FOUR LECTURES ON FINITE SYMMETRY IN QFT

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These notes are based on lectures given at the *Global Categorical Symmetries Summer School* at the Perimeter Institute, June 13–17, 2022. The ideas discussed here about symmetry in field theory represent joint work with Constantin Teleman and Greg Moore; large parts of these notes are adapted from a forthcoming paper.[1] That said, these notes are quite rough in part and are intended to be exactly that: notes. I welcome feedback, corrections, suggestions, etc.

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[1] The paper will include many references; these notes only give minimal direction for further exploration.

Lecture 1: Variations on the Theme of Symmetry

Introduction

We give a conceptual framework for some of the developments of the past few years around "global categorical symmetries". A few immediate comments to give some perspective.

Remark 1.1.

(1) The word 'global' can be dropped: we are discussing symmetries of a field theory which are analogous to symmetries of any other mathematical structure. The word 'global' is often used in contradistinction to 'gauge' symmetries, but gauge symmetries are not symmetries of theories: they are encoded in the (higher) groupoid structure of fields.
(2) The word 'categorical' can also be dropped. Mathematics is traditionally expressed in the language of sets and functions, and when mathematical objects have internal symmetries they organize into (higher) categories rather than sets. The symmetries we discuss often have this higher structure, so naturally involve categories.
(3) There is a large amount of work over the past few years on the topic of symmetry in quantum field theory, and in particular the application of "global categorical symmetries" to dynamics and other questions. We give some excerpts from this literature to illustrate how the framework we develop here applies. However, we emphasize that the topic of symmetry in quantum field theory is a large one with many facets, and the framework here does not nearly apply to all of it.
(4) We restrict to the analog of *finite* group symmetry, including homotopical versions;[2] it will be interesting to generalize to the analog of Lie group symmetry. It is more natural in quantum theory to have *algebras* of symmetries, rather than *groups* of symmetries, and so in particular we encounter non-invertible symmetries.

(1.2) *Main idea.* The motivating thought is simple:

(1.3) Separate out the abstract structure of symmetry
from its concrete manifestations as actions or representations

Historically, the concept of an abstract group was introduced to synthesize and further develop diverse instances of group symmetry in geometry (Klein), in algebra (Galois), in number theory (Gauss), etc. Perhaps it is Arthur Cayley in 1854 who first articulated the definition of an abstract group—I'm no historian—and now every student of mathematics learns this concept early on. The structure of groups is then used to study representations—linear and nonlinear. Similar comments apply to algebras. The elements of an algebra act as linear operators on any module. In the context of field theory, the analog of an algebra of symmetries is a *topological* field theory together with a boundary theory. The analog of elements of an algebra are *defects* in the topological field theory, which act on any quantum field theory which is a "module" over the topological field theory.

[2]which include "higher form symmetries" (though we do not use this term: there are no differential forms in the finite case) and "2-group symmetries"

In this lecture we begin with a brief discussion of groups and algebras before turning to symmetry in field theory. We return at the end to further aspects of groups and algebras whose analogs in field theory play an important role in later lectures.

Groups

(1.4) *Taxonomy.* The simplest dichotomy is between (1) discrete groups, and (2) groups which have a nontrivial topology. For the former we distinguish according to the trichotomy of cardinalities: finite, countable, uncountable. The nature of finite groups is quite different from that of infinite discrete groups; the phrase 'discrete group' evokes very different images from the phrase 'finite group', even though finite groups are discrete. For example, linear representations of a finite group are rigid—they do not deform—whereas, for example, the infinite cyclic group $\mathbb{Z}$ has a continuous family of 1-dimensional complex unitary representations $n \mapsto e^{ixn}$ parametrized by $x \in \mathbb{R}$. Among topological groups, the nicest are Lie groups. Here too there is an important dichotomy: compact vs. noncompact. Compact Lie groups, which include finite groups, have a well-established structure theory and representation theory; again, representations are rigid. Noncompact Lie groups, which include countable discrete groups, also enjoy a robust structure and representation theory, but of a very different nature. Moving on, there are infinite dimensional Lie groups as well as topological groups which do not admit a manifold structure. In a different direction, there are homotopical groups—*H-groups*—which may be (homotopy) finite or infinite. To wit, if $\mathcal{X}$ is any pointed topological space, then the space $\Omega\mathcal{X}$ of loops at the basepoint has the composition law of concatenation of loops, and this makes $\Omega\mathcal{X}$ a group up to homotopy.

The field theory symmetry structure we study is analogous to that of a *finite* group, or, more generally, to a homotopical group that is π*-finite* in the sense that it has only finitely many nonzero homotopy groups, each of which is finite.

(1.5) *Fibering over BG.* Let G be a finite group. A *classifying space* BG is derived from a contractible topological space EG equipped with a free G-action by taking the quotient; the homotopy type of BG is independent of choices. If X is a topological space equipped with a G-action, then the *Borel construction* is the total space of a fiber bundle

$$\begin{array}{c} X_G = EG \times_G X \\ \Big\downarrow{\scriptstyle \pi} \\ BG \end{array} \tag{1.6}$$

with fiber X. If $* \in BG$ is a chosen point, and we choose a basepoint in the G-orbit in EG labeled by $*$, then the fiber $\pi^{-1}(*)$ is canonically identified with X. We say the *abstract symmetry data* (in the sense of groups) is the pair $(BG, *)$, and a *realization* of the symmetry $(BG, *)$ on X is a pair consisting of a fiber bundle (1.6) over BG together with an identification of the fiber over $* \in BG$ with X.

Remark 1.7. We are already moving to homotopy theory, and it is more natural to take the *homotopy fiber*, which is a special case of a *homotopy fiber product*. For continuous maps f, g we can realize the homotopy fiber product as the space F and dotted maps indicated in the diagram

(1.8)
$$\begin{tikzcd} & F \arrow[dl, dashed] \arrow[dr, dashed] & \\ X \arrow[dr, "f"'] & & Z \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

A point of F is a triple (x, z, γ) in which $x \in X$, $z \in z$, and γ is a path in Y from $f(x)$ to $g(z)$. The homotopy fiber Z over the basepoint in (1.6) is the homotopy fiber product

(1.9)
$$\begin{tikzcd} & Z \arrow[dl, dashed] \arrow[dr, dashed] & \\ * \arrow[dr] & & X_G \arrow[dl] \\ & BG & \end{tikzcd}$$

Exercise 1.10. Construct a homotopy equivalence $X \xrightarrow{\simeq} Z$. (You may want to know the *homotopy lifting property* for the fiber bundle $X_G \to BG$.)

(1.11) *Homotopical groups*[3]. A pair $(\mathcal{X}, *)$ consisting of a π-finite topological space $\mathcal{X}$ and a basepoint $* \in \mathcal{X}$ is a generalization of $(BG, *)$. Here is the formal definition of the finiteness we assume.

Definition 1.12.

(1) A topological space $\mathcal{X}$ is π-*finite* if (i) $\pi_0\mathcal{X}$ is a finite set, (ii) for all $x \in \mathcal{X}$, the homotopy group $\pi_q(\mathcal{X}, x)$, $q \geq 1$, is finite, and (iii) there exists $Q \in \mathbb{Z}^{>0}$ such that $\pi_q(\mathcal{X}, x) = 0$ for all $q > Q$, $x \in \mathcal{X}$. (For a fixed bound Q we say that $\mathcal{X}$ is Q-finite.)

(2) A continuous map $f\colon \mathcal{Y} \to \mathcal{Z}$ of topological spaces is π-*finite* if for all $z \in \mathcal{Z}$ the homotopy fiber[4] over z is a π-finite space.

(3) A spectrum[5] E is π-*finite* if each space in the spectrum is a π-finite space.

Example 1.13. An Eilenberg-MacLane space $K(\pi, q)$ is π-finite if π is a finite group. We use notation which emphasizes the role of $\mathcal{X}$ as a classifying space: if $q = 1$ we denote $K(\pi, 1)$ by $B\pi$, and if $q \geq 1$ and A is a finite abelian group, we denote $K(A, q)$ by B^qA. Just as there are group

[3]By 'homotopical group' we mean an *H-group*. This nomenclature pertains if $\mathcal{X}$ is path connected; see Remark 1.15 below. But we allow disconnected spaces too.

[4]As in (1.9), the homotopy fiber over $z \in \mathcal{Z}$ consists of pairs (y, γ) of a point $y \in \mathcal{Y}$ and a path γ in $\mathcal{Z}$ from z to $f(y)$.

[5]A spectrum is a sequence of pointed topological spaces $\{E_q\}_{q\in\mathbb{Z}}$ and maps $\Sigma E_q \to E_{q+1}$.

extensions of ordinary groups, so too there are group extensions of homotopical groups. These are sometimes Postnikov towers. For example, let G be a finite group and let A be a finite abelian group. Then extensions of the form

$$1 \longrightarrow B^2A \longrightarrow \mathcal{X} \longrightarrow BG \longrightarrow 1 \tag{1.14}$$

are classified by group actions of G on A together with a cohomology class in $H^3(BG; A)$, where the coefficients A are twisted by the group action. Thus $\mathcal{X}$ is a topological space with only two nonzero homotopy groups: $\pi_1\mathcal{X} = G$, $\pi_2\mathcal{X} = A$. This class of spaces was studied long ago by George Whitehead. Nowadays we might say that $\mathcal{X}$ is the classifying space of a *2-group*.

Remark 1.15. If $\mathcal{X}$ is a path connected topological space with basepoint $* \in \mathcal{X}$, then $\mathcal{X}$ is the classifying space of its based loop space $\Omega\mathcal{X}$, where the latter is a H-group by composition of based loops.

Remark 1.16.

(1) A topological space $\mathcal{X}$ gives rise to a sequence of higher groupoids $\pi_0\mathcal{X}$, $\pi_{\le 1}\mathcal{X}$, $\pi_{\le 2}\mathcal{X}$,. . . , or indeed to an ∞-groupoid. There is a classifying space construction which passes in the opposite direction from higher groupoids to topological spaces. An ∞-groupoid is π-finite if it satisfies the conditions in Definition 1.12(1), which hold iff the corresponding topological space is π-finite.

(2) In a similar way, one can define π-finiteness for a simplicial set.

Algebras

(1.17) *The sandwich.* Let A be an algebra, and for definiteness suppose that the ground field is $\mathbb{C}$. Partly for simplicity, and partly by the analogy with *finite* groups, assume that A and the modules that follow are finite dimensional. Let R be the *regular* right A-module, i.e., the vector space A furnished with the right action of A by multiplication. The pair (A, R) is *abstract symmetry data* (in the sense of algebras): the *realization* of (A, R) on a vector space V is a pair (L, θ) consisting of a left A-module L together with an isomorphism of vector spaces

$$\theta\colon R \otimes_A L \xrightarrow{\cong} V. \tag{1.18}$$

The tensor product in (1.18)—an algebra sandwiched between a right and left module—is a general structure that recurs in these lectures.

Remark 1.19. It may seem pedantic to introduce the module R here; one usually simply talks about a left module over A. But I want to emphasize the distinction between the abstract symmetry structure and its concrete action on a vector space, and for this we need to be able to recover the underlying vector space from the left module.

Observe that the right regular module satisfies the algebra isomorphism

$$\operatorname{End}_A(R) \cong A, \tag{1.20}$$

where the left hand side is the algebra of linear maps $R \to R$ that commute with the right A-action.

(1.21) *Left vs. right.* The choice of left vs. right in a given situation is made by choice or convention. My convention puts *structural* actions on the right and *geometric* actions on the left. Here the right module R is part of the symmetry structure, and the left module is the geometric action on the vector space. As another example, if V is a finite dimensional real vector space of dimension n, then the space of bases $\mathcal{B}(V)$—the set of isomorphisms $\mathbb{R}^n \to V$—carries a right structural action of $\mathrm{GL}_n\mathbb{R}$ and a left geometric action of $\mathrm{Aut}(V)$.

(1.22) *The group algebra.* Let G be a finite group. The *group algebra* $A = \mathbb{C}[G]$ is the free vector space on the set G, which is then a linear basis of A; multiply basis elements according to the group law in G. A left A-module L is a linear representation of G. The tensor product in (1.18) recovers the vector space which underlies the representation. In the setup of **(1.5)**, take $X = L$ to construct a vector bundle $L_G \to BG$ whose fiber over $* \in BG$ is L.

There is an inclusion $G \subset \mathbb{C}[G]$ whose image consists of *units*, i.e., of invertible elements in the algebra. But the typical element of $\mathbb{C}[G]$ is not invertible. For example, the sum $g_1 + \cdots + g_k$ over a conjugacy class in G is not invertible unless $k = 1$. In general it is a central element. In fact, the center of $\mathbb{C}[G]$ is generated by these elements.

Noninvertible elements in $\mathbb{C}[G]$ play an important role in the study of G-symmetry. For example, when G is the symmetric group on n letters, then the theory of irreducible representations and their associated Young tableaux is developed in terms of certain projectors in $\mathbb{C}[G]$.

Example 1.23. Consider the Lie algebra $\mathfrak{su}_3$, and let $A = U(\mathfrak{su}_3)$ be its universal enveloping algebra (over $\mathbb{C}$). The center of A is isomorphic to a polynomial algebra in 2 variables; it is generated by the Casimir elements $x_2, x_3 \in A$. These Casimirs act as linear operators on any A-module—i.e., on a representation of SU_3—and by Schur's lemma they act by a scalar if the module is irreducible. So these operators can be used to decompose an arbitrary A-module into a direct sum of isotypical submodules. This is simply another illustration of: (1) the importance of noninvertible elements in an algebra, and (2) the use particular elements in an abstract algebras (here central elements) in concrete realizations.

(1.24) *Higher algebra.* The higher versions of finite groups in **(1.11)** have an analog in algebras as well. For example, a *fusion category* $\mathcal{A}$ is a "once higher" version of a finite dimensional semisimple algebra, and there is a well-developed theory of modules over a fusion category. In particular, $\mathcal{A}$ is a right module over itself, the right regular module. A finite group G gives rise to the fusion category $\mathcal{A} = \mathrm{Vect}[G]$ of finite rank vector bundles over G with convolution product.

Main Definitions

These are Definition 1.35 and Definition 1.38 below.

(1.25) *Remarks about field theory.* We begin with a few general remarks, deferring to Lecture 2 a more in-depth discussion.

Perhaps the first point to make is the metaphor of a field theory as a representation of a Lie group, or better

$$\text{field theory} \sim \text{module over an algebra} \tag{1.26}$$

This is of course only a very rough analogy, but nonetheless it provides useful guidance and language. (Our language sometimes seems to assume the module has an algebra structure, but that is not an assumption we make.) We do always work in the Wick-rotated context, so for a quantum field theory we work on Riemannian manifolds rather than Lorentz manifolds. As pioneered by Graeme Segal, a Wick-rotated theory is a linear representation of a bordism category; it is the bordism category which plays the role of the algebra in (1.26).

The field theories which encode finite symmetries are *topological*, and we bring to bear the mathematical development of topological field theory. In particular, we work with *fully local* (or *fully extended*) topological field theories. In the axioms this is realized by having the theory defined on a higher bordism category of manifolds with corners of all codimension. The field theories on which the symmetry acts are typically not topological, and for general quantum field theories the fully local aspect has yet to be fully developed. Nonetheless, our exposition often implicitly assumes full locality.

