand side and ‘tor’ on the right hand side. We have in mind a more precise statement—a specific map which exhibits the isomorphism. To formulate it we introduce *differential cohomology.*

[79]The (temporary) hybrid notation evokes the bordism category of oriented 0- and 1-manifolds equipped with a smooth map to M. To be consistent with (3.2) we would use ‘$\mathrm{Bord}_{\langle 0,1\rangle}(BSO_1 \times M)$’.

Let us pick up the discussion in Example 9.30, only replace $\mathbb{C}^\times$ with the circle group $\mathbb{T}$. If M is a smooth manifold, let $\mathrm{Bun}^\nabla_{\mathbb{T}}(M)$ denote the groupoid of principal $\mathbb{T}$-bundles $P \to M$ with connection; a morphism is a flat bundle isomorphism $\varphi\colon P' \to P$ which covers id_M, that is, φ pulls the connection on P back to the connection on P'. Then $\pi_0 \mathrm{Bun}^\nabla_{\mathbb{T}}(M)$ is the abelian group of isomorphism classes of principal $\mathbb{T}$-bundles with connection. It is a discrete group. Analogous to the discussion around (6.75), we can topologize it as an infinite dimensional abelian real Lie group $\mathcal{A}$. Then

$$\begin{aligned} \pi_0\mathcal{A} &\cong H^2(M;\mathbb{Z}(1)) \\ \pi_1\mathcal{A} &\cong H^1(M;\mathbb{Z}(1)) \\ \mathrm{Lie}\,\mathcal{A} &\cong \Omega^1_M/d\Omega^0_M \end{aligned} \tag{9.36}$$

and the exact sequence (6.75) holds. The curvature is a Lie group homomorphism

$$\omega\colon \mathcal{A} \longrightarrow \Omega^2_{M,\mathrm{closed}} \tag{9.37}$$

to the vector space of closed 2-forms, and the image is a disjoint union of affine subspaces indexed by the lattice $H^2(M;\mathbb{Z}(1))/\text{torsion}$. The group $H^1(M;\mathbb{T})$ of isomorphism classes of *flat* $\mathbb{T}$-connections can be topologized as a finite dimensional Lie subgroup of $\mathcal{A}$. Its identity component $T = H^1(M;\mathbb{R}(1))/H^1(M;\mathbb{Z}(1))$ is a torus, and the curvature map (9.37) exhibits each component of $\mathcal{A}$ as a principal T-bundle over an affine translate of $d\Omega^1_M$. The curvature and path component maps fit into the commutative diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\omega"] \arrow[d, "\pi_0"'] & \Omega^2_{M,\mathrm{closed}} \arrow[d] \\ H^2(M;\mathbb{Z}(1)) \arrow[r] & H^2(M;\mathbb{R}(1)) \end{tikzcd} \tag{9.38}$$

which is *not* a pullback square: the kernel of $\pi_0 \times \omega$ can be identified with the torus T of flat connections on the trivial bundle. As an extreme example, for $M = S^1$ the diagram (9.38) is

$$\begin{tikzcd} \mathbb{T} \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd} \tag{9.39}$$

EXAMPLE 9.40. Repeat the discussion one degree lower. What replaces $\mathrm{Bun}^\nabla_{\mathbb{T}}(M)$?

The sense in which (9.38) *is* a pullback square is the starting point for Hopkins-Singer's work [**HS**] on generalized differential cohomology. Namely, (9.38) is a *homotopy* pullback square. An element of $\mathcal{A}$ can be represented by a triple (c, h, ω) in which c is an integral singular 2-cocycle, ω is a closed 2-form, and h is a real 1-cochain such that

$$\delta h = \omega - c. \tag{9.41}$$

The homotopy pullback remembers the homotopy h between c and ω, so the homotopy pullback is a categorification of the ordinary pullback. As a first generalization we can replace '2' by any degree $q \in \mathbb{Z}^{\geqslant 0}$ and so obtain *differential cohomology* groups $\check{H}^q(M)$ which generalize $\check{H}^2(M) \cong \pi_0 \mathrm{Bun}^\nabla_{\mathbb{T}}(M)$. They also have a Lie group topology with analogs of (9.36)–(9.38). These groups were first introduced by Cheeger-Simons [**CheeS**], who call their elements 'differential characters', a term which refers to the generalization (9.43) of the holonomy map to arbitrary q. The homotopy pullback viewpoint leads to:

- a cocycle theory and a Picard $(q-1)$-groupoid which generalizes $\mathrm{Bun}_{\mathbb{T}}^{\nabla}(M)$ in case $q = 2$
- generalized differential cohomology theories and higher groupoids for arbitrary spectra

These ideas have been developed in many directions; see [**BG**] and the references therein.

There is an integration theory for differential cohomology. Let R be a ring spectrum, or a module over a ring spectrum. For an appropriately[80] (differential) oriented proper fiber bundle $\pi\colon M \to S$ of smooth manifolds, there is a pushforward

$$\pi_*\colon \check{R}^q(M) \longrightarrow \check{R}^{q-n}(S), \tag{9.42}$$

where $n = \dim M/S$. The generalization of holonomy referenced above is the map

$$\pi_*\colon \check{H}^{n+1}(M) \longrightarrow \check{H}^1(\mathrm{pt}) \cong \mathbb{R}(1)/\mathbb{Z}(1), \tag{9.43}$$

where M is a closed oriented smooth n-manifold. In the case of differential K-theory, if M is a closed odd-dimensional spin Riemannian manifold, then the pushforward

$$\pi_*\colon \check{K}^{n+1}(M) \longrightarrow \check{K}^1(\mathrm{pt}) \cong \mathbb{R}(1)/\mathbb{Z}(1) \tag{9.44}$$

can be proved to equal the η-invariant of the Dirac operator modulo integers; see [**Klo**, **O**, **FL**]. (Recall that in Example 9.24 we encountered its exponential as the partition function of an invertible field theory.) These integrals may be carried out over lower dimensional manifolds; the values of such pushforwards live in higher degree differential cohomology groups [**HS**, Theorem 2.17]. If we start with a map $MTH \to \Sigma^{n+1}I\mathbb{Z}(1)$ for some symmetry type (H_n, ρ_n), then we fully expect that these integrals of a "geometric representative of a differential lift" fit together into an invertible field theory whose codomain is the differential extension of $\Sigma^{n+1}I\mathbb{Z}(1)$. This should define a map from the right hand side of (9.35) to the left hand side. It will be interesting to develop these ideas into mathematical theorems.

[80] If R is the Eilenberg-MacLane spectrum $H\mathbb{Z}$, the the appropriate orientation is a usual orientation. If R is the real K-theory spectrum KO, then the appropriate orientation is a spin structure. If R is the complex K-theory spectrum K, then the appropriate orientation is a Spin^c-structure; in the differential case it is a Spin^c-connection.

LECTURE 10

Computations for Electron Systems

We are ready to apply Conjecture 9.34 to compute the classification of gapped phases of invertible lattice systems. We remind the reader that this application relies on several heuristics to pass from lattice systems to field theories; we have no mathematical justification for this passage. Problem 2.33 with surrounding text and §9.1 are our quick summaries of the problem and of the heuristics we use to arrive at a problem in field theory. We are unaware of any rigorous mathematical framework for invertible lattice systems which allows us to pass to the low energy effective invertible field theories, as we do here. The issue is discussed at greater length in various places, such as [**Ki2**, **Ki5**, **G**]; see also the discussion in [**Sa**] from this CBMS conference as well as the examples in §2.3 and the references therein. For our purposes we simply accept that invertible lattice systems are well-approximated at long range by an invertible field theory, that every invertible field theory is so realized, and that the low energy effective field theory determines the deformation class of the lattice system. Thus empowered, for each dimension and symmetry type we can compute the right hand side of (9.35) using homotopy theory methods and compare to results in the condensed matter literature. The successful comparisons provide strong evidence for the leap from lattice systems to field theories and focus attention on the problem of building a mathematical theory for that jump. Even better, our general formula—which holds for all dimensions and symmetry types—predicts classification results for lattice systems which do not appear in the physics literature.

To better test the entire picture, we choose a class of systems—electron systems—on which there are *two* classifications and a map between them. Namely, there are both *free* systems and interacting systems. Freeness is not a concept we define in terms of Axiom System 3.1. Rather, for free fermions we use the semiclassical description in relativistic field theory, and develop a conjecture for the deformation class of the long range theory in the *massive* case, since massive free fermion theories are gapped. Note that the general theory we describe in these lectures is for *interacting* systems. The semiclassical theory we use in the free case is essentially the theory of Lorentz signature spin representations, or Clifford modules. Atiyah-Bott-Shapiro [**ABS**] pioneered the relation to K-theory and we use two aspects of their work: (i) the identification of the group of Clifford modules modulo restrictions of higher Clifford modules with K-theory groups, and (ii) the map from spin bordism to K-theory given by the symbol of the Dirac operator. We set out this picture in §10.2.

We begin in §10.1 by determining symmetry types of free electron systems. We make two simple hypotheses from which straightforward arguments with compact Lie groups result in 10 distinguished symmetry types. Here again we experience the power of Wick rotation, which transforms noncompact Lie groups to compact

Lie groups, which are rigid and lead to relatively easy classification theorems. The resulting *10-fold way* has many incarnations in quantum mechanics; the original papers are [**D**, **AZ**, **HHZ**]. Here we find a *relativistic* 10-fold way. These 10 electron symmetry groups embed into Clifford algebras, which leads to a simultaneous treatment of all cases.

In §10.3 we report on computations from [**FH1**, §§9–10] and compare with results in the condensed matter literature. Our theory is confirmative and predictive. The computations based on Conjecture 9.34 are carried out using the Adams spectral sequence, which was developed in the 1960s without any thought of statistical mechanical models. Conversely, the results in the condensed matter literature to which we compare are derived without any thought of homotopy theory. The simultaneous disjunction of techniques and concurrence of results provide a powerful test for our analysis based on Axiom System 3.1 and its variations. There have been several other confirmations of these ideas. Kapustin and collaborators made comparisons for several symmetry types, both bosonic and fermionic; see [**Ka3**, **KTTW**] for a small sample. Campbell [**Cam**] makes new computations for more elaborate symmetry types. Beaudry-Campbell [**BeCa**] give a pedagogical introduction to the Adams spectral sequence, and they and [**Cam**] include details of the computations in [**FH1**], including those discussed here in §9.3. Appendix D in [**FH1**] exposes further techniques around the Adams spectral sequence. The recent paper of Guo-Putrov-Wang [**GPW**] undertakes computations for more fermionic symmetry types related to the 10-fold way.

10.1. The 10-fold way for free electron systems

Recall that a symmetry type (H, ρ) has an associated internal symmetry group K, which is a compact Lie group, and a special element $k_0 \in K$ which satisfies $k_0^2 = 1$; see Proposition 3.16. For each positive integer n there is a canonical homomorphism $\mathrm{Spin}_n \to H_n$ which sends the central element $-1 \in \mathrm{Spin}_n$ to $k_0 \in K \subset H_n$. In relativistic field theory on Minkowski spacetime there is a super Hilbert space $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ of states and k_0 acts as the grading operator.[81] To model free fermions we impose two conditions:

(10.1)
- $K = \mathrm{O}_1$, U_1, or Sp_1
- $k_0 = -1$

These three internal symmetry groups K, which we denote $\{\pm 1\}, \mathbb{T}, \mathrm{SU}_2$, are the unit norm elements in $\mathbb{R}, \mathbb{C}, \mathbb{H}$. Anti-Wick rotating again to Minkowski spacetime, an internal symmetry group K acts on $\mathcal{H}$ preserving the grading, and it decomposes a dense subspace of $\mathcal{H}$ as a direct sum indexed by irreducible representations of K. States in an isotypical summand have a definite *charge* labeled by the representation. For $K = \mathbb{T}$ the charge is $e \in \mathbb{Z}$ if the representation of $\mathbb{T}$ is $\lambda \mapsto \lambda^e$. This is the basic case: e is the electric charge. The second condition imposes the *spin/charge relation*, emphasized in [**SeWi**]: states of even charge are bosons and states of odd charge are fermions. One can imagine motivating the internal symmetry groups $K = \{\pm 1\}, \mathrm{SU}_2$ by considering a charge conjugation

[81] In the physics literature it is often denoted $(-1)^F$ to evoke the number of fermions mod 2. In a system of *free* fermions, F *is* a well-defined unbounded self-adjoint operator, and this justifies the notation.

symmetry—implemented by the outer automorphism $\lambda \mapsto \lambda^{-1}$ of $\mathbb{T}$—which sends $e \mapsto -e$. That could "break" $\mathbb{T}$ to the fixed point subgroup $\{\pm 1\} \subset \mathbb{T}$ or enhance[82] it to SU_2. One could also contemplate arguing that the $\mathbb{R}, \mathbb{C}, \mathbb{H}$ trichotomy enters in various ways, for example as the commuting algebra of an irreducible real spin representation. We do not have a convincing argument to restrict to these internal symmetry groups to model interacting[83] fermionic systems, and indeed physicists consider more possibilities [**GPW**].

Once we settle on the restrictions (10.1), it is a routine matter to classify the possible symmetry types. The following is proved in [**FH1**, §9.1].

THEOREM 10.2. *There are 10 symmetry types (H, ρ) with internal symmetry group $K = \{\pm 1\}$, $\mathbb{T}$, or SU_2 and whose special element $k_0 \in K$ equals -1. The groups H are:*

$$\begin{aligned} K = \{\pm 1\}: &\quad \mathrm{Spin},\ \mathrm{Pin}^+,\ \mathrm{Pin}^- \\ K = \mathbb{T}: &\quad \mathrm{Spin}^c,\ \mathrm{Pin}^c,\ \mathrm{Pin}^{\tilde{c}+},\ \mathrm{Pin}^{\tilde{c}-} \\ K = \mathrm{SU}_2: &\quad \mathrm{Spin} \times_{\{\pm 1\}} \mathrm{SU}_2,\ \mathrm{Pin}^+ \times_{\{\pm 1\}} \mathrm{SU}_2,\ \mathrm{Pin}^- \times_{\{\pm 1\}} \mathrm{SU}_2 \end{aligned} \tag{10.3}$$

The exotic groups for $K = \mathbb{T}$ are

$$\mathrm{Pin}^{\tilde{c}\pm} = \mathrm{Pin}^{\pm} \ltimes_{\{\pm 1\}} \mathbb{T}, \tag{10.4}$$

where $\mathrm{Pin}^{\pm}$ acts on $\mathbb{T}$ through $\pi_0 \mathrm{Pin}^{\pm}$ via $\lambda \mapsto \lambda^{-1}$, and we divide by the common subgroup $\{\pm 1\}$ of $\mathrm{Pin}^{\pm}$ and $\mathbb{T}$ to enforce the spin/charge relation.

REMARK 10.5. The 10-fold way was introduced by Altland-Zirnbauer [**AZ**] in condensed matter physics; see also [**HHZ**]. There is a 10-fold way in the famous 1962 paper of Dyson [**Dy**], despite the title only emphasizing a quotient 3-fold way. Several quantum mechanical 10-fold ways exist (see [**FM1**, **Mo**] and the references therein), and the relations among them are not completely understood. Theorem 10.2 is yet another perspective to reconcile.

REMARK 10.6. Electron systems with $K = \mathbb{T}$ are called *topological insulators*; those with $K = \{\pm 1\}, \mathrm{SU}_2$ are called *topological superconductors.* For some background on superconductors and symmetry, see [**Wei2**].

Just as SO_n sits in the algebra $M_n(\mathbb{R})$ of real $n \times n$ matrices, its double cover Spin_n sits in a unital associative algebra, a Clifford algebra. The standard representation $\mathbb{R}^n$ of SO_n extends to an $M_n(\mathbb{R})$-module, and so too *spinor representations* of Spin_n extend to Clifford modules. (In fact, spinor representations are characterized by that property.) Each of the 10 groups in Theorem 10.2 embeds in a Clifford algebra, which gives a notion of spinor representation for each and leads to a uniform and simultaneous treatment. For $p, q \in \mathbb{Z}^{\geqslant 0}$ let $\mathrm{Cliff}_{p,q}$ be the real unital associative algebra generated by $e_1^+, \dots, e_p^+, e_1^-, \dots, e_q^-$ subject to $(e_i^{\pm})^2 = \pm 1$ and distinct generators commute up to a minus sign. (More generally, we can define the Clifford algebra of a vector space with a symmetric bilinear form.) Set

[82] One could argue that the enhancement is more straightforwardly the semidirect product $\{\pm 1\} \ltimes \mathbb{T}$.

[83] In the free case the embedding (10.9) into a Clifford algebra is tied up with the $\mathbb{R}, \mathbb{C}, \mathbb{H}$ trichotomy and provides stronger motivation for restriction to $K = \{\pm 1\}, \mathbb{T}, \mathrm{SU}_2$.

$\mathrm{Cliff}_{+p} = \mathrm{Cliff}_{p,0}$ and $\mathrm{Cliff}_{-q} = \mathrm{Cliff}_{0,q}$. Then the $10 = 3+4+3$ electron symmetry types arrange into two tables ($10 = 2 + 8$) as follows:

(10.7)

s	H^c	K	Cartan	D
0	Spin^c	$\mathbb{T}$	A	$\mathbb{C}$
1	Pin^c	$\mathbb{T}$	AIII	$\mathrm{Cliff}^{\mathbb{C}}_{-1}$

(10.8)

s	H	K	Cartan	D
0	Spin	$\{\pm 1\}$	D	$\mathbb{R}$
-1	Pin^+	$\{\pm 1\}$	DIII	Cliff_{-1}
-2	$\mathrm{Pin}^+ \ltimes_{\{\pm 1\}} \mathbb{T}$	$\mathbb{T}$	AII	Cliff_{-2}
-3	$\mathrm{Pin}^- \times_{\{\pm 1\}} \mathrm{SU}_2$	SU_2	CII	Cliff_{-3}
4	$\mathrm{Spin} \times_{\{\pm 1\}} \mathrm{SU}_2$	SU_2	C	$\mathbb{H}$
3	$\mathrm{Pin}^+ \times_{\{\pm 1\}} \mathrm{SU}_2$	SU_2	CI	Cliff_{+3}
2	$\mathrm{Pin}^- \ltimes_{\{\pm 1\}} \mathbb{T}$	$\mathbb{T}$	AI	Cliff_{+2}
1	Pin^-	$\{\pm 1\}$	BDI	Cliff_{+1}

The label s distinguishes the various cases, as does the Cartan label for a symmetric space, the latter included for comparison to the condensed matter literature.[84] The super division algebra D is used to form the embedding. Namely, in the real case (10.8), there is an embedding

$$H_n(s) \longrightarrow \mathrm{Cliff}_{+n} \otimes D(s). \tag{10.9}$$

The codomain is a Clifford algebra. In fact, (10.9) exhibits the compact Lie group $H_n(s)$ as a Lie subgroup of a spin group which is not necessarily compact. For example, we have

$$\begin{aligned} \mathrm{Pin}^-_n &\subset \mathrm{Spin}_{n+1} \\ \mathrm{Pin}^+_n &\subset \mathrm{Spin}_{n,1} \end{aligned} \tag{10.10}$$

The first spin group is compact, the second noncompact, though of course Pin^+_n is compact. We defer to [**FH1**, §9.2.1] for details and proofs as well as the analog of (10.9) in the complex case (10.7).

10.2. The long range effective theory of free fermions

As an illustration we begin with the 1-dimensional case with symmetry type[85] $H_1 = \mathrm{Spin}_1$.

EXAMPLE 10.11 (Spinor fields in dimension one). The data which defines the theory is a finite dimensional real inner product space W. If X is a spin Riemannian 1-manifold with spin bundle of frames the double cover $\widetilde{X} \xrightarrow{\{\pm 1\}} X$, a spinor field is a $\{\pm 1\}$-equivariant function $\psi\colon \widetilde{X} \to \Pi W$, where ΠW is the purely odd vector space

[84] As suggested by Remark 10.5, that comparison is not completely straightforward.

[85] We may identify a symmetry type (H_n, ρ_n) by its symmetry group H_n when the homomorphism $\rho_n\colon H_n \to \mathrm{O}_n$ is unambiguous.

$\mathbb{C}^{\mathrm{odd}} \otimes W$ parity opposite to W. A *mass term* is a nondegenerate skew-symmetric form

$$m\colon W \times W \longrightarrow \mathbb{R}, \tag{10.12}$$

which exists if and only if $\dim W$ is even. There is a unit norm positively oriented vector field ∂ on X with a unique lift to $\widetilde{X}$, and the Wick-rotated lagrangian density with mass term m is

$$L = \left\{\frac{1}{2}\langle\psi, \partial\psi\rangle + \frac{1}{2}m(\psi,\psi)\right\}|dx|. \tag{10.13}$$

Let us extract the low energy theory. Working first on real time $\mathbb{M}^1$ we identify the space of classical solutions[86] as ΠW after fixing a particular time $t_0 \in \mathbb{M}^1$; see the bosonic analog in §2.2. The Hilbert space of states is an irreducible complex $\mathrm{Cliff}(W^*)$-module $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$. Identify (10.12) with an element $T_m \in \mathfrak{o}(W)$ in the orthogonal Lie algebra. Then the Hamiltonian is, up to a factor of $\sqrt{-1}$, the action of T_m on $\mathcal{H}$. For example, let $W = \mathbb{R}^2$ with $m = \left(\begin{smallmatrix} 0 & -M \\ M & 0\end{smallmatrix}\right)$ and $T_n = \frac{M}{2}e^1e^2 \in \mathrm{Cliff}_2^{\mathbb{C}}$. Then $\mathcal{H} = \mathbb{C} \oplus \mathbb{C}$ is an irreducible Clifford module and the Hamiltonian is

$$H = \begin{pmatrix} M/2 & \\ & -M/2\end{pmatrix}. \tag{10.14}$$

We are faced with a choice: Is the vacuum line even or odd? We defer a detailed discussion of this choice to §11.4, particularly Example 11.36. For now we choose the vacuum to be odd. Next, the vacuum energy is not zero. Evaluate the Wick-rotated theory on a Riemannian interval of length τ. The result is the operator $e^{-\tau H}\colon \mathcal{H} \to \mathcal{H}$. Since $\tau \to \infty$ is the long range limit, make a constant shift $H \to H + M/2$ of the energy operator; then $e^{-\tau(H+M/2)}$ converges to projection onto the vacuum as $\tau \to \infty$. We can also perform the path integral on a spin circle of circumference τ, and the result is the (super)trace of the Wick-rotated time evolution. For the nonbounding spin structure, before shifting the energy operator we obtain $\mathrm{tr}_s\, e^{-\tau H} = -2\sinh(\frac{\tau H}{2})$, but if we shift $H \to H + M/2$ then $\mathrm{tr}_s\, e^{-\tau(H+M/2)} = e^{-\tau M} - 1$ which converges to -1 as $\tau \to \infty$. For the bounding spin structure of circumference τ, the shifted partition function is $\mathrm{tr}\, e^{-\tau(H+M/2)} = e^{-\tau M} + 1$ which converges to $+1$ as $\tau \to \infty$. (See [**W5**, Problem FP16] for a careful treatment of the quantization and path integrals, but in the massless case.)

Summarizing, the low energy theory

$$\alpha\colon \mathrm{Bord}_1(\mathrm{Spin}_1) \longrightarrow s\mathrm{Line}_{\mathbb{C}} \tag{10.15}$$

is topological, invertible, reflection positive, and has order two: $\alpha^{\otimes 2}$ is trivial. It sends

$$\begin{aligned} \mathrm{pt}_+ &\longmapsto \mathbb{C}^{\mathrm{odd}} \\ S^1_{\mathrm{bounding}} &\longmapsto -1 \\ S^1_{\mathrm{nonbounding}} &\longmapsto +1 \end{aligned} \tag{10.16}$$

as deduced above. In homotopy theoretic terms the low energy theory is a map

$$\alpha\colon MT\mathrm{Spin} \longrightarrow \Sigma^1 I\mathbb{C}^\times \tag{10.17}$$

in the nonzero homotopy class, which corresponds to the nontrivial character of $\pi_1 MT\,\mathrm{Spin} \cong \mathbb{Z}/2\mathbb{Z}$.

[86] That is, solutions to the Euler-Lagrange equation derived from the lagrangian (10.13).

There is a conjectural picture in general dimensions which applies to all 10 electron symmetry types. We outline it for the standard case $H = \mathrm{Spin}$. To begin, the classification of relativistic free fermion systems relies on special facts about spinors in Lorentz signature [**De**, §6]. Let $n \geqslant 1$ be the dimension of spacetime, $C \subset \mathbb{R}^{1,n-1}$ the cone of positive timelike vectors, and[87] $\mathrm{Spin}_{1,n-1} \subset \mathrm{Cliff}^0_{n-1,1}$ the Lorentz spin group. By definition, a real spin representation $\mathbb{S}$ is a real (ungraded) $\mathrm{Cliff}^0_{n-1,1}$-module. There are two isomorphism classes of irreducibles if $n \equiv 2 \pmod 4$ and a unique irreducible otherwise. The following theorems hold:

- If $\mathbb{S}$ is irreducible, then there is a unique-up-to-scale nonnegative $\mathrm{Spin}_{1,n-1}$-invariant symmetric bilinear pairing
$$\Gamma\colon \mathbb{S} \times \mathbb{S} \longrightarrow \mathbb{R}^{1,n-1}, \tag{10.18}$$
where nonnegativity means $\Gamma(s,s) \in \overline{C}$ for all $s \in \mathbb{S}$.
- Let S_1, S_2 be representative irreducibles ($S_2 = 0$ if $n \not\equiv 2 \pmod 4$), and let $Z = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ be the commutant of the Clifford action. Then any spin representation has the form
$$\mathbb{S} = W_1 \otimes_Z S_1 \;\oplus\; W_2 \otimes_Z S_2. \tag{10.19}$$
In addition, nonnegative pairings (10.18) are in bijection with pairs of Hermitian metrics on the Z-vector spaces W_1, W_2.
- Given a pairing Γ there is a unique compatible $\mathrm{Cliff}_{n-1,1}$-module structure on $\mathbb{S} \oplus \mathbb{S}^*$.
- Every finite dimensional $\mathrm{Cliff}_{n-1,1}$-module is of the form (10.19).

These theorems reduce the classification of relativistic free fermion systems to the study of Clifford modules.

A *mass term* for $\mathbb{S}$ is a nondegenerate skew-symmetric $\mathrm{Spin}_{1,n-1}$-invariant bilinear form
$$m\colon \mathbb{S} \times \mathbb{S} \longrightarrow \mathbb{R}. \tag{10.20}$$
The following lemma [**FH1**, Lemma 9.55] is crucial.

LEMMA 10.21. *Nondegenerate mass terms for a spinor representation $\mathbb{S}$ correspond to $\mathrm{Cliff}_{n-1,2}$-module structures on $\mathbb{S} \oplus \mathbb{S}^*$ which extend the $\mathrm{Cliff}_{n-1,1}$-module structure.*

REMARK 10.22. Free fermion theories in dimension n have an associated invertible *anomaly* theory which is $(n+1)$-dimensional. The anomaly of a massive free fermion is trivial, so the anomaly only depends on the $\mathrm{Cliff}_{n-1,1}$-module modulo $\mathrm{Cliff}_{n-1,2}$-modules, which according to [**ABS**] is a KO-group. The deformation class of the anomaly theory is computed similarly to (10.26), as we explain in Lecture 11.

Our task now is to identify the low energy theory of a massive free fermion field theory, at least up to deformation. Clifford module data defines the theory, as just explained. The low energy approximation is an invertible unitary field theory, so is realized as a map from the Thom spectrum $MT\mathrm{Spin}$ to the shifted Anderson dual of the sphere spectrum. In other words, we seek a generalization of Example 10.11 to arbitrary spacetime dimension n. Fix massive spinor

[87] $\mathrm{Cliff}^0_{n-1,1} \subset \mathrm{Cliff}_{n-1,1}$ is the even subalgebra.

data: a real $\mathrm{Cliff}_{n-1,1}$-module $\mathbb{S}$, a nonnegative pairing Γ, and a mass term m. According to Lemma 10.21 we obtain a $\mathrm{Cliff}_{n-1,2}$-module $\mathbb{S} \oplus \mathbb{S}^*$. Atiyah-Bott-Shapiro [**ABS**] identify the quotient group of $\mathrm{Cliff}_{n-1,2}$-modules by restrictions of $\mathrm{Cliff}_{n-1,3}$-modules with $KO^{(n-1)-2}(\mathrm{pt}) \cong \pi_{3-n}KO$. So the massive spinor data[88] $(\mathbb{S}, m)$ has an equivalence class

$$[\mathbb{S}, m] \in \pi_{3-n}KO \cong [S^0, \Sigma^{n-3}KO]. \tag{10.23}$$

The symbol of the Dirac operator—Clifford multiplication—defines the Atiyah-Bott-Shapiro map

$$\phi\colon M\mathrm{Spin} \longrightarrow KO \tag{10.24}$$

from spin bordism to real K-theory. Let $\mu\colon KO \wedge KO \to KO$ be multiplication on real K-theory. A choice of generator in $\pi_{-4}KO \cong \mathbb{Z}$ determines a map $S^{-4} \to KO$, up to homotopy, and dually a map $KO \to \Sigma^4 I\mathbb{Z}(1)$. We call the latter 'Pfaff'; see §11.3 for nomenclature explication. The following conjecture is in [**FH1**, §9.2.6].

CONJECTURE 10.25.

(1) *The long range approximation to the massive free fermion theory based on* $(\mathbb{S}, m)$ *is invertible, and it is trivial if* $\mathbb{S} \oplus \mathbb{S}^*$ *extends to a* $\mathrm{Cliff}_{n-1,3}$*-module.*
(2) *The deformation class of the long range approximation is the composition*

$$MT\mathrm{Spin} \xrightarrow{\phi \wedge [\mathbb{S},m]} KO \wedge \Sigma^{n-3}KO \xrightarrow{\mu} \Sigma^{n-3}KO \xrightarrow{\mathrm{Pfaff}} \Sigma^{n+1}I\mathbb{Z}(1). \tag{10.26}$$

The map (10.26) may have finite or infinite order, so may or may not represent a topological theory. There is an analog of Conjecture 10.25 for each of the 10 electron symmetry types; see [**FH1**, (9.71)]. We use this generalization to compute the tables in Examples 10.30 and 10.32 below.

REMARK 10.27. Conjecture 10.25 relies on fixing indeterminacies in the construction of the free fermion quantum field theory from the algebraic data. See §11.4 for a general discussion of these indeterminacies and Example 11.36 for the particular canonical choices available for a free massive spinor field. These choices are implicit in Conjecture 10.25.

REMARK 10.28. Many special low dimensional examples are treated in detail in [**W8**].

10.3. Computations

The group (8.76) of deformation classes of invertible n-dimensional theories[89] of symmetry type (H, ρ) sits in the exact sequence (6.82):

$$0 \longrightarrow \mathrm{Ext}^1(\pi_n MTH, \mathbb{Z}) \longrightarrow [MTH, \Sigma^{n+1}I\mathbb{Z}(1)] \longrightarrow \mathrm{Hom}(\pi_{n+1}MTH, \mathbb{Z}) \longrightarrow 0 \tag{10.29}$$

The free quotient $\mathrm{Hom}(\pi_{n+1}MTH, \mathbb{Z})$ only depends on the free quotient of $\pi_{n+1}MTH$, and at least its rank can be computed by tensoring over $\mathbb{Q}$. Now $\pi_{n+1}MTH \otimes \mathbb{Q} \cong H^{n+1}(MTH; \mathbb{Q})$, so the Thom isomorphism theorem reduces us to the (twisted) rational cohomology of the classifying space BH. In particular, this group vanishes if $n+1$ is odd, hence for n even the group $[MTH, \Sigma^{n+1}I\mathbb{Z}(1)]$

[88] Since Γ is a contractible choice, we omit it from the notation henceforth.
[89] assumed fully extended and reflection positive

is torsion. The torsion subgroup $\mathrm{Ext}^1(\pi_n MTH, \mathbb{Z})$ depends only on the torsion subgroup of $\pi_n MTH$. The Adams spectral sequence can be brought to bear to compute it. We remark that the group $[MTH, \Sigma^{n+1}I\mathbb{Z}(1)]$ of interest is a generalized cohomology group of MTH, and in some cases it can be computed more easily using the Atiyah-Hirzebruch spectral sequence. As mentioned earlier, the papers [**Cam**, **BeCa**] give extensive expositions of the computational techniques. Further information is in [**FH1**, Appendix D].

We content ourselves with just two examples of the electron system computations reported in [**FH1**, §9.3].

EXAMPLE 10.30 ($H = \mathrm{Pin}^+$). Consider first the symmetry type $H = \mathrm{Pin}^+$. This is the Wick-rotated version of a fermionic system with time-reversal symmetry T such that T^2 is the central element k_0 of the Lorentz signature spin group, which in view of the spin-statistics theorem is often expressed as $T^2 = (-1)^F$; see (3.23). Here is the table of computations:

(10.31)

n	$\ker\Phi \longrightarrow$	$FF_n(\mathrm{Pin}^+) \xrightarrow{\Phi}$	$TP_n(\mathrm{Pin}^+) \longrightarrow$	$\mathrm{coker}\,\Phi$
4	$16\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/16\mathbb{Z}$	0
3	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
2	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
1	0	0	0	0
0	$2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0

The column labeled '$FF_n(\mathrm{Pin}^+)$' is the group of *free* theories, which we argued in §10.2 is an appropriate homotopy group of KO. (The exposition in §10.2 is for $H = \mathrm{Spin}$; the other 7 real electron symmetry types lead to a shift of KO by the index s in (10.8).) We see the familiar lyrics of the Bott song in that column. The column labeled '$TP_n(\mathrm{Pin}^+)$' is the abelian group of invertible *interacting* theories, which is computed as the right hand side of (8.76) in Theorem 8.75. These groups are all finite order in this case, so they represent isomorphism classes of invertible *topological* theories. 'TP' stands for 'topological phases'. The TP groups are all torsion, so are character groups of tangential Pin^+ bordism groups. Tangential Pin^+ bordism was computed in all dimensions in [**KT2**] based on the structure of Spin and Pin^- bordism determined in [**ABP1**, **ABP2**]; see [**KT1**] for a geometric account of all of these bordism groups in low dimensions. The map Φ tells the deformation class of a free fermion system, which is the (twisted version of) (10.26) in Conjecture 10.25. The kernel of Φ is the subgroup of free theories which are deformable to the trivial phase if interactions are allowed. The cokernel of Φ is the quotient group of all phases by those which can be represented by free fermions. For this particular symmetry type there are no such.

Now we compare to the physics literature. Kitaev [**Ki3**] derived the classification of free fermions in the context of lattice systems, and he made the connection with Bott periodicity. The FF column matches those results for this, and in fact for all 10, electron symmetry types. There are many arguments in the physics literature demonstrating that 16 copies of the basic free fermion theory in 4 dimensions has a trivial phase once interactions are allowed, and that this does not occur with fewer copies. A sample of the literature includes [**Ki2**, **FCV**, **WS**, **MFCV**, **Ki4**] and [**W8**, §4]. The interacting case in 3 dimensions is investigated in [**W8**, §3], and various aspects of the invertible field theory are described explicitly. It is also discussed in [**LV**, §V B], but the nonzero element is missed within the "K-formalism" as the authors explain. Again, we emphasize that the arguments in these physics papers are completely disjoint from our reasoning based on the Axiom System for field theory, which ultimately leads us to computations in stable homotopy theory. The groups $TP_n(\mathrm{Pin}^+)$ as computed here also appear in [**KTTW**, Table 2].

EXAMPLE 10.32 ($H = \mathrm{Pin}^{\tilde{c}+}$). At the end of [**M**, §VI] Metlitski raised the question of determining $TP_4(\mathrm{Pin}^{\tilde{c}+})$ by a bordism computation and checking agreement with [**WPS**]. This symmetry type corresponds to lattice systems whose internal symmetry group from the viewpoint of space (as opposed to spacetime) is $\mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{T}$, as deduced using (9.4). The identity component $\mathbb{T}$ is the symmetry group associated with electric charge. An element in the off-component of $\mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{T}$ acts as time-reversal and squares to the identity. The classification problem for topological phases of these particular free electron systems—topological insulators with time-reversal symmetry—has attracted wide attention [**KMo**] in the physics community.

Here is the table of computations:

(10.33)

n	$\ker\Phi \longrightarrow$	$FF_n(\mathrm{Pin}^{\tilde{c}+}) \xrightarrow{\Phi}$	$TP_n(\mathrm{Pin}^{\tilde{c}+}) \longrightarrow$	$\operatorname{coker}\Phi$
4	0	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^3$	$(\mathbb{Z}/2\mathbb{Z})^2$
3	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
2	0	0	0	0
1	0	$\mathbb{Z}$	$\mathbb{Z}$	0
0	0	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

Again we see a slice of the Bott song in the free fermion column FF, now shifted from (10.31). The $\mathbb{Z}/2\mathbb{Z}$ invariant of free fermion systems in 3 and 4 spacetime dimensions was introduced in papers of Kane-Mele [**KM**] and Fu-Kane-Mele [**FKM**] and has been further studied by many authors. The interacting case in 4 dimensions is investigated in [**WPS**] and in 3 dimensions in [**W8**, §3.7]; their results agree with ours. Not only are the abstract groups correct, but so is the map from free fermions. Even more, the description of the partition function of some phases in terms of Stiefel-Whitney classes matches our bordism computations as well. Also, [**W8**, §4.7] treats the invertible topological field theory in 4 dimensions defined by the free fermion theory.

10.4. Conclusions

As stated several times in these lectures, we use heuristic arguments to pass from lattice systems to field theory; the agreement of our computations with physics arguments based more directly on lattice systems is evidence that these heuristics are valid. But much more has to work. In fact, these computations provide a serious test of Axiom System 3.1 against physics.[90] There are many aspects of this Axiom System, and its extensions, which are being tested here:

(1) *Wick-rotated field theory on compact manifolds detects long range behavior.* After all, 'long range' seems contrary to restriction to compact manifolds. In a scale-dependent theory we might imagine that scaling up the metric we can capture long range behavior without passing to noncompact manifolds. Certainly bordism groups are defined in terms of compact manifolds, and we were led to these computations from the Axiom System, which encodes Wick-rotated field theory on compact manifolds. This is a direct demonstration that the Axiom System does capture long range phenomena using compact manifolds.
(2) *Extended locality is properly encoded.* Extended locality is the subject of Lecture 5. We are led to higher categories and the Extended Axiom System 5.21. In the invertible case a theory is equivalently a map of spectra (Ansatz 6.89), and ultimately that is the framework for the computations which agree with the physics results. (Some have argued that non-extended field theory suffices—see [**Yo**], for example—and indeed since our choice of target spectrum is based on the principle that the partition function determines the theory, that is true in some sense. However, locality is one of the pillars of quantum theory, and not only is the extended version is much more powerful than the non-extended version, but it is widely used in both mathematics and physics. As we pointed out in §5.4, one manifestation in physics is extended operators. If the mathematical framework for extended locality were off, then the computations might not have come out correctly.)
(3) *Extended reflection positivity for invertible topological theories is properly implemented.* Unitarity is the other pillar of quantum field theory. In Lecture 8 we motivated and executed a definition of Wick-rotated extended unitarity for invertible topological field theories. That brought us from Madsen-Tillmann spectra to Thom spectra. The computations depend on the precise spectrum, and so provide a test of this aspect of our story as well.