Remark 1.27. Just as one specifies a Lie group to talk about its representations, one must specify a bordism category to talk about its representations: field theories. There are two sorts of "discrete parameters". First, there is a dimension n, which in the physical anti-Wick-rotated theory is the dimension of spacetime. Second there is a collection $\mathcal{F}$ of background fields. We use the terminology 'n-dimensional field theory on $\mathcal{F}$' or 'n-dimensional field theory over $\mathcal{F}$'. We define background fields in the next lecture (Definition 2.6); for today's lecture they remain in the deep background. We often work in shorthand, illustrated by the following for a gauged nonlinear σ-model:

$$\mathcal{F} = \{\text{orientation, Riemannian metric, } \mathrm{SO}_3\text{-connection, section of twisted } S^2\text{-bundle}\} \tag{1.28}$$

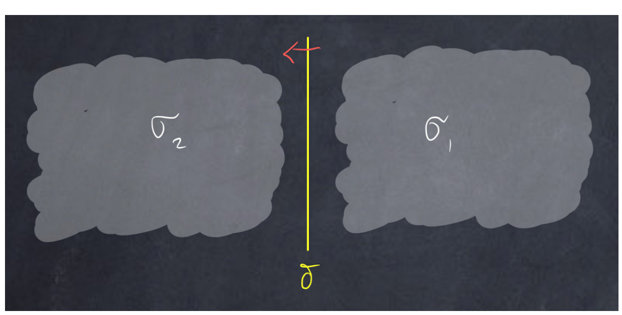

FIGURE 1. A domain wall $\delta\colon \sigma_1 \to \sigma_2$

(1.29) *Domain walls and boundary theories.* Let σ_1, σ_2 be $(n+1)$-dimensional theories on background fields $\mathcal{F}_1, \mathcal{F}_2$. A *domain wall* $\delta\colon \sigma_1 \to \sigma_2$ is the analog[6] of a "(σ_2, σ_1)-bimodule"; see Figure 1 for a depiction. We remove the scare quotes and use the convenient terminology '(σ_2, σ_1)-bimodule' for a domain wall. The triple $(\sigma_1, \sigma_2, \delta)$ is formally a functor with domain a bordism category of smooth $(n+1)$-dimensional manifolds with corners which are equipped with a partition into regions labeled '1' and '2' separated by a cooriented codimension one submanifold (with corners) which is "δ-colored". This is illustrated in Figure 2. As a special case, a domain wall from the tensor unit theory $\mathbb{1}$ to itself is an n-dimensional (absolute, standalone) theory. More generally, we can tensor any domain wall $\delta\colon \sigma_1 \to \sigma_2$ with an n-dimensional theory to obtain a new domain wall. There is a composition law on topological domain walls which are "parallel" (in the sense that they have trivial normal bundles and one is the image of a nonzero section of the normal bundle of the other, using a tubular neighborhood):

$$\begin{tikzcd} \sigma_1 \arrow[r, "\delta'"] \arrow[rr, bend right, "\delta''\circ\delta'"'] & \sigma_2 \arrow[r, "\delta''"] & \sigma_3 \end{tikzcd} \tag{1.30}$$

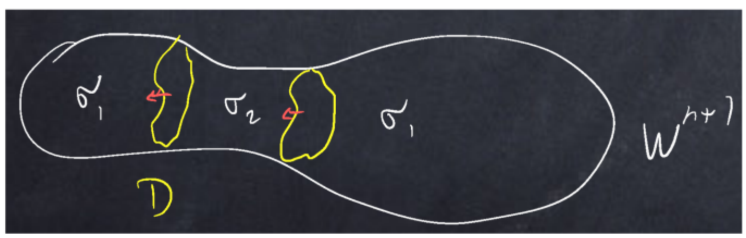

FIGURE 2. Domain walls in the manifold W

(1.31) *Boundary theories.* Following the metaphor of domain wall as bimodule, there are special cases of right or left modules. For field theory these are called *right boundary theories* or *left boundary theories*, as depicted in Figure 3. (Normally, we omit the region labeled '$\mathbb{1}$' in the drawings: it is transparent.) A right boundary theory of σ is a domain wall $\sigma \to \mathbb{1}$; a left boundary theory is a domain wall $\mathbb{1} \to \sigma$.

Remark 1.32. The nomenclature of right vs. left may at first be confusing; it does follow standard usage for modules over an algebra—the direction (right or left) is that of the action of the algebra on the module. In fact, following our general usage for domain walls, we use the terms 'right σ-module' and 'left σ-module' for right and left boundary theories. But a right boundary theory appears on the left in drawings, just as a right module R over an algebra A appears to the left of the algebra in the expression 'R_A'.

[6] We emphasize that σ_1 and σ_2 are not assumed to be algebra objects in the symmetric monoidal category of field theories.

FIGURE 3. A right boundary theory and a left boundary theory

(1.33) *Abstract finite symmetry in field theory.* We turn now to the central concept in these lectures.

Definition 1.34. Fix $n \in \mathbb{Z}^{\geq 0}$. Then *finite field-theoretic symmetry data of dimension* n is a pair (σ, ρ) in which σ is an $(n+1)$-dimensional topological field theory and ρ is a right σ-module.

The dimension n pertains to the theories on which (σ, ρ) acts, not to the dimension of the field theory σ. We do not insist on the burdensome nomenclature 'finite field-theoretic symmetry data of dimension n', but simply refer to the pair of a topological field theory and a right boundary theory. One might want to assume that ρ is nonzero, which is true for the particular boundaries in Definition 1.35 below. Our statement of Definition 1.34 has not made explicit the background fields, but they are there in the background (and there are issues to address concerning them).

This definition is extremely general. The following singles out a class of boundary theories which more closely models the discussions in **(1.5)** and **(1.17)**. We freely use the language and setting of fully local topological field theory. Recall that if $\mathcal{C}'$ is a symmetric monoidal n-category, then there is a symmetric monoidal Morita $(n+1)$-category $\mathrm{Alg}(\mathcal{C}')$ whose objects are objects in $\mathcal{C}'$ equipped with an algebra structure and whose 1-morphisms $A_0 \to A_1$ are (A_1, A_0)-bimodules.

Definition 1.35. Suppose $\mathcal{C}'$ is a symmetric monoidal n-category and σ is an $(n+1)$-dimensional topological field theory with codomain $\mathcal{C} = \mathrm{Alg}(\mathcal{C}')$. Let $A = \sigma(\mathrm{pt})$. Then A is an algebra in $\mathcal{C}'$ which, as an object in $\mathcal{C}$, is $(n+1)$-dualizable. Assume that the right regular module A_A is n-dualizable as a 1-morphism in $\mathcal{C}$. Then the boundary theory ρ determined by A_A is the *right regular boundary theory* of σ, or the *right regular* σ*-module*.

We use an extension of the cobordism hypothesis to generate the boundary theory ρ from the right regular module A_A. Observe that A_A is the value of the pair (σ, ρ) on the bordism depicted in Figure 4; the white point is incoming, so the depicted bordism maps $\mathrm{pt} \to \emptyset$.

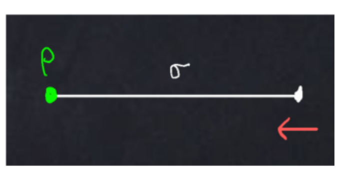

FIGURE 4. The bordism which computes A_A

Remark 1.36.

(1) The right regular σ-module ρ satisfies $\mathrm{End}_\sigma(\rho) \cong \rho$; compare (1.20).
(2) The regular boundary theory is also called a *Dirichlet boundary theory.*
(3) Not every topological field theory σ can appear in Definition 1.34. For example, the main theorem in [7] asserts that "most" 3-dimensional Reshetikhin-Turaev theories do not admit any nonzero topological boundary theory, hence they cannot act as symmetries of a 2-dimensional field theory. On the other hand, the Turaev-Viro theory σ_Φ formed from a (spherical) fusion category Φ takes values in the 3-category Alg(Cat) for a suitable 2-category Cat of linear categories. Thus σ_Φ admits the right regular σ-module defined by the right regular module Φ_Φ.
(4) Let G be a finite group. Then G-symmetry in an n-dimensional quantum field theory is realized via $(n+1)$-dimensional finite gauge theory. The partition function counts principal G-bundles, weighted by the reciprocal of the order of the automorphism group. The regular boundary theory has an additional fluctuating field: a section of the principal G-bundle.

(1.37) *Concrete realization of finite symmetry in field theory.* We now define a concrete realization of (σ, ρ) as symmetries of a quantum field theory.

Definition 1.38. Let σ be an $(n+1)$-dimensional topological field theory and let ρ be a right σ-module. Let F be an n-dimensional field theory. A (σ,ρ)*-module structure* on F is a pair $(\widetilde{F}, \theta)$ in which $\widetilde{F}$ is a left σ-module and θ is an isomorphism

$$\theta\colon \rho \otimes_\sigma \widetilde{F} \xrightarrow{\;\cong\;} F \tag{1.39}$$

of absolute n-dimensional theories.

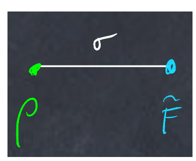

FIGURE 5. The "sandwich"

Here '$\rho \otimes_\sigma \widetilde{F}$' notates the dimensional reduction of σ along the closed interval with boundaries colored with ρ and $\widetilde{F}$; see Figure 5. The bulk theory σ with its right and left boundary theories ρ and $\widetilde{F}$ is sometimes called a "sandwich".

Remark 1.40.

(1) The theory F and so the boundary theory $\widetilde{F}$ may be topological or nontopological, and we allow it to be not fully extended (in which case we use truncations of σ and ρ).
(2) The sandwich picture of F as $\rho \otimes_\sigma \widetilde{F}$ separates out the topological part (σ, ρ) of the theory from the potentially nontopological part $\widetilde{F}$ of the theory. This is advantageous, for example in the study of defects. It allows general computations in the abstract symmetry data which apply to every realization as a symmetry of a field theory.

[7] Daniel S. Freed and Constantin Teleman, *Gapped boundary theories in three dimensions*, Comm. Math. Phys. **388** (2021), no. 2, 845–892, arXiv:2006.10200.

(3) Typically, symmetry persists under renormalization group flow, hence a low energy approximation to F should also be a (σ,ρ)-module. If F is gapped, then at low energies we expect a topological theory (up to an invertible theory), so we can bring to bear powerful methods and theorems in topological field theory to investigate *topological* left σ-modules. This leads to dynamical predictions, as we illustrate in Lecture 4.

Examples

Example 1.41 (quantum mechanics $n=1$)**.** Consider a quantum mechanical system defined by a Hilbert space $\mathcal{H}$ and a Hamiltonian H. The Wick-rotated theory F is regarded as a map with domain $\mathrm{Bord}_{\langle 0,1\rangle}(\mathcal{F})$ for

$$\mathcal{F} = \{\text{orientation, Riemannian metric}\}. \tag{1.42}$$

Roughly speaking, $F(\mathrm{pt}_+) = \mathcal{H}$ and $F(X) = e^{-\tau H/\hbar}$ for $\tau \in \mathbb{R}^{>0}$ and $X = [0,\tau]$ with the standard orientation and Riemannian metric. We refer to a recent preprint of Kontsevich-Segal [8] for more precise statements, in particular that the 0-manifold pt_+ is embedded in the germ of an oriented Riemannian 1-manifold.

Now suppose G is a finite group equipped with a unitary representation $S\colon G \to U(\mathcal{H})$, and assume that the G-action commutes with the Hamiltonian H. To express this symmetry in terms of Definition 1.34 and Definition 1.38, let σ be the 2-dimensional finite gauge theory with gauge group G. If we were only concerned with σ we might set the codomain of σ to be $\mathcal{C} = \mathrm{Alg}(\mathcal{C}')$ for $\mathcal{C}'$ the category of finite dimensional complex vector spaces and linear maps. But to accommodate the boundary theory $\widetilde{F}$ for quantum mechanics, we let $\mathcal{C}'$ be a suitable category of topological vector spaces. The pair (σ,ρ) is defined on $\mathrm{Bord}_2 = \mathrm{Bord}_{\langle 0,1,2\rangle}$ with no background fields. Then $\sigma(\mathrm{pt}) = \mathbb{C}[G]$ is the complex group algebra of G, and $\rho(\mathrm{pt})$ is its right regular module.

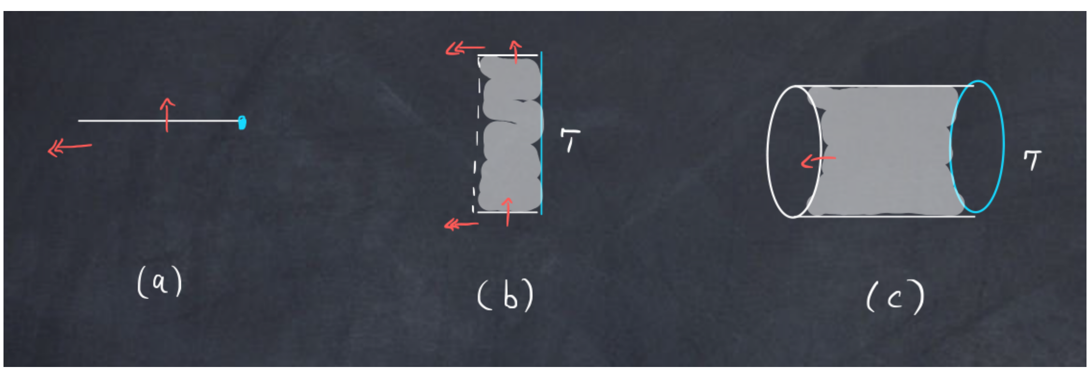

FIGURE 6. Three bordisms evaluated in (1.43) in the theory $(\sigma, \widetilde{F})$

Now we describe the left boundary theory $\widetilde{F}$, which has as background fields (1.42), as does the (absolute) quantum mechanical theory F. Observe that by cutting out a collar neighborhood it suffices to define $\widetilde{F}$ on cylinders (products with $[0,1]$) over $\widetilde{F}$-colored boundaries. The bordisms in

[8] Maxim Kontsevich and Graeme Segal, *Wick rotation and the positivity of energy in quantum field theory*, arXiv:2105.10161.

Figure 6 do not have a well-defined width since there is a Riemannian metric only on the colored boundary. That boundary has a well-defined length τ in (b) and (c). We refer to §2.1.1 of [7] for the conventions about arrows of time. Evaluation of these bordisms under $(\sigma, \widetilde{F})$ gives:

$$\begin{aligned} &\text{(a) the left module } {}_{\mathbb{C}[G]}\mathcal{H} \\ &\text{(b) } e^{-\tau H/\hbar}\colon {}_{\mathbb{C}[G]}\mathcal{H} \longrightarrow {}_{\mathbb{C}[G]}\mathcal{H} \\ &\text{(c) the central function } g \longmapsto \mathrm{Tr}_{\mathcal{H}}\bigl(S(g)e^{-\tau H/\hbar}\bigr) \text{ on } G \end{aligned} \tag{1.43}$$

Of course, this is not a complete construction of the nontopological σ-module $\widetilde{F}$, but it gives some intuition for that theory.

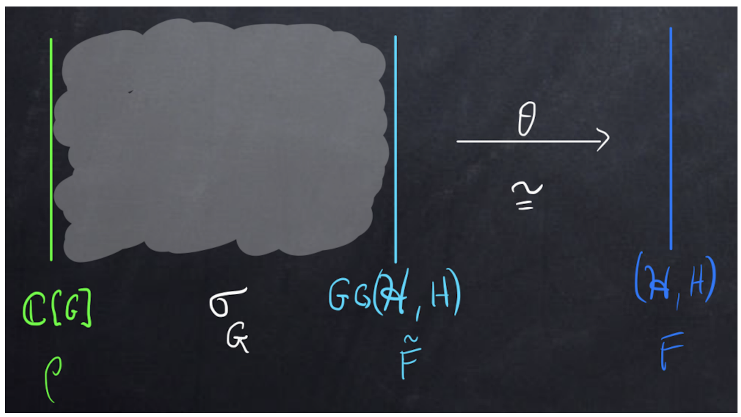

Figure 7. Quantum mechanics with G-symmetry

Figure 7 illustrates the G-action on the quantum mechanics theory F. Although we have not discussed defects yet, we illustrate how they work in this basic example. A point defect in F is what is usually called an *observable*. Think of time running up vertically, and then a point defect δ is the insertion of an observable, or operator, at a given time, as depicted in Figure 8. Also shown is the *link* of the point, which is a 0-sphere. The possible defects on the point are the elements of the topological vector space

$$\varprojlim_{\epsilon \to 0} \mathrm{Hom}\bigl(1, F(S^0_\epsilon)\bigr). \tag{1.44}$$

Here ϵ measures the size of the linking 0-sphere, and one takes an inverse limit as this size shrinks to zero. That inverse limit is a space of singular operators on $\mathcal{H}$; see [8]. In these lectures we mostly compute with defects in topological theories, and for these there is no need to take a limit. Here, for ease of notation and because we are after more formal points, we denote this space of operators as 'End($\mathcal{H}$)', even though the notation suggests bounded operators.