[90]The utility of this axiomatic viewpoint on field theory for mathematics, as opposed to physics, is well-established by its many applications to low-dimensional topology, symplectic geometry, algebraic geometry, geometric representation theory, etc.

LECTURE 11

Anomalies in Field Theory

Anomalies in quantum field theory were discovered in perturbation theory in work of Steinberger, Schwinger, Adler, Bell-Jackiw, Bardeen, Zee, Wess-Zumino, 't Hooft, and many others. We refer to the books [**B**, **FS**] for exposition as well as history and references. A more geometric point of view developed beginning in the 1980s, and that is our theme in this lecture. Formulas for anomalies were connected [**AgW**, **AgG**] to the Atiyah-Singer index formulas [**AS3**] for a *single* elliptic operator. At the same time Atiyah-Singer put forward [**AS4**] a relationship to their index theorem [**AS5**] for *families* of Dirac operators. (All of this takes place in the Wick-rotated theory on compact manifolds.) *Global* anomalies which are not detected in perturbation theory were also discovered [**DJT**, **W9**], and Witten computed [**W10**] a formula for the global anomaly in terms of Atiyah-Patodi-Singer η-invariants [**APS**]. We focus on anomalies involving spinor fields. Then one interpretation [**BF2**] of Witten's global anomaly formula and the relationship of anomalies to index theory is in terms of pfaffian and determinant line bundles of Dirac operators, as we recount in §11.1. Our treatment is for a general free spinor field; with a small modification it also applies to Rarita-Schwinger fields [**FH2**, Appendix]. In the 1990s and 2000s a more sophisticated general viewpoint on anomalies emerged: the anomaly of an n-dimensional quantum field theory F is an invertible (possibly truncated[91]) $(n+1)$-dimensional field theory α. This idea, which we take up in §11.2, crystallized from many inputs; see [**F10**, **W8**] for expositions and examples. In §11.3 we return to free spinor fields in n dimensions and formulate a precise conjecture for the associated anomaly theory. This particular anomaly theory extends to $n+1$ dimensions, and—conjecturally—it matches the low energy theory of a massive free spinor field in $n+1$ dimensions. Therefore, as we explain in Example 11.36, we can use a massless free spinor field as a boundary theory for the low energy approximation to the corresponding massive free spinor field, leading to a non-anomalous $(n+1)$-dimensional theory on manifolds with boundary. Many special cases of this general situation appear in the physics literature. Finally, in §11.4 we discuss the following assertion: *every* quantum field theory F has an associated anomaly theory α_F, and extra *structure*—a trivialization of α_F—is part of the data of a well-defined quantum field theory. That is very useful in general, and in particular illuminates the discussion in §9.1 about the form of the long range field theory approximation to a gapped quantum system; see Remark 11.33.

[91]The anomaly theory α need only be defined on a bordism category of manifolds of dimension $\leqslant n$; the value $\alpha(X)$ on a closed n-manifold X is a super line. Often α extends to a theory (we still call 'α') on a bordism category of manifolds of dimension $\leqslant n+1$, in which case the partition function $\alpha(W)$ of a closed $(n+1)$-manifold W is a nonzero complex number. An anomalous field theory is an example of what is sometimes called a *twisted field theory* [**ST**, §5] or a *relative field theory* [**FT1**].

This lecture is long on story and short on details! We encourage the reader to consult the vast literature on anomalies to explore further.

11.1. Pfaffians of Dirac operators

To begin, recall the pfaffian in finite dimensions. Let W be a finite dimensional complex vector space and

$$B\colon W \times W \longrightarrow \mathbb{C} \tag{11.1}$$

a skew-symmetric bilinear form. Identify B as a skew-symmetric map $W \to W^*$, and so an element $\omega_B \in \bigwedge^2 W^*$. A natural "integral" on the exterior algebra is the linear map[92]

$$\int_{\Pi W} : \bigwedge\nolimits^{\bullet} W^* \longrightarrow \operatorname{Det} W^*, \tag{11.2}$$

which projects a form of mixed degree to its highest degree component in $\operatorname{Det} W^* = \bigwedge^{\max} W^*$. If $\dim W = 2m$ is even, then

$$\int_{\Pi W} e^{\omega_B} = \frac{\omega_B^m}{m!} = \operatorname{pfaff} B \in \operatorname{Det} W^* \tag{11.3}$$

is by definition the *pfaffian* of B; if $\dim W$ is odd, then the integral vanishes. The pfaffian is an element of a line, not a number. It is natural to regard $\operatorname{Det} W^*$ as $\mathbb{Z}/2\mathbb{Z}$-graded by the parity of $\dim W$, which is equal to the parity of the dimension of the null space $\ker B$.

There is an infinite dimensional version of the pfaffian for W a Banach space and B a *Fredholm form*: B is Fredholm if $\ker B$ is a finite dimensional subspace. As opposed to the finite dimensional situation (11.3), the $\mathbb{Z}/2\mathbb{Z}$-graded line $\operatorname{Pfaff} B$ depends on B. As B varies we obtain a nontrivial complex line bundle over the space[93] of Fredholm skew forms, and the pfaffian elements

$$\operatorname{pfaff} B \in \operatorname{Pfaff} B \tag{11.4}$$

form a section of the pfaffian line bundle. The section vanishes identically on the component of Fredholm forms with nonzero mod 2 index; the Pfaffian line bundle is odd on that component. The explicit constructions are based on elementary functional analysis; see [**Q**], [**Se1**, Appendix B] for the following case of the Fredholm *determinant*.

REMARK 11.5. A Fredholm map $T\colon U \to V$ between complex Banach spaces has a determinant line, which can be realized as the Pfaffian line of a skew-adjoint Fredholm form as follows. Set $W = V^* \oplus U$ with the sum norm and define

$$B\big((v_1^*, u_1), (v_2^*, u_2)\big) = \langle v_1^*, Tu_2\rangle - \langle v_2^*, Tu_1\rangle. \tag{11.6}$$

Then the pfaffian line of B is canonically isomorphic[94] to the determinant line of T, and under that isomorphism $\operatorname{pfaff} B = \det T$.

---

[92] The odd vector space ΠW, the parity-reversal of W, has as its ring of functions the $\mathbb{Z}/2\mathbb{Z}$-graded exterior algebra $\bigwedge^{\bullet} W^*$. This version of fermionic integration is purely algebraic—there is no measure—and it is defined on functions rather than on forms or densities.

[93] That space has two components distinguished by the parity of $\dim\ker B$, the mod 2 index. Over each component the pfaffian line bundle represents a generator of $H^2(-;\mathbb{Z})$.

[94] The determinant line is $\mathbb{Z}$-graded; the quotient $\mathbb{Z}/2\mathbb{Z}$-grading matches that of the pfaffian line.

A Dirac operator on a closed Riemannian spin manifold is elliptic and determines a Fredholm form (see (11.7) below). There are additional geometric structures constructed using more precise linear analysis. The pfaffian line of a Dirac operator carries a natural Hermitian metric due to Quillen [**Q**]. For a smooth family of Dirac operators the pfaffian line bundle over the parameter space has a natural compatible covariant derivative [**BF1**]; see [**F8**] for a survey of the geometric and analytic ideas and some mathematical applications.[95] We remark that Quillen's work on determinants was also motivated by applications to arithmetic geometry.

Returning to relativistic quantum field theory, recall from §10.2 that a spinor field on Minkowski spacetime is instantiated by a real spinor representation $\mathbb{S}$ of $\mathrm{Spin}_{1,n-1}$ together with a symmetric nonnegative $\mathrm{Spin}_{1,n-1}$-invariant bilinear form $\Gamma\colon \mathbb{S}\times\mathbb{S}\to\mathbb{R}^{1,n-1}$. The complexification $\mathbb{S}_{\mathbb{C}}$ is a representation of the compact spin group Spin_n. On a closed Riemannian spin n-manifold X there is an associated complex vector bundle $S_X\to X$ whose sections are spinor fields ψ. Define the complex skew-symmetric form

$$B_X(\psi_1,\psi_2)=\int_X \Gamma_{\mathbb{C}}(\psi_1,\nabla\psi_2)\,|dx|, \tag{11.7}$$

where ∇ is induced from the Levi-Civita covariant derivative; $\Gamma_{\mathbb{C}}$ is the complexification of Γ, promoted to act on spinor fields; and $|dx|$ is the Riemannian measure. On appropriate function spaces B_X is Fredholm. Formally, the partition function of the free spinor field on X is the Feynman path integral of the exponential of (11.7) over ψ, which is an infinite dimensional analog of (11.3). Therefore, we *define* the result of the not-defined path integral to be (11.4), the (Fredholm) pfaffian element $\operatorname{pfaff} B_X$ of the (Fredholm) pfaffian line $\operatorname{Pfaff} B_X$. This is the partition function of a (Wick-rotated) field theory $F_{(\mathbb{S},\Gamma)}$. That the partition function is an element of a nontrivial complex line, as opposed to a complex number, signals that $F_{(\mathbb{S},\Gamma)}$ is *anomalous*. Furthermore, in parametrized families the pfaffian line bundle is equipped with a metric and compatible connection. The pfaffian line, together with its geometry in parametrized families, is called the *anomaly* on the manifold X. To obtain a well-defined field theory of a free spinor field we must trivialize the anomaly, coherently[96] for varying X. Often the total anomaly of a field theory is the tensor product of many contributions, and it is this total anomaly which must be trivialized, not each constituent.

For the free spinor field a *mass term* (10.20) trivializes the pfaffian line $\operatorname{Pfaff} B_X$, as we now explain. First, observe that in the finite dimensional situation (11.1) the analog of a mass term is a *nondegenerate* skew-symmetric form

$$M\colon W\times W\longrightarrow\mathbb{C}, \tag{11.8}$$

which only exists if $\dim W$ is even. Let $\omega_M\in{\textstyle\bigwedge}^2W^*$ be the associated 2-form. Then if $\dim W=2m$, the nonzero element

$$\frac{\omega_M^m}{m!}\in\operatorname{Det}W^* \tag{11.9}$$

[95]Missing from those papers is the formula for the moment map of a family of Dirac operators invariant under a Lie group of symmetries; the required formula is proved in [**F11**]. The moment map most directly reproduces "covariant anomaly" formulas in the physics literature.

[96]This coherence is the subject of §11.2.

trivializes the finite dimensional pfaffian line. The ratio of (11.3) and (11.9) is a complex number, the *numerical pfaffian*.

For the Wick-rotated free spinor field on a Riemannian manifold X there are Dirac operators mapping between sections of $S_X \to X$ and $S_X^* \to X$. The composition D_X^2 of these operators is the elliptic *Dirac Laplacian*, which we use to decompose the infinite dimensional space of spinor fields as the direct sum $W_a \oplus W_a^\perp$, where W_a is finite dimensional and a is a real parameter; see [**F8**, §3]. The Dirac Laplacian is invertible on $W_a^\perp$. The Weitzenböck formula expresses the Dirac Laplacian as the sum of the covariant Laplacian and scalar curvature, which implies that the appropriate mass operator, constructed from (10.20) and the Riemannian metric, commutes with D_X^2. Hence it acts separately on W_a and $W_a^\perp$. Define the numerical pfaffian to be the product of the finite dimensional numerical pfaffian on W_a and a ζ-function regularized numerical pfaffian on $W_a^\perp$.

EXERCISE 11.10. Carry out the details of this construction. In particular, verify that the result is independent of a. Recast the construction as a trivialization of the pfaffian line $\operatorname{Pfaff} B_X$.

11.2. Anomalies as invertible field theories

We continue the discussion of a free spinor field theory $F_{(\mathbb{S},\Gamma)}$ (no mass term) associated to the data $(\mathbb{S},\Gamma)$. So far we have seen that the formal path integral over a closed Riemannian spin n-manifold X is an element of the anomaly line $\alpha_{(\mathbb{S},\Gamma)}(X) = \operatorname{Pfaff} B_X$. Since $F_{(\mathbb{S},\Gamma)}$ is a quantum field theory, it obeys the locality constraints inherent in describing it as a symmetric monoidal functor (3.2) on a bordism category. It is natural, then, to expect the line $\alpha_{(\mathbb{S},\Gamma)}(X)$ to also "depend locally on X". In §5.3 we explained that locality of a vector space-valued function leads to the notion of an extended field theory, in particular to invariants of lower dimensional manifolds. Thus consider a closed Riemannian spin $(n-1)$-manifold Y. If the free spinor field theory were an ordinary (absolute) field theory, then $F_{(\mathbb{S},\Gamma)}(Y)$ would be a topological vector space. However, as was discovered in the 1980s [**Se5**, **Fa2**, **NAg**], this is not the case. Rather, only a projective space is well-defined.[97] That we obtain a projective space, as opposed to a vector space, signals an anomaly, here called the *Hamiltonian anomaly*. We defer to the references [**Se5**, **Fa2**, **NAg**] for the construction of the projective space from a Dirac operator; see also [**Lo**, **Bu**] for thorough mathematical treatments. Instead, we describe the formal structure we obtain from the projective space, and use it to motivate viewing anomalies in general as extended field theories.

Set $F = F_{(\mathbb{S},\Gamma)}$ and $\alpha = \alpha_{(\mathbb{S},\Gamma)}$. If X is a closed n-manifold, then $\alpha(X)$ is a $\mathbb{Z}/2\mathbb{Z}$-graded complex line and the anomalous theory F produces a (possibly vanishing) element of $\alpha(X)$, i.e., a linear map[98]

$$F(X)\colon \mathbb{C} \longrightarrow \alpha(X). \tag{11.11}$$

[97] The pure states in a quantum system do form a complex projective space (Example 2.11), so for a single quantum system a projective space with no vector space lift is not a problem. A family of quantum systems on Y produces a bundle of projective spaces not expressed—or perhaps not even expressible—as the projectivization of a vector bundle. This is not adequate if we want to "integrate" over the base of the family, which we do if it is a fluctuating field in the theory.

[98] Observe that if $\alpha(X)$ is odd, then $F(X)$ is necessarily the zero map.

An alternative description is in terms of the $\mathbb{C}^\times$-torsor $\alpha(X)^\times$ of nonzero elements of $\alpha(X)$. Then, $F(X)$ is a function $f\colon \alpha(X)^\times \to \mathbb{C}$ which satisfies

$$f(\ell \cdot \lambda) = \lambda^{-1} f(\ell), \qquad \lambda \in \mathbb{C}^\times, \quad \ell \in \alpha(X)^\times. \tag{11.12}$$

This is often the form of explicit descriptions in the physics literature; extra choices induce a trivialization of $\alpha(X)$ and the partition function is presented as depending on those choices. To set the stage for codimension one, we remark that a complex line is an invertible $\mathbb{C}$-module.

Similarly, we posit that for a closed $(n-1)$-manifold Y the anomaly theory produces an invertible $t\mathrm{Vect}_{\mathbb{C}}$-module[99] $\alpha(Y)$ and the anomalous theory produces a $t\mathrm{Vect}_{\mathbb{C}}$-module map

$$F(Y)\colon t\mathrm{Vect}_{\mathbb{C}} \longrightarrow \alpha(Y). \tag{11.13}$$

A complex projective space $\mathbb{P}$ gives rise to both a invertible $t\mathrm{Vect}_{\mathbb{C}}$-module $\alpha(Y)$ and a $t\mathrm{Vect}_{\mathbb{C}}$-module map (11.13) as follows. Define a $\mathbb{C}^\times$-gerbe

$$\mathcal{G}_{\mathbb{P}} = \bigl\{(V,\theta) : V \in t\mathrm{Vect}_{\mathbb{C}}, \quad \theta\colon \mathbb{P} \xrightarrow{\;\cong\;} \mathbb{P}(V)\bigr\}. \tag{11.14}$$

The action of a complex line L on (V,θ) produces[100] $(L^{-1}\otimes V, \theta')$, where θ' is the composition of θ and the canonical identification $\mathbb{P}(V) \cong \mathbb{P}(L^{-1}\otimes V)$. Let $\alpha(Y)$ be the category of functors $f\colon \mathcal{G}_{\mathbb{P}} \to t\mathrm{Vect}_{\mathbb{C}}$ such that

$$f\bigl((V,\theta)\cdot L\bigr) = L^{-1}\otimes f(V,\theta), \qquad L \in \mathrm{Line}_{\mathbb{C}}, \quad (V,\theta) \in \mathcal{G}_{\mathbb{P}}. \tag{11.15}$$

There is a canonical such functor $f = F(Y)$ defined by $f(V,\theta) = V$.

Exercise 11.16. Let X be an n-dimensional compact Riemannian spin manifold with boundary. Construct a trivialization $\alpha(X)$ of $\alpha(\partial X)$.

The basic structure of $\alpha_{(\mathbb{S},\Gamma)}(X), \alpha_{(\mathbb{S},\Gamma)}(Y)$ is common to the anomaly theory α_F of any quantum field theory F, not just to the anomaly $\alpha_{(\mathbb{S},\Gamma)}$ of the free fermion theory $F_{(\mathbb{S},\Gamma)}$. It suggests that $\alpha_F(X), \alpha_F(Y)$ fit together as part of an invertible field theory α_F, an assertion which captures the locality properties of the anomaly. (Witten's paper [**W11**] was an important clue leading to this conclusion.) The bedrock principles of locality and unitary apply to the anomaly theory α_F, which is then fully extended and reflection positive. Initially α_F is only defined on manifolds of dimension $\leqslant n$. We can ask for an extension to a full $(n+1)$-dimensional field theory. Such an extension often exists[101]—it does for free spinor fields, as we explain in §11.3—but it is not necessary for the description of an anomalous theory F, which only involves manifolds of dimension $\leqslant n$. To tell the formal structure of an anomalous theory, suppose $\mathcal{C}$ is an $(n+1)$-category whose 1-category truncation "at the top" (written $\Omega^{n-1}\mathcal{C}$) is[102] $t\mathrm{Vect}_{\mathbb{C}}$. The anomaly theory of F (without extension to dimension $n+1$) is a homomorphism

$$\alpha_F\colon \mathrm{Bord}_n(\mathcal{X}_n^\nabla) \longrightarrow \mathcal{C} \tag{11.17}$$

[99] There are gradings, which are important (see (5.37)), but we omit them here.

[100] The choice of L^{-1} rather than L is inspired by considering the equivalent category to (11.14) whose objects are tautological line bundles $\mathcal{L} \to \mathbb{P}$. The action of L sends $\mathcal{L} \to \mathbb{P}$ to $\mathcal{L}\otimes L \to \mathbb{P}$. The equivalence maps $\mathcal{L} \to \mathbb{P}$ to the space of holomorphic sections of $\mathcal{L}^{-1} \to \mathbb{P}$.

[101] The conformal anomaly of a 2-dimensional conformal field theory is an example with no extension, essentially because conformal geometry in two dimensions is radically different from conformal geometry in higher dimensions.

[102] Better: the category of *super* topological vector spaces.

which factors through the maximal subgroupoid $\mathcal{C}^\times$. Here $\mathcal{X}_n^\nabla$ is the differential tangential structure which encodes the background fields of F. The theory F is a natural transformation

$$\begin{tikzcd} \mathrm{Bord}_n(\mathcal{X}_n^\nabla) \arrow[rr, bend left, "\alpha_F"] \arrow[rr, bend right, "\mathbf{1}"'] \arrow[rr, phantom, "\Uparrow F"] & & \mathcal{C} \end{tikzcd} \tag{11.18}$$

from the tensor unit theory $\mathbf{1}$ to α_F. This notion is discussed in more detail and more generality in [**ST**, **FT2**].

REMARK 11.19. The theory F is topological if it factors through the topological bordism category $\mathrm{Bord}_n(\mathcal{X}_n)$. In some cases the anomaly α_F is topological even if F is not. This occurs for free spinor fields in dimension one, for example; see Example 11.20 below. Or, more generally, α_F as a stand-alone theory may factor through a different differential bordism category $\mathrm{Bord}_n(\mathcal{Y}_n^\nabla)$ or topological bordism category $\mathrm{Bord}_n(\mathcal{Y}_n)$ for some tangential structure $\mathcal{Y}_n$ and map $\mathcal{X}_n \to \mathcal{Y}_n$. However, α_F as the anomaly theory of F is best regarded as having the same domain as F.

11.3. The anomaly theory of a free spinor field

We begin with the simplest case: quantum mechanics.

EXAMPLE 11.20. As in Example 10.11 the datum is a finite dimensional real inner product space W. There is no mass term, lest the anomaly be trivialized. Let F_W be the resulting 1-dimensional massless spinor field, which is defined on spin Riemannian 0- and 1-manifolds. First, consider a positively oriented point. As explained in Example 10.11 the classical theory produces the real operator algebra $\mathrm{Cliff}(W^*)$, the Clifford algebra of the dual inner product space. Then $F_W(\mathrm{pt}_+)$ is a complex $\mathrm{Cliff}(W^*)$-module $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$, and basic principles of quantization demand that it be irreducible. We might like that

$$\mathrm{Cliff}(W^*) \cong \mathrm{End}(\mathcal{H}), \tag{11.21}$$

but this only happens if $\dim W$ is even. So for odd $\dim W$ we might consider the anomaly $\alpha_W(\mathrm{pt}_+)$ to be nontrivial of order two—the class of $\mathrm{Cliff}(W^*)$ in the Brauer group, which is the obstruction to (11.21). Note that pt_+ has a unique spin structure up to isomorphism, and each spin structure has a nontrivial automorphism of order two: the spin flip. It acts on spinor fields as -1, hence on $\mathrm{Cliff}(W^*)$ as the grading operator, an algebra automorphism.

Let $\mathbb{S}$ be a real spinor representation of $\mathrm{Spin}_{1,n-1}$—that is, an ungraded $\mathrm{Cliff}^0_{n-1,1}$-module—and $\Gamma\colon \mathbb{S} \times \mathbb{S} \to \mathbb{R}^{1,n-1}$ a nonnegative symmetric $\mathrm{Spin}_{1,n-1}$-invariant bilinear pairing. We formulate a precise conjecture for the anomaly theory $\alpha_{(\mathbb{S},\Gamma)}$ of the n-dimensional free quantum field theory $F_{(\mathbb{S},\Gamma)}$.

As in (10.23), drop the contractible choice Γ from the notation. Then by [**ABS**] the super Clifford module $\mathbb{S} \oplus \mathbb{S}^*$ (see §10.2) determines a KO-class

$$[\mathbb{S}] \in \pi_{2-n}KO \cong [S^0, \Sigma^{n-2}KO] \tag{11.22}$$

which measures the obstruction to a mass term (see Lemma 10.21). As in (10.9) let $\phi\colon M\mathrm{Spin} \to KO$ be the Atiyah-Bott-Shapiro map, μ multiplication on KO-theory, and $\mathrm{Pfaff}\colon KO \to \Sigma^4 I\mathbb{Z}(1)$ the map defined by a generator of $\pi_{-4}KO$.

CONJECTURE 11.23 ([**FH1**, §9.2.5]). *The $(n+1)$-dimensional anomaly theory associated to the n-dimensional free spinor field theory $F_{(\mathbb{S},\Gamma)}$ is*

(11.24)

$$\alpha_{(\mathbb{S},\Gamma)}\colon MT\mathrm{Spin} \xrightarrow{\phi\wedge[\mathbb{S}]} KO \wedge \Sigma^{n-2}KO \xrightarrow{\mu} \Sigma^{n-2}KO \xrightarrow{\mathrm{Pfaff}} \Sigma^{n+2}I\mathbb{Z}(1).$$

We must both explain the meaning of 'is' and justify the conjecture.

REMARK 11.25. Observe that, by Lemma 10.21, a mass term extends the $\mathrm{Cliff}_{n-1,1}$-module structure on $\mathbb{S}\oplus\mathbb{S}^*$ to a $\mathrm{Cliff}_{n-1,2}$-module structure, and so by [**ABS**] trivializes the class $[\mathbb{S}]$. In turn, by (11.24) this trivializes the anomaly theory $\alpha_{(\mathbb{S},\Gamma)}$. This is the extended field theory version of the trivialization of the "top piece"—the pfaffian line—that we described at the end of §11.1.

In these lectures we have suggested two invertible field theory interpretations of a map such as (11.24). The first, in Ansatz 6.99, is a *topological* theory of spin manifolds of dimension $\leqslant n+2$. Its value on a closed spin $(n+2)$-manifold M is an integer (up to $2\pi i$). For the theory (11.24) this integer vanishes if $n+2$ is not a multiple of 4. If $4 \mid n+2$ then there are two irreducible real spin representations $\mathbb{S}_1, \mathbb{S}_2$ and $\mathbb{S} \cong \mathbb{S}_1^{\oplus n_1} \oplus \mathbb{S}_2^{\oplus n_2}$; see (10.19). The value of α is, up to $2\pi i$,

$$\alpha(M) = c(n_1 - n_2)\big\langle \hat{A}(M), [M]\big\rangle, \tag{11.26}$$

where $c = 1/2$ if $n \equiv 2 \pmod 8$ and $c = 1$ if $n \equiv 6 \pmod 8$. (The sign depends on the ordering of $\mathbb{S}_1, \mathbb{S}_2$.) In this scenario α is a *primary topological invariant.* The second interpretation, described in §9.4, promotes α to a *differential* theory $\check{\alpha}$ of Riemannian spin manifolds of dimension $\leqslant n+1$. Its value on a closed $(n+1)$-manifold is the exponentiated η-invariant related to (11.26). In this scenario $\check{\alpha}$ is a *secondary geometric invariant.* The relationship between primary and secondary invariants is old. Extended topological field theory enhances both to fully local invariants, and the relationship between them persists.

The description we gave of the partition function as a secondary geometric invariant made a jump we now make explicit. Namely, from (11.24) what $\check{\alpha}$ computes on a closed Riemannian spin $(n+1)$-manifold W is the image of a class[103] in $\widetilde{KO}^{n-2}(W)$ under the sequence of maps

$$\widetilde{KO}^{n-2}(W) \xrightarrow{\pi_*} \widetilde{KO}^{-3}(\mathrm{pt}) \xrightarrow{\mathrm{Pfaff}} \widetilde{I\mathbb{Z}(1)}^{1}(\mathrm{pt}) \cong \mathbb{R}/\mathbb{Z}(1), \tag{11.27}$$

where $\pi\colon W \to \mathrm{pt}$. A differential index theorem [**Klo, O, FL**] identifies $\check{\alpha}(W)$ as the η-invariant of the Dirac operator, which is the description we gave in the previous paragraph.

An anomalous n-dimensional theory F with anomaly α is only evaluated on manifolds of dimension $\leqslant n$. Hence suppose $\pi\colon X \to S$ is a family of closed Riemannian n-manifolds. Then in the secondary geometric scenario, $\check{\alpha}(X)$ is the

[103]It has a geometric representative constructed from $\mathbb{S}$ and the Levi-Civita connection.

image of a class[104] in $\widetilde{KO}^{n-2}(X)$ under the sequence of maps

$$\widetilde{KO}^{n-2}(X) \xrightarrow{\pi_*} \widetilde{KO}^{-2}(S) \xrightarrow{\text{Pfaff}} \widetilde{I\mathbb{Z}(1)}^{2}(S). \tag{11.28}$$

The codomain $\widetilde{I\mathbb{Z}(1)}^{2}(S)$ is the abelian group of isomorphism classes of $\mathbb{Z}/2\mathbb{Z}$-graded complex line bundles with metric and compatible connection over S. A differential index theorem[105] identifies $\check{\alpha}(X)$ with the isomorphism class of the pfaffian line bundle of the Dirac forms (11.7), including the metric and connection described after Remark 11.5. The agreement of (11.24) with the anomaly line bundle is a major motivation for Conjecture 11.23.

11.4. Anomalies everywhere

We completely depart the world of mathematical solidity in this section.

THESIS 11.29. [106]

(1) Every n-dimensional quantum field theory F has an associated anomaly theory[107] α_F; and
(2) To define an absolute field theory we must provide a trivialization τ of α_F.

An “absolute” field theory is one which satisfies Axiom System 3.1; it is not anomalous. Thesis 11.29 is hardly radical, especially if one allows the possibility that α_F has a canonical trivialization which can be set equal to τ; then neither α_F nor τ need be mentioned. However, the thesis here is that with rare exceptions (including Example 1.23, Example 11.36 below) there is no canonical trivialization and, furthermore, some of the choices which appear in quantum field theory are encoded in the choice of τ. Whereas τ is involves a choice, the anomaly α_F is canonically determined by F, which in the absence of a trivialization is only a relative theory (11.18). Finally, α_F provides powerful information about a theory, especially if one considers maximal sets of parameters. (This includes relaxing coupling constants to background scalar fields, an often profitable move.) Such information can often be used to make deductions about the long range behavior of F. We will illustrate Thesis 11.29 with several examples, but we first enhance diagram (11.18)

---

[104] As in footnote [103] there is a differential cocycle (in some model), not just its differential cohomology class.

[105] [**FL**, Proposition 8.3] is the version for determinants in place of pfaffians.

[106] I believe the point of view articulated in Thesis 11.29 is widely shared. For me, [**FM2**] was an important input, as were conversations with Nathan Seiberg about [**STY**] and related topics. See [**CFLS**] for more along these lines.

[107] In general α_F is a differential theory; in this section we do not use the embellished $\check{\alpha}_F$ as we did in §11.3.

to incorporate τ:

$$\begin{tikzcd}[column sep=huge] \mathrm{Bord}_n(\mathcal{X}_n^{\nabla}) \arrow[r, bend left=50, "\mathbf{1}"] \arrow[r, "\alpha_F" description] \arrow[r, bend right=50, "\mathbf{1}"'] \arrow[r, phantom, shift left=4ex, "\Uparrow\tau"] \arrow[r, phantom, shift right=4ex, "\Uparrow F"] & \mathcal{C} \end{tikzcd} \tag{11.30}$$

The composition $\tau \circ F$ is an absolute n-dimensional field theory.

A change of the trivialization τ of the anomaly α_F is an invertible n-dimensional field theory δ, which in the notation of (11.30) is a symmetric monoidal functor

$$\delta\colon \mathrm{Bord}_n(\mathcal{X}_n^{\nabla}) \longrightarrow (\Omega\mathcal{C})^{\times}. \tag{11.31}$$

The codomain Picard n-groupoid $(\Omega\mathcal{C})^{\times}$ has 1-categorical truncation $s\mathrm{Line}_{\mathbb{C}}$ at the top. Tensoring by δ maps (F,τ) to $(F,\tau\otimes\delta)$.

Remark 11.32. Invertible theories (11.31) form a categorical abelian group which acts on the (mythical) space of field theories (F,τ) by tensoring. Theories in the same orbit may be regarded as closely related. If one works in the framework of lagrangians and path integral quantization, then tensoring by an invertible theory amounts to adding (counter)terms to the lagrangian which only depend on the *background* fields. For example, if $\mathcal{X}_n^{\nabla} = \mathrm{SO}_n^{\nabla}$—the background fields are an orientation and Riemannian metric—then such terms vanish on flat manifolds such as $\mathbb{E}^n$ and $\mathbb{M}^n$.

Remark 11.33. Thesis 11.29 clarifies the appearance of topological* in §9.1. Suppose a theory (F,τ) is gapped. Then the long range approximation to $(F,\tau\otimes\delta)$ is the long range approximation to (F,τ) tensored with δ, and even if (F,τ) is purely topological its product with δ need not be.

Example 11.34 (Quantum mechanics). A 1-dimensional invertible field theory

$$\delta\colon \mathrm{Bord}_1(\mathrm{SO}_1^{\nabla}) \longrightarrow \mathrm{Line}_{\mathbb{C}} \tag{11.35}$$

with domain the bordism category of 0- and 1-dimensional oriented Riemannian manifolds assigns a complex line L to pt_+ (better: to a germ of an oriented Riemannian 1-manifold) and, for some constant $E_0 \in \mathbb{R}$, assigns multiplication by $e^{-\tau E_0}$ to a closed interval of length τ. So tensoring a 1-dimensional theory (F,τ) with δ amounts to tensoring the state space $\mathcal{H}$ by L, which preserves the underlying projective space $\mathbb{P}(\mathcal{H}) \cong \mathbb{P}(\mathcal{H}\otimes L)$, and shifting the Hamiltonian by the constant E_0.

As already mentioned, there are field theories F whose anomaly α_F has a *canonical* trivialization τ. The canonical trivialization in the following example is implicitly used in Conjecture 10.25.

Example 11.36 (Free massive spinor field). Let $\mathbb{S}$ be a real spinor representation of $\mathrm{Spin}_{1,n-1}$ and $m\colon \mathbb{S}\times\mathbb{S} \to \mathbb{R}$ a mass term. As explained in Remark 11.25 the mass term produces a canonical trivialization $\tau_{(\mathbb{S},m)}$ of the anomaly $\alpha_{(\mathbb{S},m)}$ of the massive free theory $F_{(\mathbb{S},m)}$. We have not made $\tau_{(\mathbb{S},m)}$ very explicit, however, and it would be interesting to do so. One observation is that the proof of Lemma 10.21, which is carried out explicitly in [**FH1**, Lemma 9.55], uses the mass term to construct a $\mathrm{Cliff}_{n-1,2}$-module structure on $\mathbb{S} \oplus \mathbb{S}^*$. Write $\mathrm{Cliff}_{n-1,2} \cong \mathrm{Cliff}_{n-2,1} \otimes \mathrm{Cliff}_{1,1}$ and fix an isomorphism $\mathrm{Cliff}_{1,1} \cong \mathrm{End}(M)$, where $M \cong \mathbb{R}^{1|1}$. Then

$$\mathbb{S}' \oplus \widetilde{\mathbb{S}}' := \mathrm{Hom}_{\mathrm{Cliff}_{1,1}}(M, \mathbb{S} \oplus \mathbb{S}^*) \tag{11.37}$$

is a $\mathrm{Cliff}_{n-2,1}$-module, so $\mathbb{S}'$ is a real spinor representation of $\mathrm{Spin}_{1,n-2}$. Therefore, $\mathbb{S}'$ determines an $(n-1)$-dimensional field theory $\beta_{\mathbb{S}'}$ of massless free spinor fields. The n-dimensional anomaly theory $\alpha_{\mathbb{S}'}$ is canonically determined, according to Thesis 11.29; its isomorphism class is (11.24). On the other hand, it follows from the Morita maneuver (11.37) that the isomorphism class $[\mathbb{S}'] \in \pi_{3-n}KO$ equals the isomorphism class $[\mathbb{S}, m] \in \pi_{3-n}KO$; see (10.23) for the latter. Combine (10.26) and (11.24) to conclude: the long range approximation $\lambda_{(\mathbb{S},m)}$ of $F_{(\mathbb{S},m)}$ is isomorphic to the anomaly theory of $\alpha_{\mathbb{S}'}$. Choose an isomorphism. Then there is a combined system[108] on manifolds with boundary which has $\lambda_{(\mathbb{S},m)}$ in the bulk and $\beta_{\mathbb{S}'}$ on the boundary. Many examples are described in [**W8**].

We need not use the canonical trivialization τ: we can replace $(F_{(\mathbb{S},m)}, \tau_{(\mathbb{S},m)})$ by the closely related theory $(F_{(\mathbb{S},m)}, \tau_{(\mathbb{S},m)} \otimes \delta)$ for any invertible theory

$$delta\colon\ \mathrm{Bord}_n(\mathrm{Spin}_n^{\nabla}) \to (\Omega\mathcal{C})^{\times}.$$

This is usually done if we study the family of free spinor theories parametrized by the mass term. In the simplest 1-dimensional case (Example 10.11) the parameter $M \in \mathbb{R}$ appears in $m = \begin{pmatrix} 0 & -M \\ M & 0 \end{pmatrix}$ and in the Hamiltonian (10.14). It is clear from the latter that, if we fix a $\mathrm{Cliff}_2^{\mathbb{C}}$-module $\mathcal{H} = \mathbb{C} \oplus \mathbb{C}$ independent of M, then the vacuum switches from even to odd when M flips sign. In other words, we choose a (parametrized) trivialization $\tau(M)$ so that for $M \neq 0$ one of $\tau(M), \tau(-M)$ is the canonical trivialization and the other differs by tensoring with the nontrivial invertible 1-dimensional topological spin theory δ which assigns an odd line to the positively oriented spin point. This is a natural choice: the partition function of that parametrized family of theories $\big(F(M), \tau(M)\big)$ on a spin circle is an analytic function of M. That would not be the case if we use the canonical trivialization for all nonzero M.

Remark 11.38. Observe that when Riemannian metrics are present there is a zoo of possible invertible theories

$$\delta\colon\ \mathrm{Bord}_n(\mathrm{SO}_n^{\nabla}) \longrightarrow \Sigma^n I\mathbb{C}^{\times} \tag{11.39}$$

that can modify a given absolute theory (F, τ). For example, the partition function on a closed oriented Riemannian n-manifold (X, g) can be

$$\delta(X, g) = \exp\left(-\int_X \varphi(R_g)\,|d\mu_g|\right), \tag{11.40}$$

where $\varphi\colon \mathbb{R} \to \mathbb{R}$ is a smooth function, R_g is the scalar curvature, and $|d\mu_g|$ is the Riemannian measure. We say δ is *infrared trivial* if

$$\lim_{\lambda \to \infty} \delta(X, \lambda g) = 1 \tag{11.41}$$

[108] The formal structure of the combined system is described—in the context of topological field theory—in [**L1**, Example 4.3.22].

for all (X, g). Tensoring by an infrared trivial invertible theory does not change the long range physics, so we may decide to work modulo the subgroup of infrared trivial invertible theories. For example, if $\mathcal{X}_4 = \mathrm{SO}_4^\nabla$ then invertible reflection positive theories modulo infrared trivial theories are represented by three families of partition functions:

$$\exp\left(-c_1 \int_X R_g\,|d\mu_g|\right), \qquad \exp\Bigl(ic_2\,\mathrm{Sign}(X)\Bigr), \qquad \exp\Bigl(c_3\,\mathrm{Euler}(X)\Bigr), \tag{11.42}$$

where $c_1, c_2, c_3 \in \mathbb{R}$. The first is the Wick-rotated Einstein action, the latter two are topological. The constants c_1, c_2, c_3 are (changes of) choices one must make in passing from theories on $\mathbb{M}^4$ or $\mathbb{E}^4$ to theories on compact Riemannian manifolds ("background gravity").

'Infrared trivial' can be expressed in terms of a local scaling restriction. A closely related scaling condition is crucial in Gilkey's proof of the index theorem for Dirac operators [**AtBoPa**].

We have only illustrated Thesis 11.29 in rather simple quantum field theories. The thesis has strong consequences for nontrivial theories which are of interest; see [**GKSW**, **GKKS**, **CHS**, **STY**] for a small sample. It would be fun to tell more, but we regrettably and regretfully stop.

APPENDIX A

Review of Categories

The book [**R**] is a modern introduction to category theory and contains many examples.