Now we look at defects in the sandwich picture in Figure 7 and transport to point defects in the theory F; see Figure 8. First consider a point defect on the $\widetilde{F}$-colored boundary, as in Figure 9. The link of the point is depicted, and its value under the pair $(\sigma, \mathcal{F})$ is computed at the bottom of the figure. (There should be an inverse limit, which is omitted.) The result is that such a defect

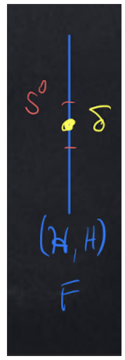

FIGURE 8. A point observable in quantum mechanics

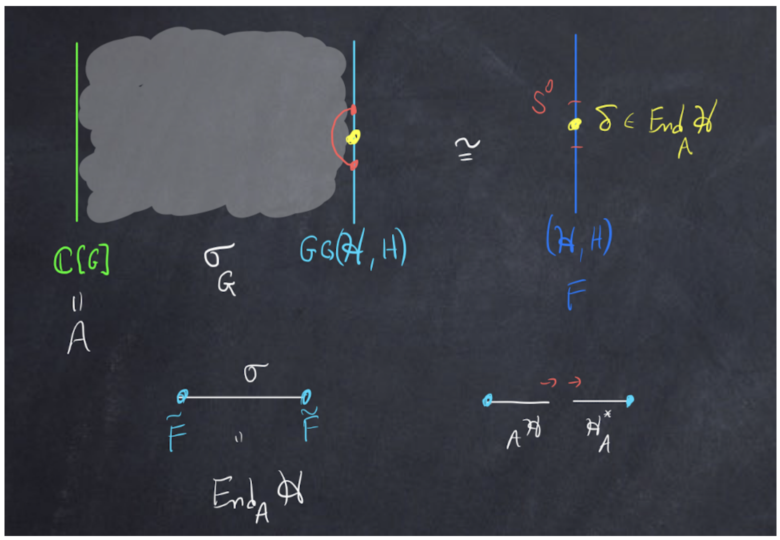

FIGURE 9. A point defect on the $\widetilde{F}$-boundary

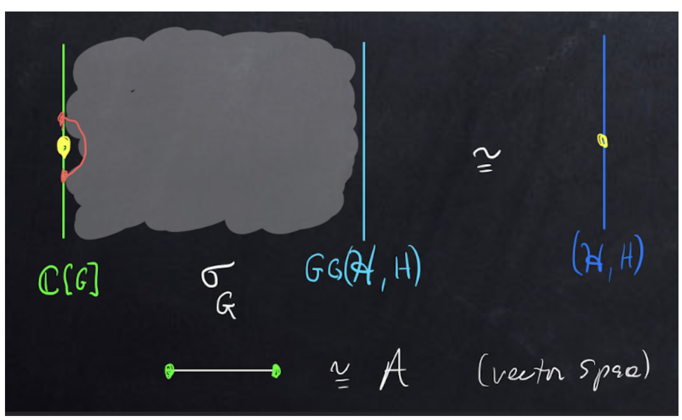

FIGURE 10. A point ρ-defect

is an operator on $\mathcal{H}$ which commutes with the G-action. Of course, such an operator need not be invertible. Also, since $\widetilde{F}$ is not topological, this is not a topological defect.

At the other extreme is a point defect on the ρ-colored boundary, which we call a *point ρ-defect*; see Figure 10. Now the link is in the topological field theory (σ, ρ); that is, the bulk topological theory σ with topological boundary theory ρ. The value of the link is the vector space which underlies the group algebra $A = \mathbb{C}[G]$. Hence the point defect is labeled by an element of A. This may be an element of the group, which is a unit in A, or it may be a nonunit (noninvertible element) in A. This is a topological defect, as it is a defect in the topological field theory. Now imagine having both of these point defects. Since the point ρ-defect is topological, it can be moved in time without changing any correlation functions. Visibly it commutes with the point defect on the $\widetilde{F}$-boundary, which recall is an operator that commutes with A.

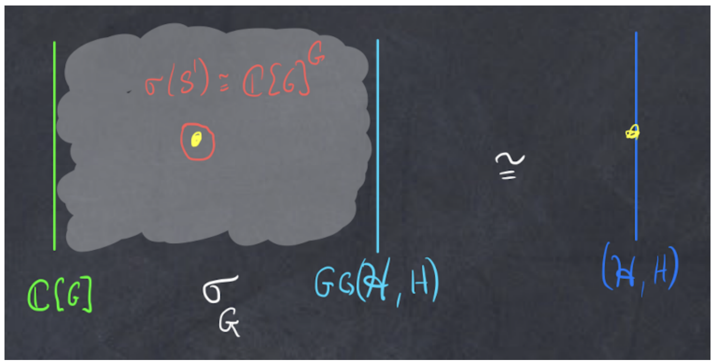

FIGURE 11. A central point defect

One can have a point defect in the bulk theory, as in Figure 11. The link of this point is a circle, and the value $\sigma(S^1)$ of the finite gauge theory on the circle is a vector space which may be identified with the center of the group algebra $\mathbb{C}[G]$. As stated earlier, it has a basis labeled by conjugacy classes: the central element of $\mathbb{C}[G]$ is the sum of elements in the conjugacy class. These topological defects commute with the point defects in Figures 9 and 10.

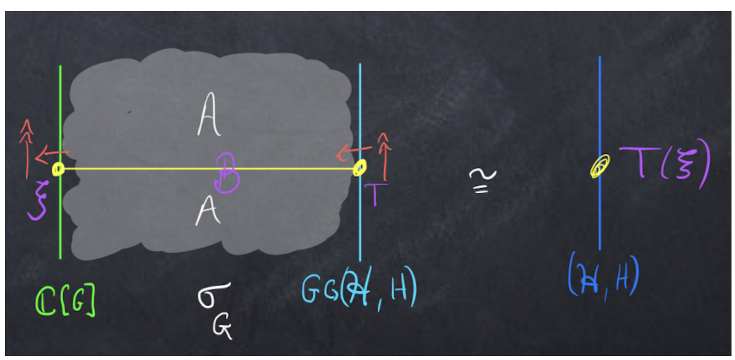

FIGURE 12. A general point defect

The general point defect in F can be realized by a defect on the closed interval depicted in Figure 12. When working on a stratified manifold, we evaluate on links working from higher dimensional strata to lower dimensional strata. The space of labels on a given stratum may depend on the labels chosen on higher dimensional strata, as they do at the endpoints in this case. We do not explain the evaluation of the links in detail here, but simply report that: (1) the label in the interior of the defect is an (A, A)-bimodule B; (2) the label at the endpoint on the ρ-colored boundary is a vector $\xi \in B$; and (3) the label at the endpoint on the $\widetilde{F}$-colored boundary is an

(A,A)-bimodule map $B \to \mathrm{End}(\mathcal{H})$. Under the isomorphism θ, this maps to the point defect labeled by $T(\xi)$ in the theory F.

Exercise 1.45. What happens if $B = A$? (This is the transparent defect in the interior.)

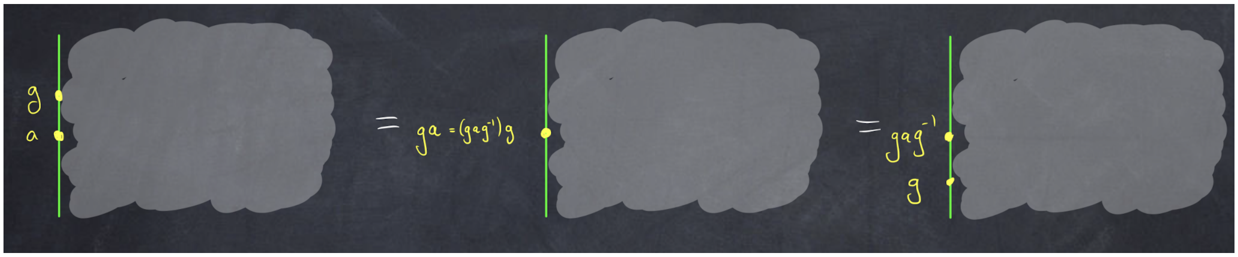

FIGURE 13. Commuting point ρ-defects

We compute the action of the point ρ-defect labeled by $g \in G \subset \mathbb{C}[G]$ on a general point defect in F. This is a computation in the topological field theory (σ,ρ); it applies universally to any left (σ,ρ)-module. First, consider the special case of a topological point defect labeled by $a \in A = \mathbb{C}[G]$, as illustrated in Figure 13. Passing from the first picture to the second is the fusion of point defects, which we will compute later is multiplication in A. The same fusion applies when passing from the third picture to the second. So the effect of moving the g-defect past the a-defect is conjugation of a. In Figure 14 we illustrate how the g-defect moves past a general (nontopological) point defect. Now the label at the ρ-colored boundary is a vector $\xi \in B$, where B is an (A,A)-bimodule, and again the effect is to conjugate ξ by g. Applying the isomorphism θ from the sandwich theory to the theory F, we pass from $T(\xi)$ to $T(g\xi g^{-1}) = gT(\xi)g^{-1}$, since T is an (A,A)-bimodule map. This is the expected action on point defects.

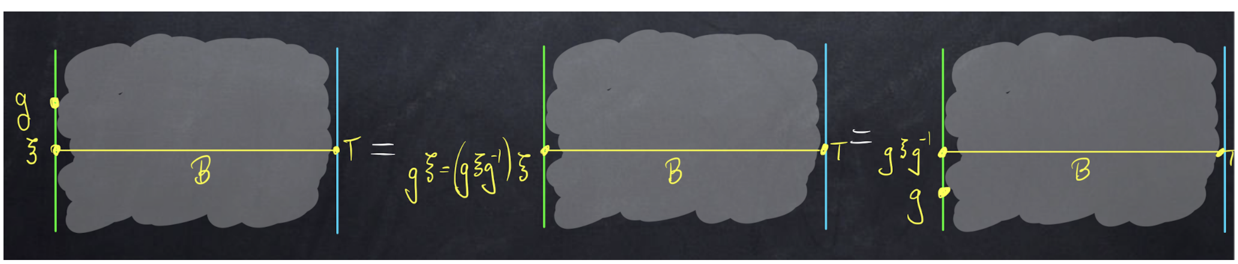

FIGURE 14. Action of a topological point defect on a general point defect

[**Fix label in middle:** $(g\xi^{-1}g^{-1})g$] ⇐

Remark 1.46. The finite gauge theory σ can be constructed via a finite path integral from the π-finite space BG; we discuss finite path integrals in the next lecture. Similarly, the boundary theory ρ can be constructed from a basepoint $* \to BG$: the principal G-bundles are equipped with a trivialization on ρ-colored boundaries. A traditional picture of the G-symmetry of the theory F uses this *classical* picture: the sheaf of background fields $\mathcal{F}$ is augmented to the sheaf $\widetilde{\mathcal{F}} = \{\text{orientation, Riemannian metric, } G\text{-bundle}\}$, which fibers over the sheaf $B_{\bullet}G = \{G\text{-bundle}\}$, so in that sense "spreads over BG" as in §(**1.5**). There is an absolute field theory on $\widetilde{\mathcal{F}}$ which is the "coupling of F to a background gauge field" for the symmetry group G. The framework we are advocating here of F as a (σ,ρ)-module uses the *quantum* finite gauge theory σ: we sum over principal G-bundles.

Remark 1.47. The finite path integral construction of the regular (Dirichlet) boundary theory makes the isomorphism θ in (1.39) apparent. Namely, to evaluate (σ,ρ) we sum over G-bundles equipped with a trivialization on ρ-colored boundaries. Since the trivialization propagates across an interval, the sandwich theory (Figure 5) is the original theory F without the explicit G-symmetry.

Example 1.48 (a homotopical symmetry)**.** Let H be a connected compact Lie group, and suppose $A \subset H$ is a finite subgroup of the center of H. Let $\overline{H} = H/A$. Then a H-gauge theory in, say, 4 dimensions—for example, pure Yang-Mills theory—has a BA symmetry. In this case we take $\sigma = \sigma^{(3)}_{B^2A}$ to be the 3-dimensional A-gerbe theory based on the π-finite space B^2A, and we take ρ to be the regular boundary theory constructed from a basepoint $* \to B^2A$. The left σ-module $\widetilde{F}$ is a $\overline{H}$-gauge theory. (Aspects of this example are discussed in more detail in [9].)

Quotients

For the remainder of this lecture we turn back to a general discussion of symmetry to remind about two aspects that we will take up in field theory in subsequent lectures, particularly Lecture 3: quotients and projective symmetries.

If X is a set equipped with the action of a group G, then there is a quotient set X/G: a point of X/G is a G-orbit in X. We now give analogs in the homotopy and algebra settings.

(1.49) *The homotopy quotient.* In the topological setting of **(1.5)**, the total space X_G of the Borel construction plays the role of the quotient space X/G. Indeed, if G acts freely on X, then there is a homotopy equivalence $X_G \simeq X/G$; in general, X_G is the *homotopy quotient*.

For any map $f\colon Y \to BG$ of topological spaces we form the *homotopy pullback* (see (1.8))

(1.50)
$$\begin{tikzcd} & Z \arrow[dl, dashed] \arrow[dr, dashed] & \\ Y \arrow[dr, "f"'] & & X_G \arrow[dl, "\pi"] \\ & BG & \end{tikzcd}$$

If Y is path connected and pointed, then there is a homotopy equivalence $Y \simeq B(\Omega Y)$. If BG also has a basepoint, and if the map $f\colon Y \to BG$ is basepoint-preserving, then f is the classifying map of a homomorphism $\Omega Y \to G = \Omega BG$, at least in the homotopical sense. In this case Z is the homotopy quotient of X by the action of ΩY. As a special case, if $G' \subset G$ is a subgroup, and $Y = BG' \to BG$ is the classifying map of the inclusion, then Z is homotopy equivalent to the total space of the Borel construction $X_{G'}$. Hence (1.50) is a generalized quotient construction. For $G' = \{e\}$ we have $Y = *$ and we recover $Z = X$, as in (1.9): no quotient at all.

[9] Daniel S. Freed and Constantin Teleman, *Relative quantum field theory*, Comm. Math. Phys. **326** (2014), no. 2, 459–476, arXiv:1212.1692.

(1.51) *Augmentations of algebras.* There is an analogous story in the setting (**1.17**) of algebras. An *augmentation* of an algebra A is an algebra homomorphism $\epsilon\colon A \to \mathbb{C}$. Use ϵ to endow the scalars $\mathbb{C}$ with a right A-module structure: set $\lambda \cdot a = \lambda\epsilon(a)$ for $\lambda \in \mathbb{C}, a \in A$. If L is a left A-module, the vector space

$$Q = \mathbb{C} \otimes_A L = \mathbb{C} \otimes_\epsilon L \tag{1.52}$$

plays the role of the "quotient" of L by A.

Example 1.53. For the group algebra $\mathbb{C}[G]$ of a finite group G, there is a natural augmentation

$$\begin{aligned} \epsilon\colon \mathbb{C}[G] &\longrightarrow \mathbb{C} \\ \sum_{g\in G} \lambda_g g &\longmapsto \sum_{g \in G} \lambda_g \end{aligned} \tag{1.54}$$

where $\lambda_g \in \mathbb{C}$. The augmentation is the pushforward on functions under the map $G \to *$. If L is a representation of G, extended to a left $\mathbb{C}[G]$-module, then the tensor product (1.52) is the vector space of *coinvariants*:

$$1 \otimes \ell = 1 \otimes g \cdot \ell = 1 \otimes g' \cdot \ell, \qquad \ell \in L, \quad g, g' \in G, \tag{1.55}$$

in the tensor product with the augmentation. As a particular case, let S be a finite set equipped with a left G-action, and let $L = \mathbb{C}\langle S\rangle$ be the free vector space generated by S. Then $\mathbb{C} \otimes_\epsilon L$ can be identified with $\mathbb{C}\langle S/G\rangle$, the free vector space on the quotient set.

Exercise 1.56. Prove this last assertion.

Exercise 1.57. For the finite group G acting on the finite set S, consider the Borel construction $S_G \to BG$. Construct an isomorphism $S/G \to \pi_0(S_G)$. Compare the information content of S_G and $\mathbb{C}\otimes_\epsilon \mathbb{C}\langle S\rangle$. Which has more information? How can you alter the other to recover more information?

Remark 1.58. Recall the fusion category $\mathcal{A}$ in (**1.24**). The analog of an augmentation is a *fiber functor* on $\mathcal{A}$: a homomorphism $\mathcal{A} \to \mathrm{Vect}$. For $\mathcal{A} = \mathrm{Vect}[G]$ the natural choice is pushforward under the map $G \to *$ to a point.

(1.59) *Quotient by a subgroup.* We can form the "sandwich" (1.52) with any right A-module in place of the augmentation. For $A = \mathbb{C}[G]$, if $G' \subset G$ is a subgroup, then $\mathbb{C}\langle G'\backslash G\rangle$ is a right G-module; for $G' = G$ it reduces to the augmentation module (1.54). If L is a G-representation, then the tensor product

$$\mathbb{C}\langle G'\backslash G\rangle \otimes_{\mathbb{C}[G]} L \cong \mathbb{C} \otimes_{\mathbb{C}[G']} L \tag{1.60}$$

is the vector space of coinvariants of the restricted G'-representation. This represents the quotient by the subgroup G'.

Central extensions and projective actions

(1.61) *Projective representations.* There are many situations in which one encounters projective representations of groups. For example, suppose A is an algebra and L is an *irreducible* left module. Let G be a finite group that acts on A by algebra automorphisms, i.e., via a group homomorphism $\alpha\colon G \to \operatorname{Aut} A$. They, typically, we can implement these symmetries on the module L: if $g \in G$ then we can find a linear automorphism $t\colon L \to L$ such that

$$t(a\xi) = \bigl(\alpha(g)a\bigr)t(\xi), \qquad a \in A, \quad \xi \in L. \tag{1.62}$$

The map t exists if the twisted A-module L^α is isomorphic to L, and by Schur's lemma t is determined up to a scalar. In other words, each $g \in G$ determines a $\mathbb{C}^\times$-torsor T_g, and the torsors depend multiplicatively on G. They fit together into a group G^τ which is a central extension of G by $\mathbb{C}^\times$:

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow G^\tau \longrightarrow G \longrightarrow 1 \tag{1.63}$$

A familiar example[10] has A a Clifford algebra, L an irreducible module, and G is the orthogonal group. Then G acts projectively on the Clifford module, and one obtains the (s)pin central extension of the orthogonal group.