A.1. Categories and groupoids

Definition A.1. A (small) *category* C consists of a collection of objects, for each pair of objects y_0, y_1 a set of morphisms $C(y_0, y_1)$, for each object y a distinguished morphism $\mathrm{id}_y \in C(y,y)$, and for each triple of objects y_0, y_1, y_2 a composition law

$$\circ\colon C(y_1, y_2) \times C(y_0, y_1) \longrightarrow C(y_0, y_2) \tag{A.2}$$

such that $\circ$ is associative and id_y is an identity for $\circ$. A morphism $f \in C(y_0, y_1)$ is *invertible* (an *isomorphism*) if there exists $g \in C(y_1, y_0)$ such that $g \circ f = \mathrm{id}_{y_0}$ and $f \circ g = \mathrm{id}_{y_1}$. If every morphism in C is invertible, then we call C a *groupoid*.

To emphasize that a category is an algebraic structure like any other, we indicate how to formulate the definition in terms of sets[109] and functions. Then a category C consists of a set C_0 of objects, a set C_1 of functions, and structure maps

$$\begin{aligned} i\colon C_0 &\longrightarrow C_1 \\ s,t\colon C_1 &\longrightarrow C_0 \\ c\colon C_1 \times_{C_0} C_1 &\longrightarrow C_1 \end{aligned} \tag{A.3}$$

which satisfy axioms expressing identity maps and associativity. The map i attaches to each object y the identity morphism id_y, the maps s,t assign to a morphism $(f\colon y_0 \to y_1) \in C_1$ the source $s(f) = y_0$ and target $t(f) = y_1$, and c is the composition law. The fiber product $C_1 \times_{C_0} C_1$ is the set of pairs $(f_2, f_1) \in C_1 \times C_1$ such that $t(f_1) = s(f_2)$. A sample axiom: for all $(f\colon y_0 \to y_1) \in C_1$ the identity $c\big(f, i(y_0)\big) = c\big(i(y_1), f\big) = f$ holds.

Example A.4 (Fundamental groupoid). Let S be a topological space. The simplest invariant is the *set* $\pi_0 S$. It is defined by imposing an equivalence relation on the set underlying the topological space: points y_0 and y_1 in S are equivalent if there exists a continuous path which connects them, i.e., a continuous map $\gamma\colon [0,1] \to S$ which satisfy $\gamma(0) = y_0$, $\gamma(1) = y_1$.

The *fundamental groupoid* $C = \pi_{\leqslant 1} S$ is defined as follows. The objects $C_0 = S$ are the points of S. The Hom-set $C(y_0, y_1)$ is the set of homotopy classes of maps $\gamma\colon [0,1] \to S$ which satisfy $\gamma(0) = y_0$, $\gamma(1) = y_1$. The homotopies are taken "rel

[109] ignoring set-theoretic complications

boundary", which means that the endpoints are fixed in a homotopy. Explicitly, a homotopy is a map

$$\Gamma\colon [0,1] \times [0,1] \longrightarrow S \tag{A.5}$$

such that $\Gamma(s,0) = y_0$ and $\Gamma(s,1) = y_1$ for all $s \in [0,1]$. The composition of homotopy classes of paths is associative, and every morphism is invertible. Note that the *automorphism group* $C(y,y)$ is the fundamental group $\pi_1(S,y)$. So $\pi_{\leqslant 1}S$ encodes both $\pi_0 S$ and all of the fundamental groups.

Exercise A.6. Given a groupoid C use the morphisms to define an equivalence relation on the objects and so a set $\pi_0 C$ of equivalence classes. Can you do the same for a category which is not a groupoid?

Definition A.7. Let C and D be categories.

(1) A *functor* or *homomorphism* $F\colon C \to D$ is a pair of maps $F_0\colon C_0 \to D_0$, $F_1\colon C_1 \to D_1$ which commute with the structure maps (A.3).
(2) Suppose $F, G\colon C \to D$ are functors. A *natural transformation* $\eta\colon F \Rightarrow G$ is a map of sets $\eta\colon C_0 \to D_1$ such that for all morphisms $(f\colon y_0 \to y_1) \in C_1$ the diagram

$$\begin{tikzcd} Fy_0 \arrow[r, "Ff"] \arrow[d, "\eta(y_0)"'] & Fy_1 \arrow[d, "\eta(y_1)"] \\ Gy_0 \arrow[r, "Gf"] & Gy_1 \end{tikzcd} \tag{A.8}$$

commutes. We write $\eta\colon F \to G$.
(3) A natural transformation $\eta\colon F \to G$ is an *isomorphism* if $\eta(y)\colon Fy \to Gy$ is an isomorphism for all $y \in C$.

In (1) the commutation with the structure maps means that F is a homomorphism in the usual sense of algebra: it preserves compositions and takes identities to identities. A natural transformation is often depicted as in the following diagram:

$$\begin{tikzcd} C \arrow[r, bend left, "G"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Uparrow\eta"] & D \end{tikzcd} \tag{A.9}$$

Definition A.10. Let C, D be categories. A functor $F\colon C \to D$ is an *equivalence* if there exist a functor $G\colon D \to C$, and natural isomorphisms $G \circ F \to \mathrm{id}_C$ and $F \circ G \to \mathrm{id}_D$ to identity functors.

Proposition A.11. *A functor $F\colon C \to D$ is an equivalence if and only if it satisfies:*

(1) *For each $d \in D$ there exist $c \in C$ and an isomorphism $\big(f(c) \to d\big) \in D$; and*
(2) *For each $c_1, c_2 \in C$ the map of Hom-sets $F\colon C(c_1,c_2) \to D\big(F(c_1), F(c_2)\big)$ is a bijection.*

If F satisfies (i) it is said to be *essentially surjective* and if it satisfies (ii) it is *fully faithful.*

Example A.12 (Functor categories). Show that for fixed categories C, D there is a category $\mathrm{Hom}(C,D)$ whose objects are functors and whose morphisms are natural transformations.

REMARK A.13. Categories have one more layer of structure than sets. Intuitively, elements of a set have no "internal" structure, whereas objects in a category do, as reflected by their self-maps. Numbers have no internal structure, whereas sets do. Try that intuition out on each of the examples above. Anything to do with categories has an extra layer of structure. For example, homomorphisms of categories form a category rather than a set (Example A.12). In Definition A.18 below we see that a monoidal structure has an extra layer of data over a monoid.

DEFINITION A.14. Let C be a category and $C'_0 \subset C_0$ a subset of objects. Then the *full subcategory* C' with set of objects C'_0 has as Hom-sets

$$C'_1(y_0, y_1) = C_1(y_0, y_1), \qquad y_0, y_1 \in C'_0. \tag{A.15}$$

There is a natural inclusion $C'_0 \to C_0$ which is an isomorphism on Hom-sets. We can describe the entire set of morphisms C'_1 as a pullback:

$$\begin{tikzcd} C'_1 \arrow[r, dashed] \arrow[d, dashed] & C_1 \arrow[d, "s\times t"] \\ C'_0 \times C'_0 \arrow[r, "j\times j"] & C_0 \times C_0 \end{tikzcd} \tag{A.16}$$

where s, t are the source and target maps (A.3) and $j\colon C'_0 \hookrightarrow C_0$ is the inclusion.

Recall that if M is a monoid, then the *group of units* $M^{\sim} \subset M$ is the subset of invertible elements. For example, if M is the monoid of $n \times n$ matrices under multiplication, then $M^{\sim}$ is the subset of invertible matrices, which forms a group.

DEFINITION A.17. Let C be a category. Its *groupoid of units*[110] is the groupoid $C^{\sim}$ with same objects $C^{\sim}_0 = C_0$ as in the category C and with morphisms $C^{\sim}_1 \subset C_1$ the subset of invertible morphisms in C.

Notice that identity arrows are invertible and compositions of invertible morphisms are invertible, so $C^{\sim}$ is a category. Obviously, it is a groupoid.

A.2. Symmetric monoidal categories and duality

A category is an enhanced version of a set; a *symmetric monoidal category* is an enhanced version of a commutative monoid.

The *Cartesian product* $C = C' \times C''$ of categories C', C'' has set of objects the Cartesian product $C_0 = C'_0 \times C''_0$ and set of morphisms the Cartesian product $C_1 = C'_1 \times C''_1$. We leave the reader to work out the structure maps (A.3).

DEFINITION A.18. Let C be a category. A *symmetric monoidal structure* on C consists of an object

$$1_C \in C, \tag{A.19}$$

a functor

$$\otimes\colon C \times C \longrightarrow C \tag{A.20}$$

[110] usually called the *maximal groupoid*

and natural isomorphisms

$$\begin{tikzcd} C \times C \times C \arrow[r, bend left=40, "-\otimes(-\otimes-)"] \arrow[r, bend right=40, "(-\otimes-)\otimes-"'] \arrow[r, phantom, "\Uparrow\alpha"] & C \end{tikzcd}, \tag{A.21}$$

$$\begin{tikzcd} C \times C \arrow[r, bend left=40, "(-\otimes-)\circ\tau"] \arrow[r, bend right=40, "-\otimes-"'] \arrow[r, phantom, "\Uparrow\sigma"] & C \end{tikzcd}, \tag{A.22}$$

and

$$\begin{tikzcd} C \arrow[r, bend left=40, "\mathrm{id}_C"] \arrow[r, bend right=40, "1_C\otimes-"'] \arrow[r, phantom, "\Uparrow\iota"] & C \end{tikzcd}. \tag{A.23}$$

The quintuple $(1_C, \otimes, \alpha, \sigma, \iota)$ is required to satisfy the axioms indicated below.

The functor τ in (A.22) is transposition:

$$\begin{aligned} \tau\colon C \times C &\longrightarrow C \times C \\ (y_1, y_2) &\longmapsto (y_2, y_1) \end{aligned} \tag{A.24}$$

A crucial axiom is that

$$\sigma^2 = \mathrm{id}\,. \tag{A.25}$$

Thus for any $y_1, y_2 \in C$, the composition

$$y_1 \otimes y_2 \xrightarrow{\ \sigma\ } y_2 \otimes y_1 \xrightarrow{\ \sigma\ } y_1 \otimes y_2 \tag{A.26}$$

is $\mathrm{id}_{y_1 \otimes y_2}$. The other axioms express compatibility conditions among the extra data (A.19)–(A.23). For example, we require that for all $y_1, y_2 \in C$ the diagram

$$\begin{tikzcd} & (1_C \otimes y_1) \otimes y_2 \arrow[dl, "\alpha"'] \arrow[dr, "\iota"] & \\ 1_C \otimes (y_1 \otimes y_2) \arrow[rr, "\iota"] & & y_1 \otimes y_2 \end{tikzcd} \tag{A.27}$$

commutes. We can state the axioms informally as asserting the equality of any two compositions of maps built by tensoring α, σ, ι with identity maps. These compositions have domain a tensor product of objects $y_1, \dots, y_n$ and any number of identity objects 1_C—ordered and parenthesized arbitrarily—to a tensor product of the same objects, again ordered and parenthesized arbitrarily. Coherence theorems show that there is a small set of conditions which needs to be verified; then arbitrary diagrams of the sort envisioned commute. You can find precise statements and proof in [**Mac**, **JS**].

Next, we review duality in a symmetric monoidal category. Let C be a symmetric monoidal category and $x \in C$. Denote the tensor unit by $1 \in C$.

Definition A.28. Let x be an object in a symmetric monoidal category C. *Duality data* for x is a triple $(x^\vee, c, e)$ consisting of an object $x^\vee \in C$ together with morphisms $c\colon 1 \to x \otimes x^\vee$ and $e\colon x^\vee \otimes x \to 1$ such that the compositions

$$\begin{aligned} &x \xrightarrow{c\otimes \mathrm{id}} x \otimes x^\vee \otimes x \xrightarrow{id\otimes e} x \\ &x^\vee \xrightarrow{\mathrm{id}\otimes c} x^\vee \otimes x \otimes x^\vee \xrightarrow{e\otimes \mathrm{id}} x^\vee \end{aligned} \tag{A.29}$$

are identity maps. We leave the reader to define a morphism between two duality data triples.

The morphism c is called *coevaluation* and e is called *evaluation.* We say that $x^\vee$ is "the" dual to x since any two triples of duality data are uniquely isomorphic.

Definition A.30. Let C be a category.

(1) If for each pair $y_0, y_1 \in C$ the Hom-set $C(y_0, y_1)$ is either empty or contains a unique element, we say that C *is a discrete groupoid.*
(2) If for each pair $y_0, y_1 \in C$ the Hom-set $C(y_0, y_1)$ has a unique element, we say that C *is contractible.*

Proposition A.31. *Let C be a symmetric monoidal category and $y \in C$. Then the category of duality data for y is either empty or is contractible.*

Definition A.32. Let $y_0, y_1 \in C$ be dualizable objects in a symmetric monoidal category and $f\colon y_0 \to y_1$ a morphism. The *dual morphism* $f^\vee\colon y_1^\vee \to y_0^\vee$ is the composition

$$y_1^\vee \xrightarrow{\mathrm{id}_{y_1^\vee}\otimes c_0} y_1^\vee \otimes y_0 \otimes y_0^\vee \xrightarrow{\mathrm{id}_{y_1^\vee}\otimes f \otimes \mathrm{id}_{y_0^\vee}} y_1^\vee \otimes y_1 \otimes y_0^\vee \xrightarrow{e_1 \otimes \mathrm{id}_{y_0^\vee}} y_0^\vee \tag{A.33}$$

In the definition we choose duality data $(y_0^\vee, c_0, e_0)$, $(y_1^\vee, c_1, e_1)$ for y_0, y_1.

Exercise A.34. Verify that a vector space V is dualizable iff it is finite dimensional. Check that Definition A.32 agrees with that of a dual linear map for $C = \mathrm{Vect}$.

A.3. Symmetric monoidal functors

Definition A.35. Let C, D be symmetric monoidal categories. A *symmetric monoidal functor* $F\colon C \to D$ is a functor with two additional pieces of data, namely an isomorphism

$$1_D \longrightarrow F(1_C) \tag{A.36}$$

and a natural isomorphism

$$\begin{tikzcd} C \times C \arrow[r, bend left, "F(-\otimes-)"] \arrow[r, bend right, "F(-)\otimes F(-)"'] \arrow[r, phantom, "\Uparrow\psi"] & C \end{tikzcd}\ . \tag{A.37}$$

There are many conditions on this data.

The first condition expresses compatibility with the associativity morphisms: for all $y_1, y_2, y_3 \in C$ the diagram

$$\begin{tikzcd} \big(F(y_1) \otimes F(y_2)\big) \otimes F(y_3) \arrow[r, "\psi"] \arrow[d, "\alpha_D"'] & F(y_1 \otimes y_2) \otimes F(y_3) \arrow[d, "\psi"] \\ F(y_1) \otimes \big(F(y_2) \otimes F(y_3)\big) \arrow[d, "\psi"'] & F\big((y_1 \otimes y_2) \otimes y_3\big) \arrow[d, "F(\alpha_C)"] \\ F(y_1) \otimes F(y_2 \otimes y_3) \arrow[r, "\psi"] & F\big(y_1 \otimes (y_2 \otimes y_3)\big) \end{tikzcd} \tag{A.38}$$

is required to commute. Next, there is compatibility with the identity data ι: for all $y \in C$ we require that

$$\begin{tikzcd} F(1_C) \otimes F(y) \arrow[r, "F(\psi)"] & F(1_C \otimes y) \arrow[d, "F(\iota)"] \\ 1_D \otimes F(y) \arrow[u, "(A.36)"] \arrow[r, "\iota"] & F(y) \end{tikzcd} \tag{A.39}$$

commute. The final condition expresses compatibility with the symmetry σ: for all $y_1, y_2 \in C$ the diagram

$$\begin{tikzcd} F(y_1) \otimes F(y_2) \arrow[r, "\sigma_D"] \arrow[d, "\psi"'] & F(y_2) \otimes F(y_1) \arrow[d, "\psi"] \\ F(y_1 \otimes y_2) \arrow[r, "F(\sigma_C)"] & F(y_2 \otimes y_1) \end{tikzcd} \tag{A.40}$$

Definition A.41. Let C, D be symmetric monoidal categories and $F, G\colon C \to D$ symmetric monoidal functors. Then a *symmetric monoidal natural transformation* $\eta\colon F \to G$ is a natural transformation such that the diagrams

$$\begin{tikzcd} & F(1_C) \arrow[dd, "\eta(1_C)"] \\ 1_D \arrow[ur] \arrow[dr] & \\ & G(1_C) \end{tikzcd} \tag{A.42}$$

and

$$\begin{tikzcd} F(y_1) \otimes F(y_2) \arrow[r, "\psi"] \arrow[d, "\eta\otimes\eta"'] & F(y_1 \otimes y_2) \arrow[d, "\eta"] \\ G(y_1) \otimes G(y_2) \arrow[r, "\psi"] & G(y_1 \otimes y_2) \end{tikzcd} \tag{A.43}$$

commute for all $y_1, y_2 \in C$.

Proposition A.44. *Let B, C be symmetric monoidal categories, $F, G\colon B \to C$ symmetric monoidal functors, and $y \in B$ dualizable. Then*

(1) *$F(y) \in C$ is dualizable.*
(2) *If $\eta\colon F \to G$ is a symmetric monoidal natural transformation, then $\eta(y)\colon F(y) \to G(y)$ is an isomorphism.*

Proof. If $(y^\vee, c, e)$ is duality data for y, then $\bigl(F(y^\vee), F(c), F(e)\bigr)$ is duality data for $F(y)$. This proves (1).

For (2) we claim that $\eta(y^\vee)^\vee$ is inverse to $\eta(y)$. Note that by Definition A.32, $\eta(y^\vee)^\vee$ is a map $G(y^\vee)^\vee \to F(y^\vee)^\vee$, and since $G(y^\vee) = G(y)^\vee$ it may be interpreted as a map $G(y) \to F(y)$. Let $c\colon 1_B \to y \otimes y^\vee$ and $e\colon y^\vee \otimes y \to 1_B$ be coevaluation and evaluation. Consider the diagram

(A.45)

$$\begin{tikzcd} G(y) \arrow[r, "\mathrm{id} \otimes F(c)"] \arrow[drr, bend right, "\mathrm{id} \otimes G(c)"'] & G(y)\otimes F(y^\vee)\otimes F(y) \arrow[r, "\mathrm{id}\otimes\eta(y^\vee)\otimes\mathrm{id}"] \arrow[dr, "\mathrm{id}\otimes\eta(y^\vee)\otimes\eta(y)"] & G(y)\otimes G(y^\vee)\otimes F(y) \arrow[r, "G(e)\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mathrm{id}\otimes\eta(y)"] & F(y) \arrow[d, "\eta(y)"] \\ & & G(y)\otimes G(y^\vee)\otimes G(y) \arrow[r, "G(e)\otimes\mathrm{id}"] & G(y) \end{tikzcd}$$

We claim it commutes. The left triangle commutes due to the naturality of η applied to the coevaluation $c\colon 1_B \to y \otimes y^\vee$. The next triangle and the right square commute trivially. Now starting on the left, the composition along the top and then down the right is the composition $\eta(y) \circ \eta(y^\vee)^\vee$. The composition diagonally down followed by the horizontal map is the identity, by G applied to the S-diagram relation (A.29) (and using (A.36)). A similar diagram proves that $\eta(y^\vee)^\vee \circ \eta(y) = \mathrm{id}$. ☐

A.4. Picard groupoids

Definition A.46. A *Picard groupoid* is a symmetric monoidal category in which all objects and morphisms are invertible.

Example A.47. Given a field k, there is a Picard groupoid Line_k whose objects are k-lines (one-dimensional vector spaces over k) and whose morphisms are isomorphisms of k-lines. Given a space X, there are Picard groupoids $\mathrm{Line}_{\mathbb{R}}(X)$ and $\mathrm{Line}_{\mathbb{C}}(X)$ of real and complex line bundles over X.

Definition A.48. Let C be a symmetric monoidal category. An *underlying Picard groupoid* is a pair $(C^\times, i)$ consisting of a Picard groupoid $C^\times$ and a functor $i\colon C^\times \to C$ which satisfies the universal property: If D is any Picard groupoid and $j\colon D \to C$ a symmetric monoidal functor, then there exists a unique functor $\tilde{j}\colon D \to C^\times$ such that the diagram

(A.49)
$$\begin{tikzcd} C^\times \arrow[rr, "i"] & & C \\ & D \arrow[ul, dashed, "\tilde{j}"] \arrow[ur, "j"'] & \end{tikzcd}$$

commutes.

We obtain $C^\times$ from C by discarding all non-invertible objects and non-invertible morphisms. The universal property implies that $C^\times$ is unique up to unique isomorphism.

Associated to a Picard groupoid D are abelian groups $\pi_0 D$, $\pi_1 D$ and a k-*invariant*

$$\pi_0 D \otimes \mathbb{Z}/2\mathbb{Z} \to \pi_1 D. \tag{A.50}$$

Define objects $y_0, y_1 \in D$ to be equivalent if there exists a morphism $y_0 \to y_1$. Then $\pi_0 D$ is the set of equivalence classes. The group law on $\pi_0 D$ is induced from the monoidal structure $\otimes$, and it is abelian since $\otimes$ is symmetric. Define $\pi_1 D = D(1_D, 1_D)$ as the automorphism group of the tensor unit. If $y \in D$ then there is an isomorphism

$$-\otimes \mathrm{id}_y\colon \mathrm{Aut}(1_D) \longrightarrow \mathrm{Aut}(y), \tag{A.51}$$

where we write $\mathrm{Aut}(y) = D(y, y)$. The k-invariant on y is the symmetry $\sigma\colon y \otimes y \to y \otimes y$, which is an element of $\mathrm{Aut}(y \otimes y) \cong \mathrm{Aut}(1_D) = \pi_1 D$. We leave the reader to verify that the k-invariant determines a homomorphism $\pi_0 D \otimes \mathbb{Z}/2\mathbb{Z} \to \pi_1 D$.

A.5. Involutions

DEFINITION A.52. Let C be a category.

(1) An *involution* of C is a pair (τ, η) of a functor $\tau\colon C \to C$ and a natural isomorphism $\eta\colon \mathrm{id}_C \to \tau^2$ such that for any $y \in C$ we have $\tau\eta_y = \eta_{\tau y}$ as morphisms $\tau y \to \tau^3 y$.

(2) A *fixed point* of τ is a pair (y, θ) of an object $y \in C$ and an isomorphism $y \xrightarrow{\theta} \tau y$ such that $\tau\theta \circ \theta = \eta_y$ as morphisms $y \to \tau^2 y$.

If C is a symmetric monoidal category, then the involution τ is required to be a symmetric monoidal functor: for $y_1, y_2 \in C$ there is given an isomorphism $\tau y_1 \otimes \tau y_2 \xrightarrow{\cong} \tau(y_1 \otimes y_2)$ and these isomorphisms are compatible with the symmetry and with η.

EXAMPLE A.53. Let $C = \mathrm{Vect}_{\mathbb{C}}$ be the category of complex vector spaces and linear maps. Define $\tau\colon C \to C$ to be the functor which takes complex vector spaces and linear maps to their complex conjugates. (The complex conjugate vector space is the same underlying real vector space with the sign of multiplication by $\sqrt{-1} \in \mathbb{C}$ reversed; the complex conjugate of a linear map is the same map of sets.) Then there is a canonical identification of τ^2 with id_C. A fixed point is a complex vector space with a real structure. As a variation, if $C = s\mathrm{Vect}_{\mathbb{C}}$ is the category of super ($\mathbb{Z}/2\mathbb{Z}$-graded) vector spaces and τ complex conjugation as above, but now η is composed with the exponentiated grading automorphism (denoted '$(-1)^F$' in the physics literature), then a fixed point is a super vector space with a real structure on its even part and a quaternionic structure on its odd part. If we restrict to the subgroupoid $C^\times$ of super lines and isomorphisms, then all fixed points are even.

DEFINITION A.54. Let (τ, η) be an involution on a category C. The *fixed point category* C^τ has as objects fixed points (y, θ), and a morphism $(y, \theta) \to (y', \theta')$ in C^τ is a morphism $(y \xrightarrow{f} y') \in C$ such that the diagram

$$\begin{tikzcd} y \arrow[r, "f"] \arrow[d, "\theta"'] & y' \arrow[d, "\theta'"] \\ \tau y \arrow[r, "\tau f"] & \tau y' \end{tikzcd} \tag{A.55}$$

commutes. There is a *forgetful functor* $C^\tau \to C$ which maps $(y, \theta) \mapsto y$.

EXAMPLE A.56. Let $\mathcal{C}$ be the *groupoid* of $\mathbb{Z}(1)$-torsors:[111] an object T is a set with a simply transitive action of the additive group $\mathbb{Z}(1)$ and a morphism $T \to T'$

[111] Recall that $\mathbb{Z}(1) = 2\pi\sqrt{-1}\mathbb{Z} \subset \mathbb{C}$.

is an isomorphism which commutes with the $\mathbb{Z}(1)$-actions. Let τ be the involution which sends a torsor T to its dual $\operatorname{Hom}_{\mathbb{Z}(1)}(T, \mathbb{Z}(1))$ and sends a morphism to its inverse transpose. The dual of T may be identified with T as a set; the dual $\mathbb{Z}(1)$ action by $\zeta \in \mathbb{Z}(1)$ is the original action by $\bar{\zeta}$. The fixed point category $\mathcal{C}^\tau$ is equivalent to the set $\mathbb{Z}/2\mathbb{Z}$: there are two isomorphism classes of objects and no nontrivial automorphisms. The first, which we call 'Type P', is the torsor $\mathbb{Z}(1)$ with complex conjugation θ as a map to the dual torsor. The second, which we call 'Type N', is the torsor $\pi\sqrt{-1} + \mathbb{Z}(1)$ with complex conjugation θ. Observe that in the Type P case the involution θ has a fixed point whereas in the Type N case it does not. Also, $\mathbb{Z}(1)$-torsors form a Picard groupoid, as do torsors for any abelian group, and the fixed point category is a Picard groupoid as well. The Type P torsor is the tensor unit; the square of a Type N torsor has Type P. The names derive from the family $\exp\colon \mathbb{C} \to \mathbb{C}^\times$ of $\mathbb{Z}(1)$-torsors with complex conjugation acting. There are two components $\mathbb{R}^{>0}$ and $\mathbb{R}^{<0}$ of fixed points in the base. The fiber of exp has Type P over positive real numbers and Type N over negative real numbers; the representatives described above are $\exp^{-1}(+1)$ and $\exp^{-1}(-1)$, respectively.

Definition A.57. Let B, C be categories with involutions and $F\colon B \to C$ a functor. Then *equivariance data* for F is an isomorphism $\phi\colon F\tau_B \xrightarrow{\cong} \tau_C F$ of functors $B \to C$ such that for every object $y \in B$ the diagram

$$\begin{tikzcd} Fy \arrow[r, "F\eta_B"] \arrow[dr, "\eta_C"'] & F\tau_B^2 y \arrow[d, "\phi^2"] \\ & \tau_C^2 F y \end{tikzcd} \tag{A.58}$$

commutes.

There are additional compatibilities for a symmetric monoidal functor between symmetric monoidal categories; we do not spell them out. We often loosely say that "F is an equivariant functor", but it is important to remember that equivariance is data+condition, not simply a condition.

Recall Definition A.28 of duals in a symmetric monoidal category. Assuming all objects have duals, we can make choices of duality data for all objects at once and so obtain a duality involution δ on C, but δ does not satisfy Definition A.52 since the direction of morphisms is reversed (A.33); in other words, δ is a functor to the *opposite* category.

Definition A.59. Let C be a category.

(1) A *twisted involution* of C is a pair (δ, η) of a functor $\delta\colon C \to C^{\mathrm{op}}$ and a natural isomorphism $\eta\colon \mathrm{id}_C \to \delta^{\mathrm{op}} \circ \delta$ such that for any $y \in C$ we have $\delta\eta_y \circ \eta_{\delta y} = \mathrm{id}_{\delta y}$.

(2) A *fixed point* of δ is a pair (y, θ) of an object $y \in C$ and an isomorphism $y \xrightarrow{\theta} \delta y$ such that $\delta\theta \circ \eta_y = \theta$ as morphisms $y \to \delta y$.

Definition A.54 applies with a single change: the direction of the bottom arrow in (A.55) is reversed.

Example A.60. For $C = \mathrm{Vect}_{\mathbb{C}}$ the duality involution $\delta\colon C \to C^{\mathrm{op}}$ maps a vector space V to its dual V^* and a linear map $f\colon V \to W$ to $f^*\colon W^* \to V^*$.

A fixed point of δ is a vector space V equipped with a nondegenerate symmetric bilinear form; a linear map $f\colon V \to W$ in C^{δ} preserves the bilinear forms. A fixed point for the composite of duality and complex conjugation (Example A.53) is a complex vector space V with a nondegenerate Hermitian form; a linear map $f\colon V \to W$ in the fixed point category is a partial isometry—an injective map which preserves the Hermitian forms.

REMARK A.61. There is a higher categorical context for Definition A.59. Let Cat denote the 2-category of categories. There is an involution $\alpha\colon$ Cat $\to$ Cat which sends a category C to its opposite C^{op}. (There is an extra categorical layer over Definition A.52 since Cat is a 2-category: an involution on Cat is a triple (α, η_1, η_2) of data and a single condition.) A twisted involution in the sense of Definition A.59 is fixed point data for α.

DEFINITION A.62. Let (τ, η) be an involution on a symmetric monoidal category C. A *Hermitian structure* on an object $y \in C$ is an isomorphism $h\colon \tau y \to y^{\vee}$ such that the composition

$$\tau y \cong \tau\big((y^{\vee})^{\vee}\big) \xrightarrow{\tau(h^{\vee})} \tau\big((\tau y)^{\vee}\big) \cong \tau^2(y^{\vee}) \xrightarrow{\eta^{-1}} y^{\vee} \tag{A.63}$$

is equal to h.

Proposition 7.27 asserts that every object in the bordism category of H_n-manifolds has a canonical Hermitian structure. Observe that if $F\colon B \to C$ is an equivariant symmetric monoidal functor between symmetric monoidal categories with involution, as in Definition 1.43, then the image of a Hermitian structure on an object $b \in B$ is a Hermitian structure on Fb.

Bibliography

[A1] M. Atiyah, *Topological quantum field theories*, Inst. Hautes Études Sci. Publ. Math. **68** (1988), 175–186 (1989). MR1001453

[A2] M. Atiyah, *On framings of 3-manifolds*, Topology **29** (1990), no. 1, 1–7, DOI 10.1016/0040-9383(90)90021-B. MR1046621

[A3] M. Atiyah, *The logarithm of the Dedekind η-function*, Math. Ann. **278** (1987), no. 1-4, 335–380, DOI 10.1007/BF01458075. MR909232

[Ab] L. Abrams, *Two-dimensional topological quantum field theories and Frobenius algebras*, J. Knot Theory Ramifications **5** (1996), no. 5, 569–587, DOI 10.1142/S0218216596000333. MR1414088

[ABHH] J. E. Andersen, H. U. Boden, A. Hahn, and B. Himpel (eds.), *Chern-Simons gauge theory: 20 years after*, AMS/IP Studies in Advanced Mathematics, vol. 50, American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2011. Papers from the workshop held in Bonn, August 3–7, 2009. MR2798328

[ABP1] D. W. Anderson, E. H. Brown Jr., and F. P. Peterson, *The structure of the Spin cobordism ring*, Ann. of Math. (2) **86** (1967), 271–298, DOI 10.2307/1970690. MR0219077

[ABP2] D. W. Anderson, E. H. Brown Jr., and F. P. Peterson, *Pin cobordism and related topics*, Comment. Math. Helv. **44** (1969), 462–468, DOI 10.1007/BF02564545. MR0261613

[ABS] M. F. Atiyah, R. Bott, and A. Shapiro, *Clifford modules*, Topology **3** (1964), no. suppl. 1, 3–38, DOI 10.1016/0040-9383(64)90003-5. MR0167985

[AF] D. Ayala and J. Francis, *The cobordism hypothesis*, `arXiv:1705.02240`.

[AFG] D. Ayala, D. S. Freed, and R. E. Grady (eds.), *Topology and quantum theory in interaction*, Contemporary Mathematics, vol. 718, American Mathematical Society, Providence, RI, 2018. NSF-CBMS Regional Conference in the Mathematical Sciences Topological and Geometric Methods in QFT, July 31–August 4, 2017, Montana State University, Bozeman, Montana. MR3869638

[AFR] D. Ayala, J. Francis, and N. Rozenblyum, *Factorization homology I: Higher categories*, Adv. Math. **333** (2018), 1042–1177, DOI 10.1016/j.aim.2018.05.031. MR3818096

[AgG] L. Alvarez-Gaumé and P. Ginsparg, *The structure of gauge and gravitational anomalies*, Ann. Physics **161** (1985), no. 2, 423–490, DOI 10.1016/0003-4916(85)90087-9. MR793821

[AgVm] L. Álvarez-Gaumé and M. Á. Vázquez-Mozo, *An invitation to quantum field theory*, Lecture Notes in Physics, vol. 839, Springer, Heidelberg, 2012. MR3014473

[AgW] L. Alvarez-Gaumé and E. Witten, *Gravitational anomalies*, Nuclear Phys. B **234** (1984), no. 2, 269–330, DOI 10.1016/0550-3213(84)90066-X. MR736803

[AH] M. F. Atiyah and F. Hirzebruch, *Riemann-Roch theorems for differentiable manifolds*, Bull. Amer. Math. Soc. **65** (1959), 276–281, DOI 10.1090/S0002-9904-1959-10344-X. MR0110106

[AlNa] A. Alexeevski and S. Natanzon, *Noncommutative two-dimensional topological field theories and Hurwitz numbers for real algebraic curves*, Selecta Math. (N.S.) **12** (2006), no. 3-4, 307–377, `arXiv:math/0202164`.

[ALW] D. Aasen, E. Lake, and K. Walker, *Fermion condensation and super pivotal categories*, `arXiv:1709.01941`.

[An] D. W. Anderson, *Universal coefficient theorems for K-theory*. mimeographed notes, University of California at Berkeley (1969).

[APS] M. F. Atiyah, V. K. Patodi, and I. M. Singer, *Spectral asymmetry and Riemannian geometry. I*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 43–69, DOI 10.1017/S0305004100049410. MR0397797

[Ar] V. I. Arnol′d, *Mathematical methods of classical mechanics*, 2nd ed., Graduate Texts in Mathematics, vol. 60, Springer-Verlag, New York, 1989. Translated from the Russian by K. Vogtmann and A. Weinstein. MR997295

[AS1] M. F. Atiyah and I. M. Singer, *The index of elliptic operators. V*, Ann. of Math. (2) **93** (1971), 139–149, DOI 10.2307/1970757. MR0279834

[AS2] M. F. Atiyah and I. M. Singer, *The index of elliptic operators. I*, Ann. of Math. (2) **87** (1968), 484–530, DOI 10.2307/1970715. MR0236950

[AS3] M. F. Atiyah and I. M. Singer, *The index of elliptic operators. III*, Ann. of Math. (2) **87** (1968), 546–604, DOI 10.2307/1970717. MR0236952

[AS4] M. F. Atiyah and I. M. Singer, *Dirac operators coupled to vector potentials*, Proc. Nat. Acad. Sci. U.S.A. **81** (1984), no. 8, , Phys. Sci., 2597–2600, DOI 10.1073/pnas.81.8.2597. MR742394

[AS5] M. F. Atiyah and I. M. Singer, *The index of elliptic operators. IV*, Ann. of Math. (2) **93** (1971), 119–138, DOI 10.2307/1970756. MR0279833

[AtBoPa] M. Atiyah, R. Bott, and V. K. Patodi, *On the heat equation and the index theorem*, Invent. Math. **19** (1973), 279–330, DOI 10.1007/BF01425417. MR0650828

[Ay] D. Ayala, *Geometric cobordism categories*, ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)–Stanford University. MR2713365

[AZ] A. Altland and M. R. Zirnbauer, *Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures*, Phys.Rev. **B55** (1997), 1142–1161.

[B] R. A. Bertlmann, *Anomalies in quantum field theory: dispersion relations and differential geometry*, Nuclear Phys. B Proc. Suppl. **39BC** (1995), 482–487, DOI 10.1016/0920-5632(95)00122-P. QCD 94 (Montpellier, 1994). MR1373240

[BB] G. Birkhoff and M. K. Bennett, *Felix Klein and his "Erlanger Programm"*, History and philosophy of modern mathematics (Minneapolis, MN, 1985), Minnesota Stud. Philos. Sci., XI, Univ. Minnesota Press, Minneapolis, MN, 1988, pp. 145–176.

[BBBDN] C. Beem, D. Ben-Zvi, M. Bullimore, T. Dimofte, and A. Neitzke, *Secondary products in supersymmetric field theory*, `arXiv:1809.00009`.

[BC] E. H. Brown Jr. and M. Comenetz, *Pontrjagin duality for generalized homology and cohomology theories*, Amer. J. Math. **98** (1976), no. 1, 1–27, DOI 10.2307/2373610. MR0405403

[BD] J. C. Baez and J. Dolan, *Higher-dimensional algebra and topological quantum field theory*, J. Math. Phys. **36** (1995), no. 11, 6073–6105, DOI 10.1063/1.531236. MR1355899

[BeCa] A. Beaudry and J. A. Campbell, *A guide for computing stable homotopy groups*, Topology and Quantum Theory in Interaction (Ryan E. Grady David Ayala, Daniel S. Freed, ed.), Contemporary Mathematics, vol. 718, American Mathematical Society, 2018, pp. 89–136. `arXiv:1801.07530`.

[BeDr] A. Beilinson and V. Drinfeld, *Chiral algebras*, American Mathematical Society Colloquium Publications, vol. 51, American Mathematical Society, Providence, RI, 2004. MR2058353

[BF1] J.-M. Bismut and D. S. Freed, *The analysis of elliptic families. I. Metrics and connections on determinant bundles*, Comm. Math. Phys. **106** (1986), no. 1, 159–176. MR853982

[BF2] J.-M. Bismut and D. S. Freed, *The analysis of elliptic families. II. Dirac operators, eta invariants, and the holonomy theorem*, Comm. Math. Phys. **107** (1986), no. 1, 103–163. MR861886

[BG] U. Bunke and D. Gepner, *Differential function spectra, the differential Becker-Gottlieb transfer, and applications to differential algebraic K-theory*, `arXiv:1306.0247`.

[BGN] D. Ben-Zvi, S. Gunningham, and D. Nadler, *The character field theory and homology of character varieties*, `arXiv:1705.04266` (2017); Mathematical Research Letters (to appear).

[BHMV] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel, *Topological quantum field theories derived from the Kauffman bracket*, Topology **34** (1995), no. 4, 883–927, DOI 10.1016/0040-9383(94)00051-4. MR1362791

[BJS] A. Brochier, D. Jordan, and N. Snyder, *On dualizability of braided tensor categories*, `arXiv:1804.07538`.