Remark 1.64. For a group extension (1.63) one considers representations $\rho\colon G^\tau \to \operatorname{Aut}(V)$ for which $\rho\big|_{\mathbb{C}^\times}$ is scalar multiplication.

(1.65) *The twisted group algebra.* Suppose G in (1.63) is a finite group. Let $L^\tau \to G$ be the complex line bundle associated to the principal $\mathbb{C}^\times$-bundle (1.63). Define the *twisted group algebra*

$$A^\tau = \bigoplus_{g\in G} L^\tau_g. \tag{1.66}$$

Then A^τ inherits an algebra structure from the group structure of G. Furthermore, $G^\tau \subset A^\tau$ is the group of units. An A^τ-module restricts to a linear representation of G^τ on which the center $\mathbb{C}^\times$ acts by scalar multiplication, and vice versa. Observe that there is no analog of the augmentation (1.54) unless we are given a splitting of the central extension (1.63). More generally, if $H \subset G$ is a subgroup, then a splitting of the restriction of (1.63) over H induces an A^τ-module structure on $\mathbb{C}\langle H\backslash G\rangle$, and we can use this to define the quotient by H, as in (1.60). Absent the splitting, the projectivity obstructs the quotient construction.

Exercise 1.67. Given an algebra homomorphism $\epsilon\colon A^\tau \to \mathbb{C}$, construct a splitting of the central extension (1.63).

Remark 1.68. In field theory, the analog of an action by the central extension of a group is called an ('t Hooft) *anomalous symmetry*, and the central extension (1.63) is called the *anomaly.* In that context too, the central extension obstructs the quotient construction (often called *gauging*). See the discussion in the second half of Lecture 3.

[10]In this example one uses $\mathbb{Z}/2\mathbb{Z}$-gradings everywhere.

Some additional problems

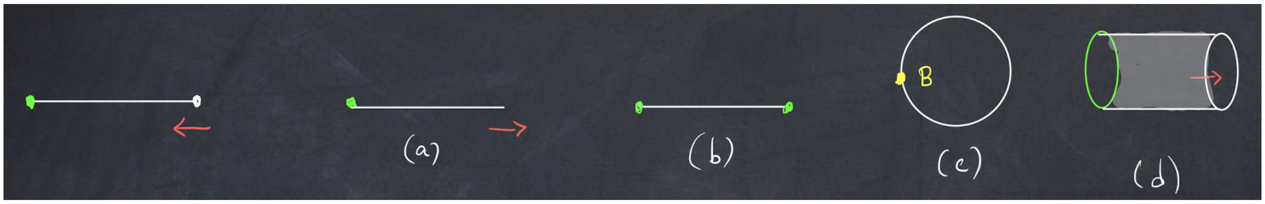

FIGURE 15. Some bordisms in the topological field theory (σ, ρ)

Problem 1.69. Let G be a finite group, and let $\sigma\colon \mathrm{Bord}_2 \to \mathcal{C}$ be the 2-dimensional finite gauge theory. You can take $\mathcal{C} = \mathrm{Alg}_1(\mathrm{Vect})$ the Morita 2-category of complex algebras, or $\mathcal{C} = \mathrm{Cat}$ a 2-category of complex linear categories. In the former case $\sigma(\mathrm{pt}) = \mathbb{C}[G]$ is the group algebra of G; in the latter case $\sigma(\mathrm{pt}) = \mathrm{Rep}(G)$ is the category of linear representations of G. Let ρ be the right regular boundary theory; then the first bordism in Figure 15 evaluates to the right regular module A_A (or to the functor $\mathrm{Rep}(G) \to \mathrm{Vect}$ which maps a G-module to its underlying vector space.) Let B be an (A, A)-bimodule. The red arrow indicates incoming vs. outgoing boundary components. Compute the value of (σ, ρ) on the bordisms (a), (b), (c), and (d) in Figure 15. You may need to specify more data to achieve an unambiguous answer.

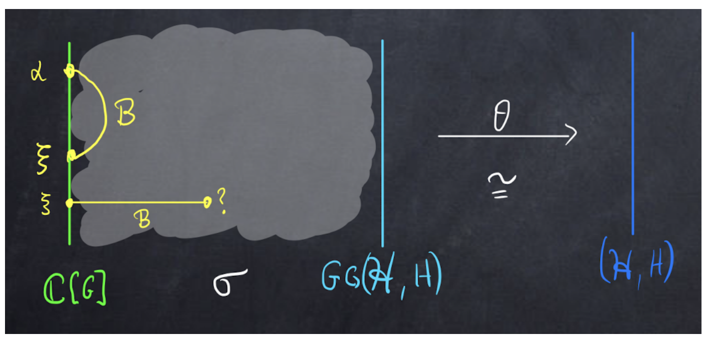

FIGURE 16. Two defects in quantum mechanics

Problem 1.70. Figure 16 uses the pictorial notation from the lecture. Here we are working with the 2-dimensional finite G-gauge theory of the previous problem, which acts on a quantum mechanical system given by a Hilbert space $\mathcal{H}$ and a Hamiltonian H. In the figure, B is a (dualizable) (A, A)-bimodule, $\xi \in B$ is a vector, and $\alpha \in B^*$ is a functional. Take the vertical line to be imaginary time. The bottom defect is at a fixed time, but data is missing at the right endpoint. What data goes there? (It is an element of $\mathrm{Hom}\big(1, \sigma(L)\big)$, where L is the link of the point.) What is the image of that defect under θ? (Note that it is a (σ, ρ)-defect.) For the top defect, suppose that ξ, α are at some times t_1, t_2. Compute the image of this defect under θ.

Lecture 2: Formal structures in field theory; finite homotopy theories

In the first part of this lecture we recall some general structures in field theory. Our framework is that introduced by Segal, adapted by Atiyah for topological theories, and more recently exposed for general quantum field theories by Kontsevich-Segal in [8]. This framework is sometimes referred to as "functorial field theory" (but see Remark 1.1(2), which strongly suggests dropping 'functorial'). In the literature what we say is developed most for *topological* field theories, though the structures should exist in some form for general field theories. In particular, the theory of fully local—or fully extended—field theories is only at its incipient stages in the nontopological case. We introduce a class of topological field theories called *finite homotopy theories*. These offer a kind of semiclassical calculus of defects which is quite convenient and powerful when it applies. These theories are also a nice playground for general ideas in field theory. In the last part of the lecture we return to finite symmetry in field theory and the central Definitions 1.34 and 1.38.

In the next lecture we take up one more general piece of structure—quotients—and then illustrate these ideas in many examples.

Basics

(2.1) *A metaphor.* As with all analogies, not a perfect one:

$$\text{field theory} \sim \text{representation of a Lie group} \tag{2.2}$$

One imperfection can be improved if we take a module over an algebra instead of a representation of a group, but we use the word 'module' below in a different sense. Despite its drawbacks, this metaphor can guide us in some limited way.

(2.3) *Discrete parameters.* The discrete parameters on the right of (2.2) might be considered to be a dimension n and a Lie group G of dimension n. One would not just fix a dimension and, say, study representations of a Lie group of dimension 8! Surely, one would distinguish between different Lie groups of that dimension to fix the "type" of representation. In field theory, too, one can consider there to be two discrete parameters to fix the "type" of field theory: (1) the dimension n of *spacetime*, and (2) the collection $\mathcal{F}$ of *background fields*. We can then speak of an n-dimensional field theory on/over $\mathcal{F}$.

Remark 2.4. The word 'discrete parameter' is potentially confusing, since the choice of background fields may include non-discrete fields, such as a Riemannian metric. It is the *choice* of which background fields to include which is discrete. It may help to observe that discrete parameters are what we fix in geometry to construct moduli spaces, for example the moduli space of curves (of a fixed genus=discrete parameter).

(2.5) *Background fields.* Informally, background fields of dimension n are sets assigned to n-dimensional manifolds X together with a pullback under local diffeomorphisms $f\colon X' \to X$; they are required to depend locally on X. However, we need more than set-valued fields, since fields—such as connections—may have internal symmetries. In the case of "B-fields" there are higher internal symmetries: automorphisms of automorphisms. Thus we could take higher groupoids as the codomain for fields. Instead we choose simplicial sets. Formally, let Man_n be the category whose objects are smooth n-manifolds and whose morphisms are local diffeomorphisms of n-manifolds. There is a Grothendieck topology of open covers.

Definition 2.6. A *field* in dimension n is a sheaf $\mathcal{F}\colon \mathrm{Man}_n \to \mathrm{Set}_\Delta$ with values in the category of simplicial sets.

We have not spelled out the sheaf condition, which is the encoding of locality. Here $\mathcal{F}$ could be a single field or a collection of fields; we do not attempt to define irreducibility here. A field on an n-manifold X is a 0-simplex in $\mathcal{F}(X)$. An example of a collection of fields is (1.28), which we repeat here for convenience:

$$\mathcal{F} = \{\text{orientation, Riemannian metric, } \mathrm{SO}_3\text{-connection, section of twisted } S^2\text{-bundle}\} \tag{2.7}$$

Observe that the last two components are sheaves that extend to Man, the category of smooth manifolds and smooth maps, but the first two do not extend. See [11] for an exposition of simplicial sheaves on Man, and in particular a discussion of the sheaf condition.

Remark 2.8. One should think of $\mathcal{F}$ as a specification of type, not a choice of specific fields on a specific manifold. Rather, $\mathcal{F}(X)$ is the simplicial set of fields on a manifold X, and—as said above—a choice of specific fields on X is a 0-simplex in $\mathcal{F}(X)$.

Remark 2.9. The sheaf on Man_n tells what a field on a *single* manifold is; one also needs to know what a smooth family of fields parametrized by an arbitrary smooth manifold is, as well as how to base change such families. In other words, we must sheafify over Man. We do not dwell on this point in these lectures.

Remark 2.10. The Axiom System we use for field theory does not include fluctuating fields, nor does it include lagrangians; it supposes that all quantization has already been executed. In that sense it is a purely quantum axiom system, as are the earlier axiom systems of Wightman and Haag-Kastler in Minkowski spacetime, or for that matter those of Dirac, von Neumann, and Irving Segal, and Mackey for quantum mechanics. We can, however, contemplate fiber bundles $\mathcal{F} \to \overline{\mathcal{F}}$ of fields and a quantization process which passes from a field theory over $\mathcal{F}$ to a field theory over $\overline{\mathcal{F}}$.

Exercise 2.11. Identify the background fields (and dimension) in familiar quantum field theories, such as:

(1) A 2-dimensional σ-model with target S^2
(2) Quantum mechanics of a particle on a ring

[11] Daniel S. Freed and Michael J. Hopkins, *Chern–Weil forms and abstract homotopy theory*, Bull. Amer. Math. Soc. (N.S.) **50** (2013), no. 3, 431–468, arXiv:1301.5959.

(3) 4-dimensional QCD and its low energy approximation, the pion theory
(4) 4-dimensional pure Yang-Mills theory
(5) Supersymmetric Yang-Mills theory
(6) Dijkgraaf-Witten theory

You may want to also identify the fields in the classical theory and the fiber bundle which maps background fields in the classical theory to those in the quantum theory.

(2.12) *Field theory.* These axioms capture a *Wick-rotated* field theory, formulated on compact Riemannian manifolds (if there is a Riemannian metric among the background fields). Fix $n \in \mathbb{Z}^{\geq 1}$ and a collection $\mathcal{F}$ of n-dimensional fields.

A field theory is expressed in the language of sets and functions, but because of the layers of structure it is rather in the language of categories and functors. The domain is a bordism category $\mathrm{Bord}_n(\mathcal{F})$ of n-dimensional smooth manifolds with corners equipped with a choice of fields from $\mathcal{F}$. In the literature one finds detailed constructions for the fully extended topological case, say in works of Lurie and Calaque-Scheimbauer;[12] and [8] for the nonextended general case. We assume that all *topological* theories are fully extended downward in dimension, in which case $\mathrm{Bord}_n(\mathcal{F})$ is a symmetric monoidal n-category. One hopes a similar strong locality is possible for *nontopological* theories, but that awaits further developments. In the nonextended case, we interpret '$\mathrm{Bord}_n(\mathcal{F})$' as a 1-category $\mathrm{Bord}_{\langle n-1,n\rangle}(\mathcal{F})$ whose objects are closed $(n-1)$-manifolds and whose morphisms are bordisms between them.

The codomain $\mathcal{C}$ of a field theory is a symmetric monoidal n-category.[13] For physical applications one takes $\mathcal{C}$ to be *complex linear*: the linearity from superposition in quantum mechanics, and the complex ground field from interference. Thus we usually have $\Omega^n\mathcal{C} = \mathbb{C}$ and $\Omega^{n-1}\mathcal{C}$ equivalent to the category Vect of complex vector spaces or to the category of $\mathbb{Z}/2\mathbb{Z}$-graded complex vector spaces, discrete vector spaces for topological theories and topological vector spaces for nontopological theories.[14] These assumptions can be relaxed for applications outside of physics.

In the topological case (so $\mathcal{F}$ consists of "topological fields") we have the following.

Definition 2.13. A *topological field theory* of dimension n over $\mathcal{F}$ is a symmetric monoidal functor

$$F\colon \mathrm{Bord}_n(\mathcal{F}) \longrightarrow \mathcal{C}. \tag{2.14}$$

We would like to think that a suitable variation of this definition applies to all field theories. This is clearest in the nonextended nontopological case, in which we replace $\mathcal{C}$ by the 1-category $t\mathrm{Vect}$ of suitable complex topological vector spaces under tensor product, as in [8].

Remark 2.15.

[12] Jacob Lurie, *On the classification of topological field theories*, Current developments in mathematics, 2008, Int. Press, Somerville, MA, 2009, pp. 129–280. arXiv:0905.0465.
Damien Calaque and Claudia Scheimbauer, *A note on the* (∞, n)*-category of cobordisms*, Algebr. Geom. Topol. **19** (2019), no. 2, 533–655, arXiv:1509.08906

[13] One can replace 'n-category' with '(∞, n)-category' in our exposition. Most of our examples are quite finite and semisimple, but one should also use examples built on derived geometry.

[14] The *looping* $\Omega\mathcal{C}$ of an n-category is the $(n-1)$-category $\mathrm{Hom}_{\mathcal{C}}(1, 1)$. If $\mathcal{C}$ is symmetric monoidal, as it is in our case, then so too is $\mathrm{Hom}_{\mathcal{C}}(1, 1)$. Hence one can iterate.

(1) A field theory may be evaluated in a smooth family of manifolds and background fields parametrized by a smooth manifold S; the result is smooth and should behave well under base change. For example, typically correlation functions are written as smooth functions of parameters—it is these spaces of parameters which are the missing piece of structure. Therefore, (2.14) should be sheafified over Man, the site of smooth manifolds and smooth diffeomorphisms. (This point has been emphasized by Stolz and Teichner.[15]) We already remarked on the need to sheafify $\mathcal{F}$ over Man in Remark 2.9. We also need to sheafify the domain $\mathrm{Bord}_n(\mathcal{F})$ and the codomain $\mathcal{C}$ over Man.
(2) A topological field theory is constrained by strong finiteness properties that follow from Definition 2.13. In the metaphor (2.2), a *topological* field theory is analogous to a representation of a *finite* group, though generalizations are possible.
(3) Definition 2.13 does not incorporate unitarity, or rather its Wick-rotated form: reflection positivity. One needs extra structure to do so, and it is an open problem to formulate *extended* reflection positivity. Observe that not every representation of a semisimple Lie group is unitarizable, and similarly for field theories, so one does not want unitarity incorporated into the general definition.
(4) The notion of a *free* theory is not evident from Definition 2.13.
(5) The collection of field theories of a fixed dimension n on a fixed collection $\mathcal{F}$ of background fields has an associative composition law: juxtaposition of quantum systems with no interaction, sometimes called 'stacking'. There is a unit theory $\mathbb{1}$ for this operation. For example, if F_1, F_2 are theories, and Y is a closed $(n-1)$-manifold with background fields, then $(F_1 \otimes F_2)(Y) = F_1(Y) \otimes F_2(Y)$. The unit theory has $\mathbb{1}(Y) = \mathbb{C}$; there is a single state on every space. A unit for the composition law is called an *invertible field theory*.