[BM] M. Bökstedt and I. Madsen, *The cobordism category and Waldhausen's K-theory*, An alpine expedition through algebraic topology, Contemp. Math., vol. 617, Amer. Math. Soc., Providence, RI, 2014, pp. 39–80, DOI 10.1090/conm/617/12282. MR3243393

[BN] D. Ben-Zvi and D. Nadler, *The Character Theory of a Complex Group*, `arXiv:0904.1247`. preprint.

[BR1] J. E. Bergner and C. Rezk, *Comparison of models for (∞, n)-categories, I*, Geom. Topol. **17** (2013), no. 4, 2163–2202, DOI 10.2140/gt.2013.17.2163. MR3109865

[BR2] J. E. Bergner and C. Rezk, *Comparison of models for (∞, n)-categories, II*, `arXiv:1406.4182`.

[BT] R. Bott and L. W. Tu, *Differential forms in algebraic topology*, Graduate Texts in Mathematics, vol. 82, Springer-Verlag, New York-Berlin, 1982. MR658304

[Bu] U. Bunke, *Transgression of the index gerbe*, Manuscripta Math. **109** (2002), no. 3, 263–287, `arXiv:math/0109052`.

[C1] K. Costello, *Renormalization and effective field theory*, Mathematical Surveys and Monographs, vol. 170, American Mathematical Society, Providence, RI, 2011. MR2778558

[C2] K. Costello, *Topological conformal field theories and Calabi-Yau categories*, Adv. Math. **210** (2007), no. 1, 165–214, `arXiv:math/0412149`.

[Ca] F. Catanese, *On the moduli spaces of surfaces of general type*, J. Differential Geom. **19** (1984), no. 2, 483–515. MR755236

[Cam] J. A. Campbell, *Homotopy Theoretic Classification of Symmetry Protected Phases*, `arXiv:1708.04264`.

[Cas] D. Castelvecchi, *The strange topology that is reshaping physics*, July 2017, pp. 272–274. `https://www.nature.com/news/the-strange-topology-that-is-reshaping-physics-1.22316`.

[CaSc] D. Calaque and C. Scheimbauer, *A note on the (∞,n)–category of cobordisms*, Algebr. Geom. Topol. **19** (2019), no. 2, 533–655, DOI 10.2140/agt.2019.19.533. MR3924174

[Ce] J. Cerf, *La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudo-isotopie* (French), Inst. Hautes Études Sci. Publ. Math. **39** (1970), 5–173. MR0292089

[CFLS] H. Lam, N. Seiberg, C. Córdova, and D. S. Freed, *Anomalies in the space of coupling constants and their dynamical applications, I*, `arXiv:1905.09315` (2019); *Part II*, `arXiv:1905.13361` (2019).

[CG] K. Costello and O. Gwilliam, *Factorization algebras in quantum field theory. Vol. 1*, New Mathematical Monographs, vol. 31, Cambridge University Press, Cambridge, 2017. MR3586504

[CGW] X. Chen, Z.-C. Gu, and X.-G. Wen, *Local unitary transformation, long-range quantum entanglement, wave function renormalization, and topological order*, Phys. Rev. B **82** (2010), 155138, `arXiv:1004.3835`.

[CheeS] J. Cheeger and J. Simons, *Differential characters and geometric invariants*, Geometry and topology (College Park, Md., 1983/84), Lecture Notes in Math., vol. 1167, Springer, Berlin, 1985, pp. 50–80, DOI 10.1007/BFb0075216. MR827262

[CHS] C. Córdova, P.-S. Hsin, and N. Seiberg, *Time-Reversal Symmetry, Anomalies, and Dualities in (2+1)d*, SciPost Phys. **5** (2018), no. 1, 006, `arXiv:1712.08639`.

[CM] S. Coleman and J. Mandula, *All possible symmetries of the S matrix*, Physical Review **159** (1967), no. 5, 1251–56.

[CS] S. S. Chern and J. Simons, *Characteristic forms and geometric invariants*, Ann. of Math. (2) **99** (1974), 48–69, DOI 10.2307/1971013. MR0353327

[D] P. A. M. Dirac, *The Principles of Quantum Mechanics*, Oxford, at the Clarendon Press, 1947. 3d ed. MR0023198

[DaFr] X. Dai and D. S. Freed, *η-invariants and determinant lines*, J. Math. Phys. **35** (1994), no. 10, 5155–5194, DOI 10.1063/1.530747. Topology and physics. MR1295462

[Dav] O. Davidovich, *State sums in 2-dimensional fully extended topological field theories*, Ph.D. thesis, University of Texas at Austin, 2011. `http://repositories.lib.utexas.edu/handle/2152/ETD-UT-2011-05-3139`.

[De] P. Deligne, *Notes on spinors*, Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997), Amer. Math. Soc., Providence, RI, 1999, pp. 99–135. MR1701598

[Deb] A. Debray, *The low-energy TQFT of the generalized double semion model*, arXiv:1811.03583.

[DeGu] A. Debray and S. Gunningham, *The Arf-Brown TQFT of pin$^-$ surfaces*, Topology and quantum theory in interaction, Contemp. Math., vol. 718, Amer. Math. Soc., Providence, RI, 2018, pp. 49–87. arXiv:1803.11183.

[DeM] P. Deligne and J. W. Morgan, *Notes on supersymmetry (following Joseph Bernstein)*, Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997), Amer. Math. Soc., Providence, RI, 1999, pp. 41–97. MR1701597

[DF1] P. Deligne and D. S. Freed, *Classical field theory*, Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997), Amer. Math. Soc., Providence, RI, 1999, pp. 137–225. MR1701599

[DF2] P. Deligne and D. S. Freed, *Sign manifesto*, Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997), Amer. Math. Soc., Providence, RI, 1999, pp. 357–363.

[Dij] R. Dijkgraaf, *A geometrical approach to two-dimensional conformal field theory*, igitur-archive.library.uu.nl/dissertations/2011-0929-200347/UUindex.html. Ph.D. thesis.

[DJT] S. Deser, R. Jackiw, and S. Templeton, *Three-Dimensional Massive Gauge Theories*, Phys. Rev. Lett. **48** (1982), 975–978.

[DM] P. Deligne and J. W. Morgan, *Notes on supersymmetry (following Joseph Bernstein)*, Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997), Amer. Math. Soc., Providence, RI, 1999, pp. 41–97. MR1701597

[DSPS] C. L. Douglas, C. Schommer-Pries, and N. Snyder, *Dualizable tensor categories*, arXiv:1312.7188.

[DW] R. Dijkgraaf and E. Witten, *Topological gauge theories and group cohomology*, Comm. Math. Phys. **129** (1990), no. 2, 393–429. MR1048699

[Dy] F. J. Dyson, *The threefold way. Algebraic structure of symmetry groups and ensembles in quantum mechanics*, J. Mathematical Phys. **3** (1962), 1199–1215, DOI 10.1063/1.1703863. MR0177643

[E] J. Ebert, *A vanishing theorem for characteristic classes of odd-dimensional manifold bundles*, J. Reine Angew. Math. **684** (2013), 1–29, arXiv:0902.4719.

[EGNO] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik, *Tensor categories*, Mathematical Surveys and Monographs, vol. 205, American Mathematical Society, Providence, RI, 2015. MR3242743

[F1] D. S. Freed, *Commentary on "Lectures on Morse theory, old and new" [comment on the reprint of MR0663786]*, Bull. Amer. Math. Soc. (N.S.) **48** (2011), no. 4, 517–523, DOI 10.1090/S0273-0979-2011-01349-0. MR2823021

[F2] D. S. Freed, *Characteristic numbers and generalized path integrals*, Geometry, topology, & physics, Conf. Proc. Lecture Notes Geom. Topology, IV, Int. Press, Cambridge, MA, 1995, pp. 126–138. MR1358615

[F3] D. S. Freed, *Higher algebraic structures and quantization*, Comm. Math. Phys. **159** (1994), no. 2, 343–398. MR1256993

[F4] D. S. Freed, *Extended structures in topological quantum field theory*, Quantum topology, Ser. Knots Everything, vol. 3, World Sci. Publ., River Edge, NJ, 1993, pp. 162–173, DOI 10.1142/9789812796387_0008. MR1273572

[F5] D. S. Freed, *The cobordism hypothesis*, Bull. Amer. Math. Soc. (N.S.) **50** (2013), no. 1, 57–92, arXiv:1210.5100.

[F6] D. S. Freed, *Remarks on Chern-Simons theory*, Bull. Amer. Math. Soc. (N.S.) **46** (2009), no. 2, 221–254, arXiv:0808.2507.

[F7] Y.-T. Siu, *A new bound for the effective Matsusaka big theorem*, Houston J. Math. **28** (2002), no. 2, 389–409. Special issue for S. S. Chern. MR1898197

[F8] S.-T. Yau (ed.), *Mathematical aspects of string theory*, Advanced Series in Mathematical Physics, vol. 1, World Scientific Publishing Co., Singapore, 1987. MR915812

[F9] R. Kirby, V. Krushkal, and Z. Wang (eds.), *Proceedings of the Freedman Fest*, Geometry & Topology Monographs, vol. 18, Geometry & Topology Publications, Coventry, 2012. Papers from the Conference on Low-Dimensional Manifolds and High-Dimensional Categories held in honor of Michael Hartley Freedman at the University of California, Berkeley, CA, June 6–10, 2011, and the Freedman Symposium held in Santa Barbara, CA, April 15–17, 2011. MR3137657

[F10] R. Kirby, V. Kurshkal, and Z. Wang (eds.), *Anomalies and invertible field theories*, String-Math 2013, Proc. Sympos. Pure Math., vol. 88, Amer. Math. Soc., Providence, RI, 2014, pp. 25–45. `arXiv:1404.7224`.

[F11] R. Kirby, V. Kurshkal, and Z. Wang (eds.), *On equivariant Chern-Weil forms and determinant lines*, Surveys in Differential Geometry 2017 (Richard M. Schoen Shing-Tung Yau Huai-Dong Cao, Jun Li, ed.), vol. XXII, International Press, Somerville, MA, 2018, pp. 125–132. `arXiv:1606.01129`.

[Fa1] L. Faddeev, *Elementary introduction to quantum field theory*, Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997), Amer. Math. Soc., Providence, RI, 1999, pp. 513–550. MR1701606

[Fa2] L. Faddeev, *Hamiltonian approach to the theory of anomalies*, Recent Developments in Mathematical physics (H. Mitter and L. Pittner, eds.), Internationale Universitatswoche fur Kernphysik, Schladming, Austria, vol. 26, 1987, pp. 137–159.

[FCV] L. Fidkowski, X. Chen, and A. Vishwanath, *Non-Abelian Topological Order on the Surface of a 3D Topological Superconductor from an Exactly Solved Model*, Phys. Rev. **X3** (2013), no. 4, 041016, `arXiv:1305.5851`.

[FH1] D. S. Freed and M. J. Hopkins, *Reflection positivity and invertible topological phases.* `arXiv:1604.06527`.

[FH2] D. S. Freed and M. J. Hopkins, *Consistency of M-theory on unorientable manifolds.* in preparation.

[FH3] D. S. Freed and M. J. Hopkins, *Invertible phases of matter with spatial symmetry*, `arXiv:1901.06419` (2019).

[FHa] M. H. Freedman and M. B. Hastings, *Double semions in arbitrary dimension*, Comm. Math. Phys. **347** (2016), no. 2, 389–419, `arXiv:1507.05676`.

[FHLT] D. S. Freed, M. J. Hopkins, J. Lurie, and C. Teleman, *Topological quantum field theories from compact Lie groups*, A celebration of the mathematical legacy of Raoul Bott, CRM Proc. Lecture Notes, vol. 50, Amer. Math. Soc., Providence, RI, 2010, pp. 367–403. MR2648901

[FHT] D. S. Freed, M. J. Hopkins, and C. Teleman, *Consistent orientation of moduli spaces*, The many facets of geometry, Oxford Univ. Press, Oxford, 2010, pp. 395–419. `arXiv:0711.1909`.

[FKM] L. Fu, C. L. Kane, and E. J. Mele, *Topological Insulators in Three Dimensions*, Phys. Rev. Lett. **98** (2007), 106803, `arXiv:cond-mat/0607699`.

[FKS] D. S. Freed, Z. Komargodski, and N. Seiberg, *The sum over topological sectors and θ in the $2+1$-dimensional $\mathbb{CP}^1\sigma$-model*, Comm. Math. Phys. **362** (2018), no. 1, 167–183, DOI 10.1007/s00220-018-3093-0. MR3833607

[FL] D. S. Freed and J. Lott, *An index theorem in differential K-theory*, Geom. Topol. **14** (2010), no. 2, 903–966, `arXiv:0907.3508`.

[FM1] D. S. Freed and G. W. Moore, *Twisted equivariant matter*, Ann. Henri Poincaré **14** (2013), no. 8, 1927–2023, `arXiv:1208.5055`.

[FM2] D. S. Freed and G. W. Moore, *Setting the quantum integrand of M-theory*, Commun. Math. Phys. **263** (2006), 89–132, `arXiv:hep-th/0409135`.

[FMS] D. S. Freed, G. W. Moore, and G. Segal, *The uncertainty of fluxes*, Commun. Math. Phys. **271** (2007), 247–274, `arXiv:hep-th/0605198`.

[FQ] D. S. Freed and F. Quinn, *Chern-Simons theory with finite gauge group*, Comm. Math. Phys. **156** (1993), no. 3, 435–472, `arXiv:hep-th/911100`.

[Fr] H. Freudenthal, *Über die Klassen der Sphärenabbildungen I. Große Dimensionen* (German), Compositio Math. **5** (1938), 299–314. MR1556999

[FS] K. Fujikawa and H. Suzuki, *Path integrals and quantum anomalies*, International Series of Monographs on Physics, vol. 122, The Clarendon Press, Oxford University Press, New York, 2004. Translated from the 2001 Japanese original. MR2077220

[FT1] D. S. Freed and C. Teleman, *Relative quantum field theory*, Comm. Math. Phys. **326** (2014), no. 2, 459–476, `arXiv:1212.1692`.

[FT2] D. S. Freed and C. Teleman, *Topological dualities in the Ising model*, `arXiv:1806.00008`.

[FW] D. S. Freed and E. Witten, *Anomalies in string theory with D-branes*, Asian J. Math. **3** (1999), no. 4, 819–851, DOI 10.4310/AJM.1999.v3.n4.a6. MR1797580

[G] D. Gaiotto, *Gapped phases of matter vs. Topological field theories*, July, 2017. `http://pirsa.org/17070066`. Lecture at Perimeter Institute (Hopf Algebras in Kitaev's Quantum Double Models: Mathematical Connections from Gauge Theory to Topological Quantum Computing and Categorical Quantum Mechanics).

[GJ] J. Glimm and A. Jaffe, *Quantum physics*, 2nd ed., Springer-Verlag, New York, 1987. A functional integral point of view. MR887102

[GK] D. Gaiotto and A. Kapustin, *Spin TQFTs and fermionic phases of matter*, International Journal of Modern Physics A **31** (2016), no. 28n29, 1645044, `arXiv:1505.05856`.

[GKKS] D. Gaiotto, A. Kapustin, Z. Komargodski, and N. Seiberg, *Theta, time reversal and temperature*, J. High Energy Phys. **5** (2017), 091, front matter+49, DOI 10.1007/JHEP05(2017)091. MR3662840

[GKSW] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, *Generalized global symmetries*, J. High Energy Phys. **2** (2015), 172, front matter+61, DOI 10.1007/JHEP02(2015)172. MR3321281

[GPW] M. Guo, P. Putrov, and J. Wang, *Time reversal, $SU(N)$ Yang-Mills and cobordisms: interacting topological superconductors/insulators and quantum spin liquids in $3+1D$*, Ann. Physics **394** (2018), 244–293, DOI 10.1016/j.aop.2018.04.025. MR3812704

[Gu] S. Gunningham, *Spin Hurwitz numbers and topological quantum field theory*, Geom. Topol. **20** (2016), no. 4, 1859–1907, DOI 10.2140/gt.2016.20.1859. MR3548460

[H] M. W. Hirsch, *Differential topology*, Graduate Texts in Mathematics, vol. 33, Springer-Verlag, New York, 1994. Corrected reprint of the 1976 original. MR1336822

[Ha] R. Haag, *Local quantum physics*, 2nd ed., Texts and Monographs in Physics, Springer-Verlag, Berlin, 1996. Fields, particles, algebras. MR1405610

[HeSt] D. Heard and V. Stojanoska, *K-theory, reality, and duality*, J. K-Theory **14** (2014), no. 3, 526–555, `arXiv:1401.2581`.

[HHR] M. A. Hill, M. J. Hopkins, and D. C. Ravenel, *On the nonexistence of elements of Kervaire invariant one*, Ann. of Math. (2) **184** (2016), no. 1, 1–262, `arXiv:0908.3724`.

[HHZ] P. Heinzner, A. Huckleberry, and M.R. Zirnbauer, *Symmetry Classes of Disordered Fermions*, Communications in Mathematical Physics **257** (2005), no. 3, 725–771, `arXiv:math-ph/0411040`.

[HL] G. Heuts and J. Lurie, *Ambidexterity*, Topology and field theories, Contemp. Math., vol. 613, Amer. Math. Soc., Providence, RI, 2014, pp. 79–110, DOI 10.1090/conm/613/12236. MR3221291

[HS] M. J. Hopkins and I. M. Singer, *Quadratic functions in geometry, topology, and M-theory*, J. Differential Geom. **70** (2005), no. 3, 329–452. MR2192936

[HST] H. Hohnhold, S. Stolz, and P. Teichner, *From minimal geodesics to supersymmetric field theories*, A celebration of the mathematical legacy of Raoul Bott, CRM Proc. Lecture Notes, vol. 50, Amer. Math. Soc., Providence, RI, 2010, pp. 207–274. MR2648897

[JS] A. Joyal and R. Street, *Braided tensor categories*, Adv. Math. **102** (1993), no. 1, 20–78, DOI 10.1006/aima.1993.1055. MR1250465

[JW] A. Jaffe and E. Witten, *Quantum Yang-Mills theory*, The millennium prize problems, Clay Math. Inst., Cambridge, MA, 2006, pp. 129–152. MR2238278

[K] D. Kazhdan, *Introduction to QFT*, Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997), Amer. Math. Soc., Providence, RI, 1999, pp. 377–418. MR1701603

[Ka1] A. Kapustin, *Topological field theory, higher categories, and their applications*, Proceedings of the International Congress of Mathematicians. Volume III, Hindustan Book Agency, New Delhi, 2010, pp. 2021–2043. MR2827874

[Ka2] A. Kapustin, *Is quantum mechanics exact?*, J. Math. Phys. **54** (2013), no. 6, 062107, 15, `arXiv:1303.6917`.

[Ka3] A. Kapustin, *Symmetry protected topological phases, anomalies, and cobordisms: beyond group cohomology*, `arXiv:1403.1467`.

[Kap] M. Kapranov, *Supergeometry in mathematics and physics*, `arXiv:1512.07042`.

[Ke] M. Kervaire, *Courbure intégrale généralisée et homotopie* (French), Math. Ann. **131** (1956), 219–252, DOI 10.1007/BF01342961. MR0086302

[Kh] M. Khovanov, *A categorification of the Jones polynomial*, Duke Math. J. **101** (2000), no. 3, 359–426, `arXiv:math/9908171`.

[Ki1] A. Yu. Kitaev, *Fault-tolerant quantum computation by anyons*, Ann. Physics **303** (2003), no. 1, 2–30, DOI 10.1016/S0003-4916(02)00018-0. MR1951039

[Ki2] A. Yu. Kitaev, *Toward Topological Classification of Phases with Short-range Entanglement*, 2011. `http://online.kitp.ucsb.edu/online/topomat11/kitaev/`. Lecture at KITP.