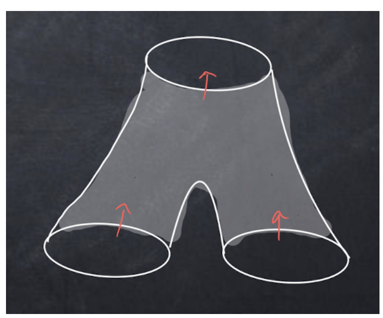

FIGURE 17. The pair of pants bordism

(2.16) *Product structures.* Let F be a field theory, as in (2.14). As with any homomorphism in algebra, structures and equations in the domain transport to the codomain. One example is the product structure on $F(S^k)$, $0 \le k \le n-1$. Assume first that F is a *topological* theory. To illustrate, let $n = 2$ and $k = 1$. Then the "pair of pants" bordism P in Figure 17 induces an algebra structure on the vector space $F(S^1)$. There is a diffeomorphism of P which exchanges the

[15]Stephan Stolz and Peter Teichner, *Supersymmetric field theories and generalized cohomology*, Mathematical foundations of quantum field theory and perturbative string theory, Proc. Sympos. Pure Math., vol. 83, Amer. Math. Soc., Providence, RI, 2011, pp. 279–340. arXiv:1108.0189.

two incoming circles, and this implies that the multiplication is commutative. For a 3-dimensional theory F', the value $F'(S^1)$ of the theory on a circle is a linear category, say, and P induces a monoidal structure. The aforementioned diffeomorphism induces *data*, not a *condition*, on $F'(S^1)$, namely the data of a *braiding* for the monoidal structure. As we go higher in category number, the data and conditions proliferate.

From a more sophisticated viewpoint, the sphere S^k is an E_{k+1}-*object* in the bordism category, and the functor (2.14) induces an E_{k+1} structure on its image.

In a nontopological theory one also has these product structures, but now they depend continuously on background fields, such as a conformal structure or Riemannian metric. So on the topological vector space attached to S^{n-1} in a limit of zero radius we obtain some version of an operator product expansion. In 2-dimensional conformal field theories this leads to vertex operator algebras.

Exercise 2.17. Draw the pictures which illustrate algebra structure on S^0 in the bordism category. Be sure to verify associativity and also to identify the unit. Include topological background fields, such as an orientation or spin structure.

Finite homotopy theories I

As mentioned in the introduction, this is a useful class of theories, both in applications and for general theory. One constructs them using a version of the Feynman path integral. It is simultaneously more special—because one has finite sums rather than integrals over infinite dimensional spaces—and more general—because one integrates in all codimensions, and in positive codimension one sums in a higher category rather than simply summing complex numbers.

This class of topological field theories was introduced by Kontsevich in 1988 and was picked up by Quinn a few years later. They are also the subject of a series of a series of papers by Turaev in the early 2000's. These *finite homotopy theories* lend themselves to explicit computation using topological techniques.

The "finite path integral" quantization in *extended* field theory was introduced in 1992 in [16]; see also §3 and §8 of [17]. The modern approach uses *ambidexterity* or *higher semiadditivity*, as introduced by Hopkins-Lurie[18]. We do not say much about quantization in these lectures; Mike Hopkins will say more in his.

In these lectures we use finite homotopy theories as examples of $(n+1)$-dimensional topological field theories σ which act on n-dimensional quantum field theories. For that reason, in this section we denote the dimension by 'm' rather than 'n'; in the application to symmetry, $m = n + 1$.

[16]Daniel S. Freed, *Higher algebraic structures and quantization*, Comm. Math. Phys. **159** (1994), no. 2, 343–398, arXiv:hep-th/9212115.

[17]Daniel S. Freed, Michael J. Hopkins, Jacob Lurie, and Constantin Teleman, *Topological quantum field theories from compact Lie groups*, A celebration of the mathematical legacy of Raoul Bott, CRM Proc. Lecture Notes, vol. 50, Amer. Math. Soc., Providence, RI, 2010, pp. 367–403. arXiv:0905.0731.

[18]Michael J. Hopkins and Jacob Lurie, *Ambidexterity in K(n)-local stable homotopy theory*. https://www.math.ias.edu/~lurie/papers/Ambidexterity.pdf. preprint. See also Gijs Heuts and Jacob Lurie, *Ambidexterity*, Topology and Field Theories, Contemp. Math., vol. 613, Amer. Math. Soc., Providence, RI, 2014, pp. 79–110 as well as Yonatan Harpaz, *Ambidexterity and the universality of finite spans*, Proc. Lond. Math. Soc. (3) **121** (2020), no. 5, 1121–1170, arXiv:1703.09764 and Shachar Carmeli, Tomer M. Schlank, and Lior Yanovski, *Ambidexterity in chromatic homotopy theory*, Invent. Math. **228** (2022), no. 3, 1145–1254, arXiv:1811.02057.

(2.18) *π-finite spaces, maps, simplicial sheaves, ...* We defined π-finite spaces and π-finite maps in Definition 1.12. I invite you to review it now. We will also use π-finite infinite loop spaces and π-finite infinite loop maps.

Remark 2.19.

(1) A simplicial sheaf is π-finite if its values are π-finite simplicial sets.
(2) These variations pertain to the relative cases of maps, as in Definition 1.12(2).

Example 2.20. Fix $m \in \mathbb{Z}^{\geq 1}$ and consider the simplicial sheaves of fields which assign to an m-manifold W:

$$\begin{aligned} \widetilde{\mathcal{F}}(W) &= \{\text{Riemannian metric, } \mathrm{SU}_2\text{-connection}\} \\ \mathcal{F}(W) &= \{\text{Riemannian metric, } \mathrm{SO}_3\text{-connection}\} \end{aligned} \tag{2.21}$$

There is a map $p\colon \widetilde{\mathcal{F}} \to \mathcal{F}$ which takes an SU_2-connection to the associated SO_3-connection. The map p is a fiber bundle of simplicial sheaves. Neither $\widetilde{\mathcal{F}}$ nor $\mathcal{F}$ is π-finite, but the map p is π-finite. The fiber over a principal SO_3-bundle $\overline{P} \to W$ is the groupoid of lifts to a principal SU_2-bundle $P \to W$ (which may be empty). These form a torsor over the groupoid of double covers of W. The groupoid of double covers is the fundamental groupoid of the mapping space $\mathrm{Map}(W, B\mu_2)$, where $\mu_2 = \{\pm 1\}$ is the center of SU_2. Observe that $B\mu_2 \simeq \mathbb{RP}^\infty$ is a π-finite space, in fact a π-finite infinite loop space. As a matter of notation, we write '$B_\nabla G$' for the sheaf of G-connections—here G is a Lie group—and the group extension $\mu_2 \to \mathrm{SU}_2 \to \mathrm{SO}_3$ leads to the fiber bundle

$$\begin{tikzcd} B_\nabla \mathrm{SU}_2 \arrow[r] & B_\nabla \mathrm{SO}_3 \arrow[d] \\ & B^2\mu_2 \end{tikzcd} \tag{2.22}$$

of simplicial sheaves on Man (which we restrict to Man_m). The vertical map is the second Stiefel-Whitney class.

(2.23) *Cocycles.* We abuse the term 'cocycle', which should be reserved for cohomology theories described in terms of cochain complexes. We use it for any 'geometric representative' of a cohomology class. Thus, if $\{E_q\}$ is a spectrum—a sequence of pointed infinite loop spaces that represent a cohomology theory—then a "cocycle" of degree q on a space X can be taken to be a continuous map $X \to E_q$. In any model there is a zero cocycle.

The data which determines a finite homotopy theory is a triple $(m, \mathfrak{X}, \lambda)$ in which $m \in \mathbb{Z}^{\geq 1}$, the space $\mathfrak{X}$ is π-finite, and λ is a cocycle on $\mathfrak{X}$ of degree m. We denote the theory as '$\sigma^{(m)}_{(\mathfrak{X},\lambda)}$' or '$\sigma^{(m)}_{\mathfrak{X}}$', the latter in case $\lambda = 0$. A nonzero cocycle λ encodes an *'t Hooft anomaly* in case $\sigma^{(m)}_{(\mathfrak{X},\lambda)}$ is part of symmetry data.

(2.24) *Examples.* I hope the following list helps to relate this account to familiar terrain.

(1) Let G be a finite group. Its classifying space $\mathcal{X} = BG$ is an Eilenberg-MacLane space, so is π-finite. Set $\lambda = 0$. Then for any m the triple $(m, BG, 0)$ gives rise to finite G-gauge theory in dimension m.
(2) Now suppose λ represents a cohomology class in $H^m(BG; \mathbb{C}^\times)$. Then $\sigma^{(m)}_{(BG,\lambda)}$ is a twisted finite G-gauge theory, a Dijkgraaf-Witten theory.
(3) Let A be a finite abelian group and $p \in \mathbb{Z}^{\ge 0}$. Set $\mathcal{X} = B^{p+1}A$, an Eilenberg-MacLane space $K(A, p+1)$. A map into $B^{p+1}A$ represents a higher A-gerbe, which is a background field for a "p-form" symmetry. More standardly, this is the symmetry group B^pA, which is a homotopical form of a group (called an *H-group*). For any $m \in \mathbb{Z}^{\ge 1}$, the theory $\sigma^{(m)}_{B^{p+1}A}$ counts these higher A-gerbes. In the context of Definitions 1.34 and 1.38 it encodes B^pA-symmetry.
(4) Example 1.13 describes the classifying space of a *2-group*, which is a (general) path-connected 2-finite space.

Exercise 2.25. Identify the triple $(m, \mathcal{X}, \lambda)$ for a spin Chern-Simons theory with finite gauge group.

Finite homotopy theories II

Fix $m \in \mathbb{Z}^{\ge 1}$ and suppose $p\colon \widetilde{\mathcal{F}} \to \mathcal{F}$ is a π-finite fiber bundle of simplicial sheaves $\mathrm{Man}_m \to \mathrm{Set}_\Delta$; see (2.22) for an example. The basic idea is that there is a finite process which takes an m-dimensional field theory $\widetilde{F}$ over $\widetilde{\mathcal{F}}$ as input and produces an m-dimensional field theory F over $\mathcal{F}$ as output. One obtains F from $\widetilde{F}$ by summing over the (fluctuating) fields in the fibers of p. Since p is π-finite, this is a finite sum—a finite version of the Feynman path integral.

Remark 2.26.

(1) It often happens that the theory $\widetilde{F}$ is[19] "classical", in which case it is an invertible field theory. Then F is its quantization.
(2) The framework is most developed for *topological* field theories, in which case we can work in *extended* field theory.

We do not give a systematic treatment of this quantization. Rather, we illustrate through an example, which brings in the semiclassical mapping spaces that are our focus. This example is relevant for many 4-dimensional gauge theories, in which case the abelian group A in the example is a subgroup of the center of the gauge group. (One can change the numbers to apply this example in any dimension.)

Example 2.27. Let A be a finite abelian group and set $\mathcal{X} = B^2A$. For definiteness fix dimension $m = 5$. Our aim is to construct a 5-dimensional topological field theory $F = \sigma^{(5)}_{B^2A}$. In the terms above: $\widetilde{\mathcal{F}}$ is the simplicial sheaf on Man_5 which assigns to a 5-manifold W the 2-groupoid $\pi_{\le 2}\,\mathrm{Map}(W, B^2A)$ (made into a simplicial set), $\widetilde{F}$ is the tensor unit theory, and $\mathcal{F}$ is the trivial simplicial sheaf which assigns a point to each 5-manifold W. (The theory F is unoriented: there

[19]One point of view, advocated by Nathan Seiberg, is: 'classical' field theory = invertible field theory.

are no background fields.) We have not specified the codomain $\mathcal{C}$ of the theory, and one has latitude in this choice. For our purposes we assume standard choices at the top three levels: $\Omega^3\mathcal{C} = \mathrm{Cat}$ is a linear 2-category of complex linear categories, from which it follows that $\Omega^4\mathcal{C} = \mathrm{Vect}$ is a linear 1-category of complex vector spaces and $\Omega^5\mathcal{C} = \mathbb{C}$.

Let M be a closed manifold. Then $F(M)$ is the quantization of the mapping space

$$\mathcal{X}^M = \mathrm{Map}(M, \mathcal{X}) \tag{2.28}$$

The nature of that quantization depends on $\dim M$.

$\dim M = 5$: The quantization is a (rational) number, a weighted sum over homotopy classes of maps $M \to \mathcal{X}$:

$$F(M) = \sum_{[\phi]\in\pi_0(\mathcal{X}^M)} \frac{\#\pi_2(\mathcal{X}^M, \phi)}{\#\pi_1(\mathcal{X}^M, \phi)} = \frac{\#H^0(M;A)}{\#H^1(M;A)}\, \#H^2(M;A). \tag{2.29}$$

$\dim M = 4$: The quantization is the vector space of locally constant complex-valued functions on $\mathcal{X}^M$:

$$F(M) = \mathrm{Fun}\big(\pi_0(\mathcal{X}^M)\big) = \mathrm{Fun}\big(H^2(M;A)\big). \tag{2.30}$$

$\dim M = 3$: The quantization is the linear category of flat vector bundles (local systems) over $\mathcal{X}^M$:

$$\begin{aligned} F(M) = \mathrm{Vect}\big(\pi_{\le 1}(\mathcal{X}^M)\big) &= \mathrm{Vect}\big(H^2(M;A)\big) \times \mathrm{Rep}\big(H^1(M;A)\big) \\ &\simeq \mathrm{Vect}\big(H^2(M;A) \times H^1(M;A)^\vee\big), \end{aligned} \tag{2.31}$$

where $\mathcal{A}^\vee$ is the Pontrjagin dual group of characters of the finite abelian group $\mathcal{A}$. (If M is oriented, there is an isomorphism $H^1(M;A)^\vee \cong H^2(M;A^\vee)$.)

Remark 2.32. In this example $\mathcal{X}$ is an infinite loop space—an Eilenberg-MacLane space—which explains the cohomological translations in (2.29)–(2.31).

As a further illustration, we describe the quantization of a bordism of top dimension, which leads to a correspondence diagram of mapping spaces.

Example 2.33 ($\mathcal{X} = B^2A$ **redux**). Suppose $M\colon N_0 \to N_1$ is a 5-dimensional bordism between closed 4-manifolds N_0, N_1. The restriction maps to incoming and outgoing boundaries

$$\begin{tikzcd} & \mathcal{X}^M \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\ \mathcal{X}^{N_0} & & \mathcal{X}^{N_1} \end{tikzcd} \tag{2.34}$$

form a correspondence diagram of mapping spaces. The quantization $F(M)\colon F(N_0)\to F(N_1)$ maps a function $f\in \mathrm{Fun}\bigl(\pi_0(\mathcal{X}^{N_0})\bigr)$ to $(p_1)_*(p_0)^*f$, where the pushforward $(p_1)_*$ is the "weighted finite homotopy sum" or "finite path integral"

$$\bigl[(p_1)_*g\bigr](\psi)=\sum_{[\phi]\in\pi_0(p_1^{-1}\psi)}\frac{\#\pi_2(p_1^{-1}\psi,\phi)}{\#\pi_1(p_1^{-1}\psi,\phi)},\qquad g\in \mathrm{Fun}\bigl(\pi_0(\mathcal{X}^M)\bigr),\quad \psi\in\mathcal{X}^{N_1}. \tag{2.35}$$

Remark 2.36. If $N_0=\emptyset$, then $\mathcal{X}^{N_0}=*$ and we obtain an element of $\mathrm{Hom}\bigl(1,F(N_1)\bigr)$, i.e., a vector in the vector space $F(N_1)$.

In terms of the paradigm at the beginning of this subsection, the example so far has trivial $\widetilde{F}$. We now give an example in which $\widetilde{F}$ is a nontrivial invertible theory. The data which defines it is a pair $(\mathcal{X},\lambda)$ consisting of a π-finite space $\mathcal{X}$ and a cocycle λ on $\mathcal{X}$. Typically we need a generalized orientation to integrate λ, depending on the generalized cohomology theory in which λ is a cocycle.

Example 2.37 (twisted $\mathcal{X}=B^2A$)**.** We continue with $\mathcal{X}=B^2A$, and now specialize to $A=\mathbb{Z}/2\mathbb{Z}$ the cyclic group of order 2. Then[20] $H^5(\mathcal{X};\mathbb{C}^\times)\cong H^6(\mathcal{X};\mathbb{Z})$ is cyclic of order 2. Let λ be a cocycle which represents this class. The quantizations in Example 2.27 are altered as follows. For $\dim M=5$ weight the sum in (2.29) by $\langle\phi^*\lambda,[M]\rangle$, where $[M]$ is the fundamental class.[21] For $\dim M=4$ the transgression of λ to $\mathcal{X}^M$ induces a flat complex line bundle (of order 2) $L\to\mathcal{X}^M$; now (2.30) becomes the space of flat sections of $L\to\mathcal{X}^M$. Similarly, for $\dim M=3$ the cocycle λ transgresses to a twisting of K-theory, and the quantization is a category of twisted vector bundles. The quantization in Example 2.33 is also altered using transgressions of λ: they produce line bundles $L_0\to\mathcal{X}^{N_0}$ and $L_1\to\mathcal{X}^{N_1}$ as well as an isomorphism $p_0^*(L_0)\xrightarrow{\cong}p_1^*(L_1)$. Then $(p_1)_*(p_0)^*$ maps sections of $L_0\to\mathcal{X}^{N_0}$ to sections of $L_1\to\mathcal{X}^{N_1}$.

Domain walls and boundaries in finite homotopy theories

(2.38) *Semiclassical domain walls.* Fix $m\in\mathbb{Z}^{\ge 1}$ and let $(\mathcal{X}_1,\lambda_1)$, $(\mathcal{X}_2,\lambda_2)$ be pairs of π-finite spaces and degree m cocycles. In the following we use trivializations of cocycles. In a model with cochain complexes, a trivialization of a degree m cocycle λ is a cochain of degree $m-1$ whose differential is λ. In a model in which λ is a map to a space in a spectrum, then a trivialization is a null homotopy of the map, i.e., a homotopy to the constant map with value the basepoint.

Definition 2.39. A *semiclassical domain wall* from $(\mathcal{X}_1,\lambda_1)$ to $(\mathcal{X}_2,\lambda_2)$ is a pair $(\mathcal{Y},\mu)$ consisting of a π-finite space $\mathcal{Y}$ equipped with a correspondence

$$\begin{tikzcd} & \mathcal{Y} \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ \mathcal{X}_1 & & \mathcal{X}_2 \end{tikzcd} \tag{2.40}$$

[20]Let $\iota\in H^2(B^2\mathbb{Z}/2\mathbb{Z};\mathbb{Z}/2\mathbb{Z})$ be the tautological class. Then $\iota\smile \mathrm{Sq}^1\iota\in H^5(B^2\mathbb{Z}/2\mathbb{Z};\mathbb{Z}/2\mathbb{Z})$ becomes the nonzero class after extending coefficients $\mathbb{Z}/2\mathbb{Z}\to\mathbb{C}^\times$.

[21]Since λ is induced from a mod 2 class, orientations are not necessary—we can proceed in mod 2 cohomology.

and a trivialization μ of $f_2^*\lambda_2 - f_1^*\lambda_1$.

Remark 2.41.

(1) We have written (2.40) to conform to standard practice for a correspondence from $\mathcal{X}_1$ to $\mathcal{X}_2$, but to fit our right/left conventions, as illustrated in Figure 2, we might have swapped $\mathcal{X}_1$ and $\mathcal{X}_2$.

(2) If $(\mathcal{Y}', \mu')$ is a π-finite space and a degree $m-1$ cocycle, then there is a new semiclassical domain wall

$$\begin{tikzcd} & (\mathcal{Y} \times \mathcal{Y}', \mu + \mu') \arrow[dl] \arrow[dr] & \\ (\mathcal{X}_1, \lambda_1) & & (\mathcal{X}_2, \lambda_2) \end{tikzcd} \tag{2.42}$$

This corresponds to tensoring with the $(m-1)$-dimensional theory $(\mathcal{Y}', \mu')$ on the domain wall.

(2.43) *Quantization of a semiclassical domain wall.* To quantize a semiclassical domain wall, we use (2.40) to construct a mapping space. Let M be a closed manifold presented as a union

$$M = M_1 \cup_Z M_2 \tag{2.44}$$

of manifolds with boundary along the boundary Z; then $Z \subset M$ is a codimension 1 cooriented submanifold. Form the mapping space

$$\mathcal{M} = \bigl\{(\phi_1, \phi_2, \psi) : \phi_i\colon M_i \to \mathcal{X}_i,\ \psi\colon Z \to \mathcal{Y},\ f_i \circ \psi = \phi_i\big|_Z\bigr\}. \tag{2.45}$$

Now quantize $\mathcal{M}$ as illustrated in Example 2.27.

(2.46) *Composition.* Composition of semiclassical domain walls proceeds by homotopy fiber product. Suppose $(\mathcal{X}_1, \lambda_1)$, $(\mathcal{X}_2, \lambda_2)$, $(\mathcal{X}_3, \lambda_3)$ are π-finite spaces and degree m cocycles, and let

$$\begin{aligned} (\mathcal{Y}', \mu')\colon (\mathcal{X}_1, \lambda_1) &\longrightarrow (\mathcal{X}_2, \lambda_2) \\ (\mathcal{Y}'', \mu'')\colon (\mathcal{X}_2, \lambda_2) &\longrightarrow (\mathcal{X}_3, \lambda_3) \end{aligned} \tag{2.47}$$

be semiclassical domain walls. Their composition

$$(\mathcal{Y}, \mu)\colon (\mathcal{X}_1, \lambda_1) \longrightarrow (\mathcal{X}_3, \lambda_3) \tag{2.48}$$

is constructed via the homotopy fiber product

$$\begin{tikzcd}[column sep=small] & & & \mathcal{Y} \arrow[dl, dashed] \arrow[dr, dashed] & & & \\ & & \mathcal{Y}' \arrow[dl] \arrow[dr] & & \mathcal{Y}'' \arrow[dl] \arrow[dr] & & \\ & \mathcal{X}_1 & & \mathcal{X}_2 & & \mathcal{X}_3 & \end{tikzcd} \tag{2.49}$$

which is the composition of correspondence diagrams (in the homotopy category); the trivialization μ of $\lambda_3 - \lambda_1$ is the sum $\mu_1 + \mu_2$. (For ease of reading, we omitted pullbacks in the previous clause.) We write (2.49) with cocycles and trivializations as follows:

$$\begin{tikzcd}[column sep=small] & & (\mathcal{Y}, \mu' + \mu'') \arrow[dl, dashed] \arrow[dr, dashed] & & \\ & (\mathcal{Y}', \mu') \arrow[dl] \arrow[dr] & & (\mathcal{Y}'', \mu'') \arrow[dl] \arrow[dr] & \\ (\mathcal{X}_1, \lambda_1) & & (\mathcal{X}_2, \lambda_2) & & (\mathcal{X}_3, \lambda_3) \end{tikzcd} \tag{2.50}$$

(2.51) *Boundaries.* As in **(1.31)** we specialize domain walls to boundary theories, here in the semiclassical world of finite homotopy theories.

Definition 2.52. Let $\mathcal{X}$ be a π-finite space and suppose λ is a cocycle of degree m on $\mathcal{X}$.

(1) A *right semiclassical boundary theory* of $(\mathcal{X}, \lambda)$ is a pair $(\mathcal{Y}, \mu)$ consisting of a π-finite space $\mathcal{Y}$, a map $f\colon \mathcal{Y} \to \mathcal{X}$, and a trivialization μ of $-f^*\lambda$.

(2) A *left semiclassical boundary theory* of $(\mathcal{X}, \lambda)$ is a pair $(\mathcal{Y}, \mu)$ consisting of a π-finite space $\mathcal{Y}$, a map $f\colon \mathcal{Y} \to \mathcal{X}$, and a trivialization μ of $f^*\lambda$.

The mapping spaces used for quantization specialize (2.45).

In this finite homotopy context there is a special form for a regular boundary theory.

Definition 2.53. Let $\mathcal{X}$ be a π-finite space and suppose λ is a cocycle of degree m on $\mathcal{X}$. A *semiclassical right regular boundary theory* of $(\mathcal{X}, \lambda)$ is a basepoint $f\colon * \to \mathcal{X}$ and a trivialization μ of $-f^*\lambda$.

In terms of Definition 2.52, the semiclassical right regular boundary theory is $(*, \mu)$. If the cocycle has positive degree in an ordinary (Eilenberg-MacLane) cohomology theory, then it vanishes on a point so we can and do take $\mu = 0$.

Remark 2.54. The regular boundary condition amounts to an extra semiclassical (fluctuating) field on the boundary which is a trivialization of the bulk field (map to $\mathcal{X}$).

Exercise 2.55. Define a domain wall between boundary theories. For any $(m, \mathcal{X}, 0)$ consider the right semiclassical boundary theory D given by a basepoint $* \to \mathcal{X}$ and the right semiclassical boundary theory D given by the identity map $\mathrm{id}_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{X}$. Construct a distinguished domain wall $\mathrm{D} \to \mathrm{N}$ and a distinguished domain wall $\mathrm{N} \to \mathrm{D}$. What is their composition (in both orders)?

Example 2.56. Let $m = 2$. Fix a finite group G and let $\mathcal{X} = BG$ with basepoint $* \to BG$. The quantization of the interval depicted in Figure 4 is the quantization of the restriction map to the right endpoint

$$\mathrm{Map}\big(([0,1], \{0\}), (BG, *)\big) \longrightarrow \mathrm{Map}\big(\{*\}, BG\big), \tag{2.57}$$

which up to homotopy is the map $* \to BG$. Choose the codomain $\mathcal{C} = \mathrm{Cat}$ so that, as in (2.31), the quantization of $\mathrm{Map}(*, BG)$ is the category $\mathrm{Vect}(BG) \simeq \mathrm{Rep}(G)$. Then the quantization of the map $* \to BG$, or better of the correspondence

$$\begin{tikzcd} & * \arrow[dl] \arrow[dr] & \\ * & & BG \end{tikzcd} \tag{2.58}$$

of mapping spaces derived from Figure 4, is the pushforward of the trivial bundle over $*$ with fiber $\mathbb{C}$ (the tensor unit). This is the regular representation of G in $\mathrm{Rep}(G)$. If, instead, we choose $\mathcal{C} = \mathrm{Alg}(\mathrm{Vect})$, then BG quantizes to the group algebra $\mathbb{C}[G]$ and $* \to BG$ quantizes to the right regular module.

Exercise 2.59. Incorporate a nonzero cocycle in the form of a central extension

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow G^\tau \longrightarrow G \longrightarrow 1 \tag{2.60}$$

(2.61) *A special sandwich.* Let $(\mathcal{X}, \lambda)$ be given and suppose $(\mathcal{Y}', \mu')$ and $(\mathcal{Y}'', \mu'')$ are right and left semiclassical boundary theories for $(\mathcal{X}, \lambda)$. Then, as a special case of the composition (2.50), the $(m-1)$-dimensional semiclassical sandwich of $(\mathcal{X}, \lambda)$ between $(\mathcal{Y}', \mu')$ and $(\mathcal{Y}'', \mu'')$ has as its semiclassical data the pair $(\mathcal{Y}' \overset{h}{\times}_{\mathcal{X}} \mathcal{Y}'', \mu' + \mu'')$, where $\mathcal{Y}' \overset{h}{\times}_{\mathcal{X}} \mathcal{Y}''$ is the homotopy fiber product; observe that $\mu' + \mu''$ is a cocycle of degree $m - 1$.

Defects

Domain walls and boundaries are special cases of the general notion of a *defect* in a field theory. Our discussion here is specifically for *topological* theories, though with modifications it applies more generally (see Remark 2.68(7) below).

(2.62) *Preliminary: The category* $\mathrm{Hom}(1, x)$. We will encounter the expression '$\mathrm{Hom}(1, x)$' in a higher symmetric monoidal category, so we begin by elucidating its meaning. Suppose Vect is the symmetric monoidal category of vector spaces, and $V \in \mathrm{Vect}$ is an object, i.e., a vector space. The tensor unit 1 in Vect is the vector space $\mathbb{C}$ of scalars. We usually identify a linear map $T \in \mathrm{Hom}(\mathbb{C}, V)$ with $T(1) \in V$, so in this case $\mathrm{Hom}(1, V)$ is the space of vectors in V. Similarly, if $C \in \mathrm{Cat}$ is a category, then $\mathrm{Hom}(1, C)$ can be identified with the objects in C. On the other hand, for the Morita 2-category $\mathrm{Alg(Vect)}$ of complex algebras, the tensor unit 1 is the algebra $\mathbb{C}$ and for any algebra A, the 1-category $\mathrm{Hom}(1, A)$ is the category of left A-modules.

Let $\mathcal{C}$ be a symmetric monoidal n-category. If $x \in \mathcal{C}$, then $\mathrm{Hom}(1, x)$ is an $(n-1)$-category. It is possible that it is empty, or, rather, in our usual linear situation that it only contains the zero object. We also use '$\mathrm{Hom}(1, x)$' when $x \in \Omega^\ell \mathcal{C}$ is in some looping of $\mathcal{C}$. Then the homs are taken in $\Omega^\ell \mathcal{C}$.

(2.63) *Definition of a defect in a topological theory.* Suppose m is a positive integer, $\mathcal{F}$ is a collection of background fields, and

$$\sigma\colon\ \mathrm{Bord}_m(\mathcal{F}) \longrightarrow \mathcal{C} \tag{2.64}$$

is a topological field theory with values in a symmetric monoidal m-category $\mathcal{C}$. We describe defects of *codimension* ℓ in a k-dimensional manifold M, where $k \in \{0, 1, \dots, m\}$, $\ell \in \{1, \dots, m\}$, and $\ell \le k$. Let $Z \subset M$ be a submanifold of codimension ℓ, and let $\nu \subset M$ be an open tubular neighborhood of $Z \subset M$; assume the closure $\bar\nu$ is the total space of a fiber bundle $\bar\nu \to Z$ with fiber the closed ℓ-dimensional disk. The fiber over $p \in Z$ is denoted $\bar\nu_p$; its boundary $\partial\bar\nu_p$ is diffeomorphic to the ℓ-dimensional sphere $S^{\ell-1}$. It is the *link* of $Z \subset M$ at p; see Figure 18.

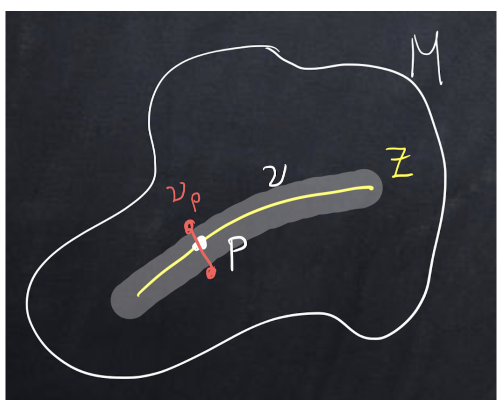

FIGURE 18. The tubular neighborhood and link of a submanifold

Definition 2.65. Assume that M is a closed manifold and $Z \subset M$ is a closed submanifold.

(1) A *local defect* at $p \in Z$ is an element

$$\delta_p \in \mathrm{Hom}\big(1, \sigma(\partial\bar\nu_p)\big). \tag{2.66}$$

(2) The *transparent (local) defect* is $\delta_p = \sigma(\bar{\nu}_p)$.
(3) A *global defect* on Z is a vector

$$\delta_Z \in \operatorname{Hom}\bigl(1, \sigma(\partial\bar{\nu})\bigr). \tag{2.67}$$

(4) The *transparent (global) defect* is $\delta_Z = \sigma(\bar{\nu})$.

The transparent defects can safely be erased.

Remark 2.68. We make several comments about this definition.

(1) $M \setminus \nu$ is a compact manifold with boundary $\partial\bar{\nu}$. Define the bordism $W\colon \partial\bar{\nu} \to \emptyset$ by letting the boundary be incoming. If δ_Z is a global defect, we evaluate the theory on (M, Z, δ_Z) as $\sigma(W)(\delta_Z)$. This is of the same type as the value $\sigma(M)$ on the closed manifold M: a complex number if $\dim M = m$, a complex vector space if $\dim M = n-1$, etc.
(2) A normal framing of $Z \subset W$ identifies each link $\partial\bar{\nu}_p$, $p \in Z$, with the standard sphere $S^{\ell-1}$. Assuming enough finiteness, $\sigma(S^{\ell-1}) \in \Omega^{\ell-1}\mathcal{C}$ defines[22] an $(m-\ell+1)$-dimensional field theory $\sigma^{(\ell-1)}$—the dimensional reduction of σ along $S^{\ell-1}$—and a local defect δ_p determines a left boundary theory $\delta^{(\ell-1)}$ for $\sigma^{(\ell-1)}$, again assuming sufficient finiteness. Using the normal framing it makes sense to assign a single local defect

$$\delta \in \operatorname{Hom}\bigl(1, \sigma(S^{\ell-1})\bigr) \tag{2.69}$$

to Z. The cobordism hypothesis computes an associated global defect

$$\delta_Z \in \operatorname{Hom}\bigl(1, \sigma^{(\ell-1)}(Z)\bigr) = \operatorname{Hom}\bigl(1, \sigma(Z \times S^{\ell-1})\bigr) \tag{2.70}$$

which we can use to evaluate the theory σ on (M, Z, δ) as in (1). Absent the normal framing, we can use a flat family of local defects and some twisted dimensional reduction. We can also use objects δ which are invariant under a subgroup of the orthogonal group. For example, see §8.1 of [23] for a discussion of topological defects using orientation in place of normal framing.
(3) A defect on Z may be tensored with a standalone field theory on Z to obtain a new defect. This corresponds to composing with an element of $\operatorname{Hom}(1,1)$ in (2.66) or (2.67). We illustrated that in a special case in Remark 2.41(2).
(4) The sheaf of background fields on a defect need not agree with the sheaf of background fields in the bulk; the former need only map to the latter. Thus we can have a spin defect in an theory of oriented manifolds.
(5) Defects are also defined for manifolds M with boundaries and corners, and we also allow boundaries, corners, and singularities in Z. In short, we allow Z to be a stratified manifold. Then different strata of Z have different links, and we compute them and assign (local) defects working from the lowest codimension to the highest. We give several illustrations in these lectures.