[Ki3] A. Yu. Kitaev, *Periodic table for topological insulators and superconductors*, AIP Conf.Proc. **1134** (2009), 22–30, `arXiv:0901.2686`.

[Ki4] A. Yu. Kitaev, *Homotopy-theoretic approach to SPT phases in action: $\mathbb{Z}/16\mathbb{Z}$ classification of three-dimensional superconductors*, January 2015. `http://www.ipam.ucla.edu/abstract/?tid=12389&pcode=STQ2015`. talk at *Symmetry and Topology in Quantum Matter*, Institute for Pure and Applied Mathematics.

[Ki5] A. Yu. Kitaev, *On the Classification of Short-Range Entangled States*, June, 2013. `http://scgp.stonybrook.edu/archives/7874`. Lecture at SCGP.

[Klo] K. R. Klonoff, *An index theorem in differential K-theory*, ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)–The University of Texas at Austin. MR2711943

[KM] C. L. Kane and E. J. Mele, *Z_2 Topological Order and the Quantum Spin Hall Effect*, Phys. Rev. Lett. **95** (2005), 146802, `arXiv:cond-mat/0506581`.

[KMo] C. Kane and J. Moore, *Topological insulators*, Physics World **24** (2011), no. 02, 32.

[KN] S. Kobayashi and K. Nomizu, *Foundations of differential geometry. Vol. I*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1996. Reprint of the 1963 original; A Wiley-Interscience Publication. MR1393940

[Ko] J. Kock, *Frobenius algebras and 2D topological quantum field theories*, London Mathematical Society Student Texts, vol. 59, Cambridge University Press, Cambridge, 2004. MR2037238

[KT1] R. C. Kirby and L. R. Taylor, Pin *structures on low-dimensional manifolds*, Geometry of low-dimensional manifolds, 2 (Durham, 1989), London Math. Soc. Lecture Note Ser., vol. 151, Cambridge Univ. Press, Cambridge, 1990, pp. 177–242. MR1171915

[KT2] R. C. Kirby and L. R. Taylor, *A calculation of* Pin^+ *bordism groups*, Comment. Math. Helv. **65** (1990), no. 3, 434–447, DOI 10.1007/BF02566617. MR1069818

[KTTW] A. Kapustin, R. Thorngren, A. Turzillo, and Z. Wang, *Fermionic symmetry protected topological phases and cobordisms*, J. High Energy Phys. **12** (2015), 052, front matter+20pp, DOI 10.1007/jhep12(2015)052. MR3464750

[Ku] H.-H. Kuo, *Introduction to stochastic integration*, Universitext, Springer, New York, 2006. MR2180429

[Ky] R. Kirby, *A calculus for framed links in S^3*, Invent. Math. **45** (1978), no. 1, 35–56, DOI 10.1007/BF01406222. MR0467753

[L1] J. Lurie, *On the classification of topological field theories*, Current developments in mathematics, 2008, Int. Press, Somerville, MA, 2009, pp. 129–280. MR2555928

[L2] J. Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR2522659

[La] R. J. Lawrence, *Triangulations, categories and extended topological field theories*, Quantum topology, Ser. Knots Everything, vol. 3, World Sci. Publ., River Edge, NJ, 1993, pp. 191–208, DOI 10.1142/9789812796387_0011. MR1273575

[LMP] G. Lusztig, J. Milnor, and F. P. Peterson, *Semi-characteristics and cobordism*, Topology **8** (1969), 357–359, DOI 10.1016/0040-9383(69)90021-4. MR0246308

[Lo] J. Lott, *Higher-degree analogs of the determinant line bundle*, Comm. Math. Phys. **230** (2002), no. 1, 41–69, `arXiv:math/0106177`.

[LV] Y.-M. Lu and A. Vishwanath, *Theory and classification of interacting integer topological phases in two dimensions: A Chern-Simons approach*, Phys. Rev. B **86** (2012), 125119, `arXiv:1205.3156`.

[M] M. A. Metlitski, *S-duality of $u(1)$ gauge theory with $\theta = \pi$ on non-orientable manifolds: Applications to topological insulators and superconductors*, `arXiv:1510.05663`.

[Ma] G. W. Mackey, *The mathematical foundations of quantum mechanics: A lecture-note volume*, W., A. Benjamin, Inc., New York-Amsterdam, 1963. MR0155567

[Mac] S. Mac Lane, *Categories for the working mathematician*, 2nd ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998. MR1712872

[MFCV] M. A. Metlitski, L. Fidkowski, X. Chen, and A. Vishwanath, *Interaction effects on 3D topological superconductors: surface topological order from vortex condensation, the 16 fold way and fermionic Kramers doublets*, arXiv:1406.3032.

[Mi1] J. W. Milnor, *Topology from the differentiable viewpoint*, Based on notes by David W. Weaver, The University Press of Virginia, Charlottesville, Va., 1965. MR0226651

[Mi2] J. Milnor, *Morse theory*, Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51, Princeton University Press, Princeton, N.J., 1963. MR0163331

[Mi3] J. Milnor, *Lectures on the h-cobordism theorem*, Notes by L. Siebenmann and J. Sondow, Princeton University Press, Princeton, N.J., 1965. MR0190942

[Mig] A. A. Migdal, *Recursion Equations in Gauge Theories*, Sov. Phys. JETP **42** (1975), 413. [Zh. Eksp. Teor. Fiz.69,810(1975)].

[Mo] G. W. Moore, *Quantum symmetries and K-theory*. http://www.physics.rutgers.edu/users/gmoore/QuantumSymmetryKTheory-Part1.pdf. notes from St. Ottilien lectures, July, 2012.

[Mos] J. Moser, *On the volume elements on a manifold*, Trans. Amer. Math. Soc. **120** (1965), 286–294, DOI 10.2307/1994022. MR0182927

[MS] P. S. Aspinwall, T. Bridgeland, A. Craw, M. R. Douglas, M. Gross, A. Kapustin, G. W. Moore, G. Segal, B. Szendrői, and P. M. H. Wilson, *Dirichlet branes and mirror symmetry*, Clay Mathematics Monographs, vol. 4, American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2009. MR2567952

[MSS] M. Markl, S. Shnider, and J. Stasheff, *Operads in algebra, topology and physics*, Mathematical Surveys and Monographs, vol. 96, American Mathematical Society, Providence, RI, 2002. MR1898414

[NAg] P. Nelson and L. Alvarez-Gaumé, *Hamiltonian interpretation of anomalies*, Comm. Math. Phys. **99** (1985), no. 1, 103–114. MR791642

[Nob] D. Tong, *The 2016 Nobel prize in physics*, Math. Today (Southend-on-Sea) **52** (2016), no. 6, 260–261. MR3587395

[O] M. L. Ortiz, *Differential equivariant K-theory*, ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)–The University of Texas at Austin. MR2713530

[OS] K. Osterwalder and R. Schrader, *Axioms for Euclidean Green's functions. II*, Comm. Math. Phys. **42** (1975), 281–305. With an appendix by Stephen Summers. MR0376002

[P] L. S. Pontryagin, *Gladkie mnogoobraziya i ikh primeneniya v teorii gomotopiĭ* (Russian), 2nd ed., Izdat. "Nauka", Moscow, 1976. MR0445517

[PT] R. S. Palais and C.-L. Terng, *Critical point theory and submanifold geometry*, Lecture Notes in Mathematics, vol. 1353, Springer-Verlag, Berlin, 1988. MR972503

[Q] D. Kvillen, *Determinants of Cauchy-Riemann operators on Riemann surfaces* (Russian), Funktsional. Anal. i Prilozhen. **19** (1985), no. 1, 37–41, 96. MR783704

[R] E. Riehl, *Category theory in context*, Courier Dover Publications, 2017.

[Re] B. L. Reinhart, *Cobordism and the Euler number*, Topology **2** (1963), 173–177, DOI 10.1016/0040-9383(63)90031-4. MR0153021

[RS] I. Runkel and L. Szegedy, *Area-dependent quantum field theory with defects*, arXiv:1807.08196.

[S] S. Sternberg, *Lectures on differential geometry*, 2nd ed., Chelsea Publishing Co., New York, 1983. With an appendix by Sternberg and Victor W. Guillemin. MR891190

[S-P1] C. J. Schommer-Pries, *The classification of two-dimensional extended topological field theories*, ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)–University of California, Berkeley. MR2713992

[S-P2] C. Schommer-Pries, *Invertible field theories*, arXiv:1712.08029.

[Sa] D. Ayala, D. S. Freed, and R. E. Grady (eds.), *Topology and quantum theory in interaction*, Contemporary Mathematics, vol. 718, American Mathematical Society, Providence, RI, 2018. NSF-CBMS Regional Conference in the Mathematical Sciences Topological and Geometric Methods in QFT, July 31–August 4, 2017, Montana State University, Bozeman, Montana. MR3869638

[Se1] G. Segal, *The definition of conformal field theory*, Topology, geometry and quantum field theory, London Math. Soc. Lecture Note Ser., vol. 308, Cambridge Univ. Press, Cambridge, 2004, pp. 421–577. MR2079383

[Se2] G. Segal, *Stanford Lectures, Lecture 1: Topological field theories*, `http://www.cgtp.duke.edu/ITP99/segal/stanford/lect1.pdf`.

[Se3] G. Segal, *Felix Klein Lectures 2011*. `http://www.mpim-bonn.mpg.de/node/3372/abstracts`.

[Se4] G. Segal, private conversations.

[Se5] G. Segal, *Faddeev's anomaly in Gauss's law*. preprint.

[SeWi] N. Seiberg and E. Witten, *Gapped boundary phases of topological insulators via weak coupling*, PTEP. Prog. Theor. Exp. Phys. **12** (2016), 12C101, 78, DOI 10.1093/ptep/ptw083. MR3628684

[Sny] N. Snyder, *Mednykh's formula via lattice topological quantum field theories*, Proceedings of the 2014 Maui and 2015 Qinhuangdao conferences in honour of Vaughan F. R. Jones' 60th birthday, Proc. Centre Math. Appl. Austral. Nat. Univ., vol. 46, Austral. Nat. Univ., Canberra, 2017, pp. 389–398. MR3635678

[ST] S. Stolz and P. Teichner, *Supersymmetric field theories and generalized cohomology*, Mathematical foundations of quantum field theory and perturbative string theory, Proc. Sympos. Pure Math., vol. 83, Amer. Math. Soc., Providence, RI, 2011, pp. 279–340, DOI 10.1090/pspum/083/2742432. MR2742432

[Str] F. Strocchi, *An introduction to the mathematical structure of quantum mechanics*, Advanced Series in Mathematical Physics, vol. 27, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005. A short course for mathematicians. MR2222127

[STY] N. Seiberg, Y. Tachikawa, and K. Yonekura, *Anomalies of duality groups and extended conformal manifolds*, PTEP. Prog. Theor. Exp. Phys. **7** (2018), 073B04, 27, DOI 10.1093/ptep/pty069. MR3842900

[SW] R. F. Streater and A. S. Wightman, *PCT, spin and statistics, and all that*, Princeton Landmarks in Physics, Princeton University Press, Princeton, NJ, 2000. Corrected third printing of the 1978 edition. MR1884336

[T] R. Thom, *Quelques propriétés globales des variétés différentiables* (French), Comment. Math. Helv. **28** (1954), 17–86, DOI 10.1007/BF02566923. MR0061823

[Ta] L. A. Takhtajan, *Quantum mechanics for mathematicians*, Graduate Studies in Mathematics, vol. 95, American Mathematical Society, Providence, RI, 2008. MR2433906

[Te1] C. Teleman, *Five lectures on topological field theory*, Geometry and quantization of moduli spaces, Adv. Courses Math. CRM Barcelona, Birkhäuser/Springer, Cham, 2016, pp. 109–164. MR3675464

[tH] G. 't Hooft, *On the phase transition towards permanent quark confinement*, Nuclear Phys. B **138** (1978), no. 1, 1–25, DOI 10.1016/0550-3213(78)90153-0. MR0489444

[TT] V. Turaev and P. Turner, *Unoriented topological quantum field theory and link homology*, Algebr. Geom. Topol. **6** (2006), 1069–1093, `arXiv:math/0506229`.

[Tu] V. Turaev, *Homotopy quantum field theory*, EMS Tracts in Mathematics, vol. 10, European Mathematical Society (EMS), Zürich, 2010. Appendix 5 by Michael Müger and Appendices 6 and 7 by Alexis Virelizier. MR2674592

[V] E. Verlinde, *Fusion rules and modular transformations in 2D conformal field theory*, Nuclear Phys. B **300** (1988), no. 3, 360–376, DOI 10.1016/0550-3213(88)90603-7. MR954762

[vN1] J. von Neumann, *Mathematical foundations of quantum mechanics*, Princeton University Press, Princeton, 1955. Translated by Robert T. Beyer. MR0066944

[vN2] J. v. Neumann, *Zur Algebra der Funktionaloperationen und Theorie der normalen Operatoren* (German), Math. Ann. **102** (1930), no. 1, 370–427, DOI 10.1007/BF01782352. MR1512583

[W1] E. Witten, *Dynamics of quantum field theory*, Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997), Amer. Math. Soc., Providence, RI, 1999, pp. 1119–1424. MR1701615

[W2] E. Witten, *The "parity" anomaly on an unorientable manifold*, Physical Review B **94** (2016), no. 19, 195150, `arXiv:1605.02391`.

[W3] E. Witten, *On quantum gauge theories in two dimensions*, Comm. Math. Phys. **141** (1991), no. 1, 153–209. MR1133264

[W4] E. Witten, *The Verlinde algebra and the cohomology of the Grassmannian*, Geometry, topology, & physics, Conf. Proc. Lecture Notes Geom. Topology, IV, Int. Press, Cambridge, MA, 1995, pp. 357–422. MR1358625

[W5] E. Witten, *Homework*, Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997), Amer. Math. Soc., Providence, RI, 1999, pp. 609–717.

[W6] E. Witten, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. **121** (1989), no. 3, 351–399. MR990772

[W7] E. Witten, *What one can hope to prove about three-dimensional gauge theory.* `http://scgp.stonybrook.edu/video_portal/video.php?id=563`. Talk at *Mathematical Foundations of Quantum Field Theory Workshop*, Simons Center for Geometry and Physics, January 2012.

[W8] E. Witten, *Fermion Path Integrals And Topological Phases*, Rev. Mod. Phys. **88** (2016), no. 3, 035001, `arXiv:1508.04715`.

[W9] E. Witten, *An SU(2) Anomaly*, Phys. Lett. **B117** (1982), 324–328.

[W10] E. Witten, *Global gravitational anomalies*, Commun. Math. Phys. **100** (1985), 197.

[W11] E. Witten, *World sheet corrections via D instantons*, JHEP **02** (2000), 030, `arXiv:hep-th/9907041`.

[We] F. J. Wegner, *Duality in generalized Ising models and phase transitions without local order parameters*, J. Mathematical Phys. **12** (1971), 2259–2272, DOI 10.1063/1.1665530. MR0289087

[Wei1] S. Weinberg, *The quantum theory of fields. Vol. II*, Cambridge University Press, Cambridge, 1996. Modern applications. MR1411911

[Wei2] S. Weinberg, *Superconductivity for Particular Theorists*, Prog. Theor. Phys. Suppl. **86** (1986), 43.

[Wi] K. G. Wilson, *Confinement of Quarks*, Phys. Rev. **D10** (1974), 2445–2459.

[WPS] C. Wang, A. C Potter, and T Senthil, *Classification of interacting electronic topological insulators in three dimensions*, Science **343** (2014), no. 6171, 629–631, `arXiv:1306.3238`.

[WS] C. Wang and T Senthil, *Interacting fermionic topological insulators/superconductors in 3D*, Physical Review B **89** (2014), no. 19, 195124, `arXiv:1401.1142`.

[WW] K. Walker and Z. Wang, *(3+ 1)-TQFTs and topological insulators*, Frontiers of Physics **7** (2012), no. 2, 150–159, `arXiv:1104.2632`.

[Yo] K. Yonekura, *On the cobordism classification of symmetry protected topological phases*, IPMU-18-0040 (2018), `arXiv:1803.10796`.

Index

Selected Published Titles in This Series

133 **Daniel S. Freed,** Lectures on Field Theory and Topology, 2019
131 **Nalini Joshi,** Discrete Painlevé Equations, 2019
130 **Alice Guionnet,** Asymptotics of Random Matrices and Related Models, 2019
129 **Wen-Ching Winnie Li,** Zeta and L-functions in Number Theory and Combinatorics, 2019
128 **Palle E.T. Jorgensen,** Harmonic Analysis, 2018
127 **Avner Friedman,** Mathematical Biology, 2018
126 **Semyon Alesker,** Introduction to the Theory of Valuations, 2018
125 **Steve Zelditch,** Eigenfunctions of the Laplacian on a Riemannian Manifold, 2017
124 **Huaxin Lin,** From the Basic Homotopy Lemma to the Classification of C^*-algebras, 2017
123 **Ron Graham and Steve Butler,** Rudiments of Ramsey Theory, Second Edition, 2015
122 **Carlos E. Kenig,** Lectures on the Energy Critical Nonlinear Wave Equation, 2015
121 **Alexei Poltoratski,** Toeplitz Approach to Problems of the Uncertainty Principle, 2015
120 **Hillel Furstenberg,** Ergodic Theory and Fractal Geometry, 2014
119 **Davar Khoshnevisan,** Analysis of Stochastic Partial Differential Equations, 2014
118 **Mark Green, Phillip Griffiths, and Matt Kerr,** Hodge Theory, Complex Geometry, and Representation Theory, 2013
117 **Daniel T. Wise,** From Riches to Raags: 3-Manifolds, Right-Angled Artin Groups, and Cubical Geometry, 2012
116 **Martin Markl,** Deformation Theory of Algebras and Their Diagrams, 2012
115 **Richard A. Brualdi,** The Mutually Beneficial Relationship of Graphs and Matrices, 2011
114 **Mark Gross,** Tropical Geometry and Mirror Symmetry, 2011
113 **Scott A. Wolpert,** Families of Riemann Surfaces and Weil-Petersson Geometry, 2010
112 **Zhenghan Wang,** Topological Quantum Computation, 2010
111 **Jonathan Rosenberg,** Topology, C^*-Algebras, and String Duality, 2009
110 **David Nualart,** Malliavin Calculus and Its Applications, 2009
109 **Robert J. Zimmer and Dave Witte Morris,** Ergodic Theory, Groups, and Geometry, 2008
108 **Alexander Koldobsky and Vladyslav Yaskin,** The Interface between Convex Geometry and Harmonic Analysis, 2008
107 **Fan Chung and Linyuan Lu,** Complex Graphs and Networks, 2006
106 **Terence Tao,** Nonlinear Dispersive Equations, 2006
105 **Christoph Thiele,** Wave Packet Analysis, 2006
104 **Donald G. Saari,** Collisions, Rings, and Other Newtonian N-Body Problems, 2005
103 **Iain Raeburn,** Graph Algebras, 2005
102 **Ken Ono,** The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-series, 2004
101 **Henri Darmon,** Rational Points on Modular Elliptic Curves, 2004
100 **Alexander Volberg,** Calderón-Zygmund Capacities and Operators on Nonhomogeneous Spaces, 2003
99 **Alain Lascoux,** Symmetric Functions and Combinatorial Operators on Polynomials, 2003
98 **Alexander Varchenko,** Special Functions, KZ Type Equations, and Representation Theory, 2003
97 **Bernd Sturmfels,** Solving Systems of Polynomial Equations, 2002
96 **Niky Kamran,** Selected Topics in the Geometrical Study of Differential Equations, 2002
95 **Benjamin Weiss,** Single Orbit Dynamics, 2000

For a complete list of titles in this series, visit the
AMS Bookstore at **www.ams.org/bookstore/cbmsseries/**.