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[22]The looping $\Omega\mathcal{C}$ of the symmetric monoidal n-category $\mathcal{C}$ is the symmetric monoidal $(n-1)$-category $\operatorname{Hom}(1,1)$ of endomorphisms of the tensor unit. We can iterate this construction. The theory $\sigma^{\ell-1}$ takes values in $\Omega^{\ell-1}\mathcal{C}$. The cobordism hypothesis, both with and without singularities, is used to define the theory and its boundary theory.

[23]Daniel S. Freed and Constantin Teleman, *Topological dualities in the Ising model*, arXiv:1806.00008.

(6) If M is closed with no boundary or corners, then $V=\sigma(M)$ is a vector space and $\sigma\bigl([0,1]\times M\bigr)$ is the identity map id_V. A defect supported in the interior of $[0,1]\times M$ evaluates under σ to a linear operator on V. In this situation the terms 'operator' and 'observable' are often used in place of with 'defect'.
(7) There are also (nontopological) defects in nontopological theories, but then unless we are in maximal codimension an extension beyond a two-tier theory is perhaps implicit. In nontopological theories local defects take values in a limit as the radius of the linking sphere shrinks to zero. For $\dim M=m$ and $\dim Z=0$ the resulting *point defects* are often called 'local operators'. For $\dim Z=1$ they are *line defects*.[24] It is increasingly common in the physics literature to consider a *1-category* of line defects; higher dimensional defects and higher categories of defects also appear.

(2.71) *Composition law on local defects.* The value of a topological field theory σ on $S^{\ell-1}$ is an E_ℓ-algebra. By **(2.16)**, this leads to a composition law on defects, either for local defects (2.66), (2.69) or global defects (2.67), (2.70). If Z is normally framed, one can consider two parallel copies Z',Z'', and then a normal slice of the complement of open tubular neighborhoods of Z',Z'' inside a closed tubular neighborhood of Z is the "pair of pants" which defines the composition law of the E_ℓ-structure. The composition law on topological point defects is a topological version of the usual operator product expansion. The composition law gives rise to the dichotomy between *invertible defects* and *noninvertible defects.*

Semiclassical defects in finite homotopy theories

Let F be an m-dimensional finite homotopy theory based on a π-finite space $\mathcal{X}$ and a cocycle λ of degree m on $\mathcal{X}$.

(2.72) *Preliminary on transgression.* Fix $\ell\in\{2,\dots,m\}$. Define the iterated free loop space

$$\mathcal{L}^{\ell-1}\mathcal{X}=\mathrm{Map}(S^{\ell-1},\mathcal{X}). \tag{2.73}$$

Consider the diagram

$$\begin{tikzcd} \mathcal{L}^{\ell-1}\mathcal{X}\times S^{\ell-1} \arrow[r, "e"] \arrow[d, "\pi_1"'] & \mathcal{X} \\ \mathcal{L}^{\ell-1}\mathcal{X} \end{tikzcd} \tag{2.74}$$

in which e is evaluation and π_1 is projection onto the first factor. The composition $(\pi_1)_*(e)^*\lambda$ is a cocycle of degree $m-\ell+1$ on $\mathcal{L}^{\ell-1}\mathcal{X}$ called the *transgression* of λ. Note that the stable framing of the sphere is used to execute the pushforward.

[24]And for $\dim Z=2$ they are called *surface defects.* The progression point-line-surface is an uncomfortable mishmash of point-line-plane (affine geometry) and point-curve-surface (differential geometry).

(2.75) *Semiclassical local defects.* For a defect on a submanifold of codimension $\ell \in \mathbb{Z}^{\geq 1}$, the link is $S^{\ell-1}$—canonically if the normal bundle is framed—and so the mapping space on the link is[25] $\mathcal{L}^{\ell-1}\mathcal{X}$, the iterated free loop space of $\mathcal{X}$ defined in (2.73). By **(2.72)** the cocycle λ transgresses to a cocycle $\tau^{\ell-1}\lambda$ on $\mathcal{L}^{\ell-1}\mathcal{X}$ with a drop of degree by $\ell - 1$. Recall the definition of a local defect in Definition 2.65(1) .

Definition 2.76. Fix $m, \ell \in \mathbb{Z}^{\geq 2}$ with $\ell \leq m$. Let $\mathcal{X}$ be a π-finite space and suppose λ is a cocycle of degree m on $\mathcal{X}$. A *semiclassical local defect* of codimension ℓ for $(\mathcal{X}, \lambda)$ is a π-finite map

$$\delta\colon \mathcal{Y} \longrightarrow \mathcal{L}^{\ell-1}\mathcal{X} \tag{2.77}$$

and a trivialization μ of $\delta^*(\tau^{\ell-1}\lambda)$.

Since $\mathcal{L}^{\ell-1}\mathcal{X}$ is π-finite, (2.77) amounts to a π-finite space $\mathcal{Y}$ and a continuous map δ. The local quantum defect in $\operatorname{Hom}\bigl(1, F(S^{\ell-1})\bigr)$ is the quantization of the map (2.77); see Remark 2.36 for an analogous quantization.

(2.78) *Semiclassical global defects.* To pass from local to global we use a tangential structure. As an example, if M is a closed manifold and $Z \subset M$ is a *normally framed* codimension ℓ submanifold on which the defect (2.77) is placed, the value of the theory F on M with the defect on Z is the quantization of the mapping space

$$\operatorname{Map}\bigl((M, Z), (\mathcal{X}, \mathcal{Y})\bigr) \tag{2.79}$$

consisting of pairs of maps $\phi\colon M \to \mathcal{X}$ and $\psi\colon Z \to \mathcal{Y}$ which satisfy a compatibility condition: if $Z \times S^{\ell-1} \hookrightarrow M$ is the inclusion of the boundary of a tubular neighborhood of $Z \subset M$, and $\phi'\colon Z \to \mathcal{L}^{\ell-1}\mathcal{X}$ is the transpose of the composition

$$Z \times S^{\ell-1} \hookrightarrow M \xrightarrow{\phi} \mathcal{X}, \tag{2.80}$$

then the diagram

$$\begin{tikzcd} & \mathcal{Y} \arrow[d, "\delta"] \\ Z \arrow[ur, "\psi"] \arrow[r, "\phi'"'] & \mathcal{L}^{\ell-1}\mathcal{X} \end{tikzcd} \tag{2.81}$$

is required to commute.

Remark 2.82.

[25]The iterated free loop space notation holds for $\ell \geq 2$. The case $\ell = 1$ is a domain wall; see **(2.38)**

(1) One should use instead a mapping space of triples (ϕ,ψ,γ) where instead of demanding that (2.81) commute we specify a homotopy $\gamma\colon \delta\circ\psi\to\phi'$. This sort of derived mapping space should in principle replace all of the strict mapping spaces we write throughout the paper. However, the homotopy can be incorporated into a tubular neighborhood of Z, so in fact nothing is lost by using the strict mapping space.
(2) There are many variations of this basic scenario. The defect may have support on a manifold with boundary or corners, or more generally on a stratified manifold. Such is the case for the ρ-defects in Definition 2.92 below; a further example is in Figure 24.

(2.83) *Composition of semiclassical local defects.* The general composition law on local defects is constructed using the higher dimensional pair of pants or, in the case of ρ-defects as in Figure 20, using the higher dimensional pair of chaps. Here we state the semiclassical version of the first.

Resume the setup of Definition 2.76: $m,\ell\in\mathbb{Z}^{\ge 2}$ are integers with $\ell\le m$, and $(\mathcal{X},\lambda)$ is the finite homotopy data for an m-dimensional theory F. Let P be the ℓ-dimensional pair of pants: as a manifold with boundary,

$$P = D^\ell \ \setminus\ B^\ell\amalg B^\ell, \tag{2.84}$$

where $B^\ell\amalg B^\ell$ are embedded balls in the interior of D^ℓ. As a bordism,

$$P\colon S^{\ell-1}\amalg S^{\ell-1}\longrightarrow S^{\ell-1}, \tag{2.85}$$

where the domain spheres are the inner boundaries of P and the codomain sphere is the outer boundary. The cocycle λ on $\mathcal{X}$ transgresses to an isomorphism

$$\mu\colon \pi_1^*(\tau^{\ell-1}\lambda)+\pi_2^*(\tau^{\ell-1}\lambda)\longrightarrow \tau^{\ell-1}\lambda \tag{2.86}$$

of cocycles on $\mathcal{X}^P$. Here $\pi_i\colon \mathcal{L}^{\ell-1}\mathcal{X}\times\mathcal{L}^{\ell-1}\mathcal{X}\to\mathcal{L}^{\ell-1}\mathcal{X}$ is projection onto the i^{th} factor, and in (2.86) we omit pullbacks under the source and target maps in the correspondence (2.87) below. Then the composition law on $F(S^{\ell-1})$ is the quantization of the correspondence

$$\begin{tikzcd}[column sep=small] & \bigl(\mathcal{X}^P,\,\mu\bigr) \arrow[dl, "r_0"'] \arrow[dr, "r_1"] & \\ \bigl(\mathcal{L}^{\ell-1}\mathcal{X}\times\mathcal{L}^{\ell-1}\mathcal{X},\,\pi_1^*(\tau^{\ell-1}\lambda)+\pi_2^*(\tau^{\ell-1}\lambda)\bigr) & & \bigl(\mathcal{L}^{\ell-1}\mathcal{X},\,\tau^{\ell-1}\lambda\bigr)\end{tikzcd} \tag{2.87}$$

The composition law on $F(S^{\ell-1})$ induces the composition law—the fusion product—on $\operatorname{Hom}\bigl(1,F(S^{\ell-1})\bigr)$, the higher category of local codimension ℓ defects. Suppose given $(\mathcal{Y}_1,\mu_1)$ and $(\mathcal{Y}_2,\mu_2)$ semiclassical local defects of codimension ℓ, as in Definition 2.76. Then their product in $\operatorname{Hom}\bigl(1, F(S^{\ell-1})\bigr)$ is the quantization of the composition $r_1 \circ g$ in the homotopy fiber product

$$\begin{tikzcd} & \bigl(\mathcal{Y},\, \pi_1^*\mu_1 + \pi_2^*\mu_2 + \mu\bigr) \arrow[dl, dashed] \arrow[d, dashed, "g"] & \\ \bigl(\mathcal{Y}_1 \times \mathcal{Y}_2,\, \pi_1^*\mu_1 + \pi_2^*\mu_2\bigr) \arrow[d] & \bigl(\mathcal{X}^P,\, \mu\bigr) \arrow[dl, "r_0"'] \arrow[dr, "r_1"] & \\ \bigl(\mathcal{L}^{\ell-1}\mathcal{X} \times \mathcal{L}^{\ell-1}\mathcal{X},\, \pi_1^*(\tau^{\ell-1}\lambda) + \pi_2^*(\tau^{\ell-1}\lambda)\bigr) & & \bigl(\mathcal{L}^{\ell-1}\mathcal{X},\, \tau^{\ell-1}\lambda\bigr) \end{tikzcd} \tag{2.88}$$

This diagram is the general semiclassical composition law on semiclassical defects.

Remark 2.89. The identity object—the tensor unit—in $\operatorname{Hom}\bigl(1, F(S^{\ell-1})\bigr)$ is the quantization of the semiclassical defect

$$\mathcal{X}^{D^\ell} \longrightarrow \mathcal{L}^{\ell-1}\mathcal{X} \tag{2.90}$$

given by the restriction from maps out of D^ℓ to maps out of its boundary $S^{\ell-1}$.

Defects and symmetry

Now, finally, we return to the setup for finite symmetry in field theory, as in Definitions 1.34 and 1.38.

(2.91) (σ,ρ)-*defects.* Fix a positive integer n. Suppose (σ,ρ) is n-dimensional symmetry data.

Definition 2.92. A (σ,ρ)-*defect* is a topological defect in the topological field theory (σ,ρ). We call it a ρ-*defect* if its support lies entirely in a ρ-colored boundary.

Figure 19 depicts some (σ,ρ)-defects. These are defects in the abstract symmetry theory. If F is a quantum field theory equipped with an (σ,ρ)-module structure $(\widetilde{F},\theta)$, then a (σ,ρ)-defect induces a defect in the theory $(\sigma,\rho,\widetilde{F})$, and then θ maps it to a defect in the theory F. Since the defect in the sandwich picture is supported away from $\widetilde{F}$-colored boundaries, it is a *topological* defect in the theory F.

Remark 2.93. Computations with (σ,ρ)-defects, such as compositions, are carried out in the topological field theory (σ,ρ). They apply to the induced defects in any (σ,ρ)-module.

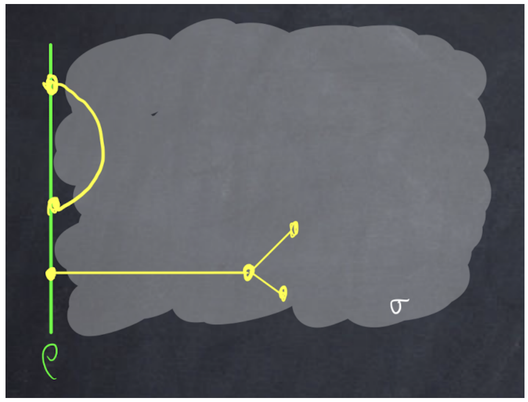

FIGURE 19. (σ, ρ)-defects

Remark 2.94. In higher dimensions, pictures such as Figure 19 are interpreted as a schematic for a tubular neighborhood of the support $Z \subset M$ of a defect on a manifold M (and its Cartesian product with $[0,1]$). Also, unless otherwise stated, for ease of exposition we often implicitly assume a normal framing to Z so that its link may be identified with a standard sphere.

The image in F of a defect in the $(\sigma, \rho, \widetilde{F})$-theory may not be apparent; this is a significant advantage of the sandwich picture of F.

Example 2.95. Let $n = 3$ and consider a 3-dimensional quantum field theory F on S^3, and assume F has an (σ, ρ)-module structure. In the corresponding $(\sigma, \rho, \widetilde{F})$-theory we can contemplate a defect supported on a 2-disk D in $[0,1) \times S^3$ whose boundary $K = \partial D \subset \{0\} \times S^3$ is a knot in the Dirichlet boundary. (Such a knot is termed 'slice'.) It is possible that K does not bound a disk in S^3—its Seifert genus may be positive. In this case the projection of the slice disk D to a defect in the theory F on S^3 is at best an immersed disk with boundary K, and it appears that such a topological defect is difficult to describe directly in the theory F.

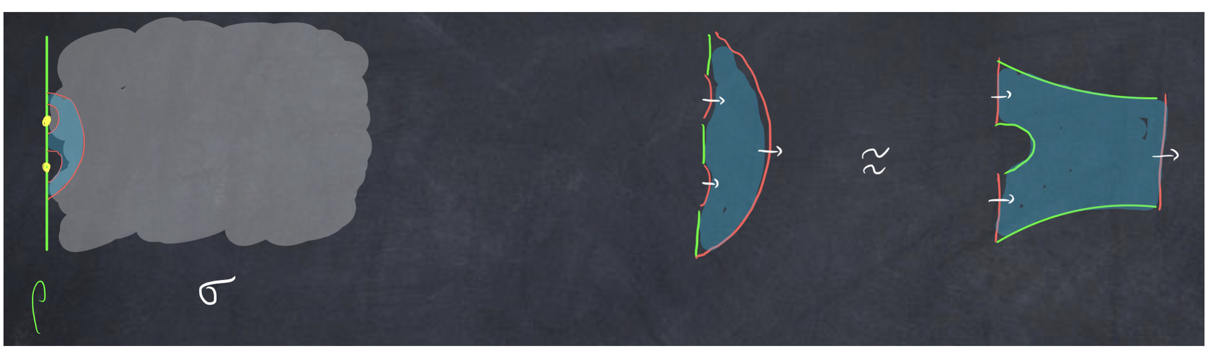

FIGURE 20. The composition law by evaluation on a pair of chaps

(2.96) *Composition of ρ-defects.* (What we say here also applies to more general (σ, ρ)-defects.) Since the labels on ρ-defects come from the topological field theory (σ, ρ) evaluated on the links, we compute the composition law by applying (σ, ρ) to a bordism whose boundary consists of links.

For concreteness, we again take up the quantum mechanical Example 1.41 from Lecture 1. The theory $\sigma = \sigma^{(2)}_{BG}$ is the 2-dimensional finite gauge theory with gauge group G a finite group which acts as symmetries of a quantum mechanical system $(\mathcal{H}, H)$. Figure 20 depicts the composition of two point ρ-defects. The link of such a defect evaluates under (σ, ρ) to the vector space which underlies the group algebra $A = \mathbb{C}[G]$. The composition law on point ρ-defects is computed by evaluating the[26] "pair of chaps" on the right in Figure 20. Picture vertical cross sections of this bordism as links of the two points as the move together and merge into a single point. The (σ, ρ)-value of the pair of chaps works out to be the multiplication map $A \otimes A \to A$ of the group algebra. In particular, on "classical labels" in $G \subset A$ it restricts to the group product $G \times G \to G$.

Exercise 2.97. Evaluate all of the bordisms in the previous paragraph using the finite path integral, as described in (**2.78**).

Remark 2.98. We make several observations that apply far beyond this particular example.

(1) This is a hint for Exercise 2.97! The mapping space of the link of a point ρ-defect is

$$\mathrm{Map}\big(([0,1], \{0,1\})\,,\,(BG, *)\big) \simeq \Omega BG \simeq G. \tag{2.99}$$

The mapping space of the pair of chaps C in Figure 20 fits into the correspondence diagram

$$\begin{tikzcd} & \mathrm{Map}\big((C, \partial C_\rho), (BG, *)\big) \arrow[dl] \arrow[dr] & \\ \Omega BG \times \Omega BG & & \Omega BG \end{tikzcd} \tag{2.100}$$

that encodes restriction to the incoming and outgoing boundaries. Here ∂C_ρ is the ρ-colored portion of ∂C. The left arrow in (2.100) is a homotopy equivalence and the right arrow is composition of loops.

(2) The computation in (2.100) generalizes to any pointed π-finite space $(\mathcal{X}, *)$ in place of $(BG, *)$. Then the correspondence is multiplication on the group $\Omega\mathcal{X}$, and the quantization is pushforward under multiplication, i.e., a convolution product. If the codomain of σ has the form $\mathrm{Alg}(\mathcal{C}')$, then compute $\sigma(\mathrm{pt})$ as follows: (1) quantize $\Omega\mathcal{X}$ to an object in $\mathcal{C}'$, and (2) induce the algebra structure from pushforward under multiplication $\Omega\mathcal{X} \times \Omega\mathcal{X} \to \Omega\mathcal{X}$.

(3) Even if we begin with a group symmetry, as in this example, there are noninvertible topological (σ, ρ)-defects. In this example, elements of the group algebra $\mathbb{C}[G]$ label point defects on the ρ-colored boundary, and the algebra $\mathbb{C}[G]$ contains noninvertible elements. This fits general quantum theory, which produces algebras rather than groups.

(4) (σ, ρ)-defects give rise to structure in any (σ, ρ)-module: linear operators on vector spaces of point defects and on state spaces, endofunctors on categories of line defects and categories of superselection sectors, etc. These can be used to explore dynamics.

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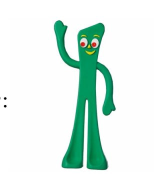
[26] This particular bordism is also known as *Gumby*: 

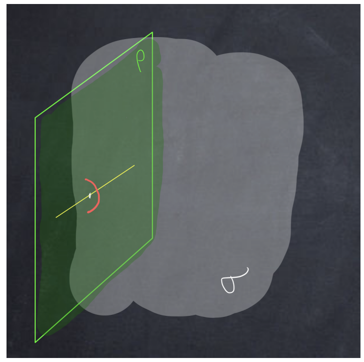


FIGURE 21. A line defect supported on the ρ-colored boundary

(2.101) *2-dimensional theories with finite symmetry.* Let G be a finite group and let $\sigma = \sigma^{(3)}_{BG}$ be finite pure 3-dimensional G-gauge theory. As an extended field theory, σ can take values in Alg(Cat), a suitable 3-category of tensor categories, in which case $\sigma(\mathrm{pt})$ is the fusion category $\mathcal{A} = \mathrm{Vect}[G]$ introduced in (**1.24**). The right regular boundary theory ρ is constructed using the right regular module $\mathcal{A}_{\mathcal{A}}$. Alternatively, in terms of Definition 2.53, choose a basepoint in BG. There are no background fields for σ or ρ: (σ, ρ) is an unoriented theory.

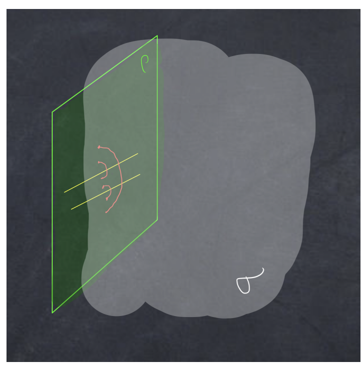


FIGURE 22. Fusion of line defects

The most familiar (σ, ρ)-defects are the codimension 1 defects supported on the ρ-colored boundary, as depicted in Figure 21. The link maps under (σ, ρ) to the quantization of the mapping space (2.99). (It is the same mapping space for the link of a codimension 1 defect in finite gauge theory of *any* dimension.) That quantization in this dimension is a linear category, the category $\mathrm{Vect}(G)$ of vector bundles over G; it is the linear category which underlies the fusion category $\mathcal{A}$. The fusion product—computed from the link in Figure 22, which is the same as the link in Figure 20—is derived from the correspondence (2.100) and is the fusion product of $\mathcal{A}$; see Remark 2.98(2). Each $g \in G$ gives rise to an *invertible* defect, labeled by the vector bundle over G whose fiber is $\mathbb{C}$ at g and is the zero vector space away from g.

Now consider a line defect supported in the bulk, as in Figure 23. The link is a circle, and so a local defect is an object in the category $\sigma(S^1) = \mathrm{Vect}_G(G)$ of G-equivariant vector bundles over G. (Here G acts on itself via conjugation.) This is the (Drinfeld) center of $\mathcal{A}$. Note that unlike the quantum mechanical situation in Figure 11, the center here is "larger" than the algebra $\mathcal{A}$. The

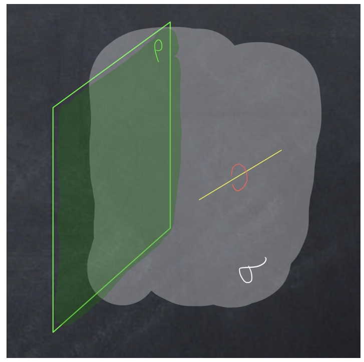


FIGURE 23. A line defect supported in the bulk

simple objects of the center are labeled by a pair consisting of a conjugacy class and an irreducible representation of the centralizer of an element in the conjugacy class. The corresponding defect is invertible iff the representation is 1-dimensional. Among these defects are the Wilson and 't Hooft lines of the 3-dimensional G-gauge theory. There is a rich set of topological defects that goes beyond those labeled by group elements.

Remark 2.102. So, even if we begin with the invertible G-symmetry, we are inexorably led to "non-invertible symmetries".

Exercise 2.103. Verify that there are no non-transparent point defects, either on the ρ-colored boundary or in the bulk.

Exercise 2.104. Evaluate $\sigma(S^1)$ using the finite path integral.

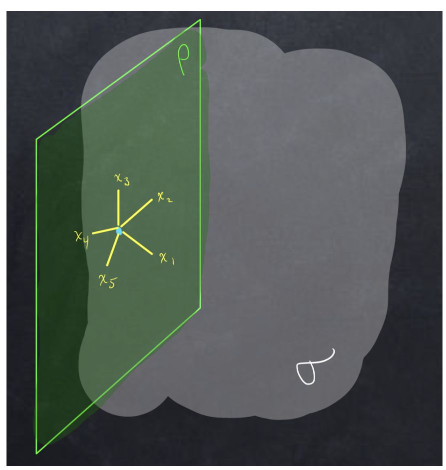


FIGURE 24. A stratified ρ-defect

(2.105) *Turaev-Viro theories.* The example of finite G-gauge theory generalizes to arbitrary Turaev-Viro theories. Let Φ be a spherical fusion category, let σ be the induced 3-dimensional topological field theory (of oriented bordisms) with $\sigma(\mathrm{pt}) = \Phi$, and define the regular boundary theory ρ via the right regular module Φ_Φ. The category of point ρ-defects is the linear category which underlies Φ, so a defect is labeled (locally) by an object of Φ. We can also have nontrivial stratified ρ-defects, such as illustrated in Figure 24. In the figure the x_i are objects of Φ and the label at the central point is a vector in $\mathrm{Hom}_\Phi(1, x_1 \otimes \cdots \otimes x_5)$.

Some additional problems

Problem 2.106. In (2.101) we computed the local line ρ-defects in 3-dimensional pure gauge theory $\sigma = \sigma^{(3)}_{BG}$ with gauge group a finite group G. Recall that σ is defined on all (unoriented) manifolds. Compute the category of line ρ-defects supported on $\mathbb{RP}^1 \subset \mathbb{RP}^2$. (Work on $[0,1) \times \mathbb{RP}^2$ with $\{0\} \times \mathbb{RP}^2$ ρ-colored.) Is there a composition law for these defects?

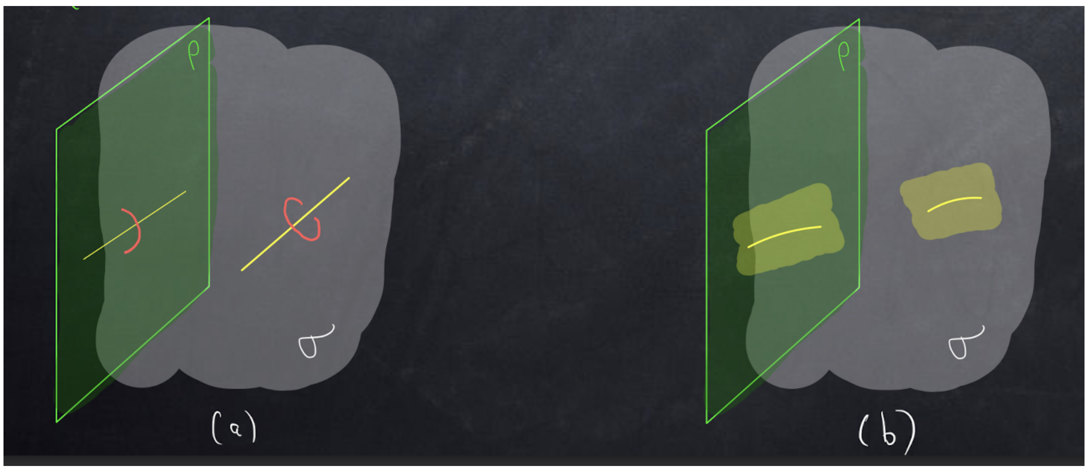


FIGURE 25. (a) A ρ-defect, a σ-defect, and their links; (b) line defects embedded in surface defects

Problem 2.107. Let G be a finite group, let A be a finite abelian group, and let $\mathcal{X}$ be a 2-finite path connected topological space which is an extension

$$B^2A \longrightarrow \mathcal{X} \longrightarrow BG$$

(This is the Postnikov tower of $\mathcal{X}$, which in this case is the classifying space of a 2-group.) Fix a basepoint $* \to \mathcal{X}$ of $\mathcal{X}$. Let $\sigma = \sigma^{(m)}_{\mathcal{X}}$ be the corresponding finite homotopy theory in dimension m, and let ρ be the right regular boundary theory defined by the basepoint. The pair (σ, ρ) is abstract symmetry data for field theories of dimension $n = m - 1$.

(a) Set $n = 3$. Compute the category of local line ρ-defects, i.e., of local line defects supported on the ρ-colored boundary. Compute the category of local line defects in the theory σ. See Figure 25(a).

(b) Repeat for surface defects of both types. Your answer may be a linear 2-category, a tensor category, ...

(c) Repeat for a line defect in a surface defect of both types; see Figure 25(b).

(d) Repeat the problem for $n = 4$.

Problem 2.108. A *crossed module* is the following data: a pair of groups (H, K), a homomorphism $d\colon H \to K$, and an action of K on H such that for all $h, h', h'' \in H$ and $k \in K$ we have

$$\begin{aligned} d(h') \cdot h'' &= h' h'' (h')^{-1} \\ d(k \cdot h) &= k\, d(h)\, k^{-1} \end{aligned}$$

Define $A = \ker d$ and $G = \operatorname{coker} d$. (First verify that $d(H)$ is normal in K.) Assume that A and G are finite groups.

(a) Prove that A is a subgroup of the center of H. In particular, A is abelian.

(b) Construct an action of G on A.

(c) Construct a cohomology class in $H^3(G; A)$.

(d) Show that a crossed module is a group object in the category of groupoids.

(e) The data heretofore produces a 2-finite path connected pointed space $\mathcal{X}$ with $\pi_1\mathcal{X} = G$ and $\pi_2\mathcal{X} = A$. In other terms, this is the classifying space of a 2-group. How, from a 2-finite path connected pointed space $\mathcal{X}$, do you extract the data (G, A, action of G on A, cohomology class)?

Problem 2.109. Suppose H is a Lie group, $A \subset H$ is a finite subgroup of the center of H, and $\widehat{H}$ is a Lie group which contains H as a normal subgroup which is a union of components of $\widehat{H}$. Define $G = \widehat{H}/H$. Assume that G is a finite group.

(a) Prove that $A \subset \widehat{H}$ is normal. Set $K = \widehat{H}/A$.

(b) Construct a crossed module $d\colon H \to K$. Let $\mathcal{X}$ be the corresponding 2-finite path connected pointed space (defined up to homotopy).

(c) Construct a fibration $BH \to BK \to \mathcal{X}$. In the world of simplicial sheaves on Man, construct a fibration $B_\nabla H \to B_\nabla G \to \mathcal{X}$.

(d) Find examples of (the result of) this construction in `arXiv:2204.06564`. In other words, find examples of the groups $H, \widehat{H}, A$. Can you find some of the higher categories of defects you computed in the previous problem in that paper?

(e) For any Lie group H, the automorphism group $\operatorname{Aut}(H)$ is a Lie group. Construct a crossed module $d\colon H \to \operatorname{Aut}(H)$. What is the data of the associated homotopy type? How/When does this 2-group act on H-gauge theory?

Lecture 3: Quotients and projectivity

This lecture has two main subjects: quotients and projectivity. We already treated these topics in the context of groups and algebras: see **(1.49)** to the end of Lecture 1. Here we take up quotients by a symmetry of a field theory. Working in the "sandwich" picture, which separates out the abstract topological symmetry from the potentially nontopological field theory on which it acts, the quotient is effected by replacing the right regular boundary theory (Dirichlet) with an augmentation (Neumann). We also introduce *quotient defects*, which amounts to executing the quotient construction on a submanifold. In the literature these are often called "condensation defects", and the quotienting process is called "gauging". Quantum theory takes place in projective geometry, not linear geometry, and so in the second half of the lecture we take up projectivity in the context of field theory. It is expressed via invertible field theories. The projectivity of an n-dimensional field theory is an $(n+1)$-dimensional invertible field theory—its *anomaly (theory)*—and trivializations of the anomaly form a torsor over the group of n-dimensional invertible field theories. In other words, the group of n-dimensional invertible field theories acts on the space of all n-dimensional field theories, and theories in the same orbit share many of the same properties. A symmetry may only act projectively, in which case it is said to enjoy an *'t Hooft anomaly*, and that obstructs the existence of an augmentation, so too obstructs the existence of a quotient. We conclude with an example of quotients and twisted quotients for theories with a BA-symmetry for a finite abelian group A, namely line operators in 4-dimensional gauge theory.

Preliminary remarks

I began the lecture with two remarks, which I include in these notes.

(3.1) *Invertibility.* Anytime we have a (higher categorical) monoid—a set with an associative composition law $*$ and unit 1—then we have a notion of invertibility: an element/object x is *invertible* if there exists y such that $x * y = 1$ (or $x * y \cong 1$ in a categorical context). Note that invertibility is a condition, not data. This applies to two situations in these lectures: (1) the composition law ("stacking") of field theories, which leads to the notion of an *invertible field theory*; and (2) the composition law ("fusion") of local topological defects, which leads to the notion of an *invertible topological defect.*

(3.2) *Symmetries of a boundary theory.* Not every left boundary theory F of a field theory α indicates that α acts as *symmetries* on F. In these lectures we define *finite* symmetry in terms of a pair (σ, ρ); simply having a left boundary theory F for a theory α is not an action of symmetry. Furthermore, in this paragraph we do not require that α be topological.[27] On the other hand, if (σ, ρ) is finite symmetry data, then there is a notion of (σ, ρ) acting by symmetries on a theory F which is a left boundary theory for α. Namely, the *left module structure* data $(\widetilde{F}, \theta)$ is a left module $\widetilde{F}$ for $\sigma \otimes \alpha$ and an isomorphism as indicated in Figure 26.

[27] Nor do we require that α be invertible; if it is, then we say F is *anomalous* with *anomaly theory* α.

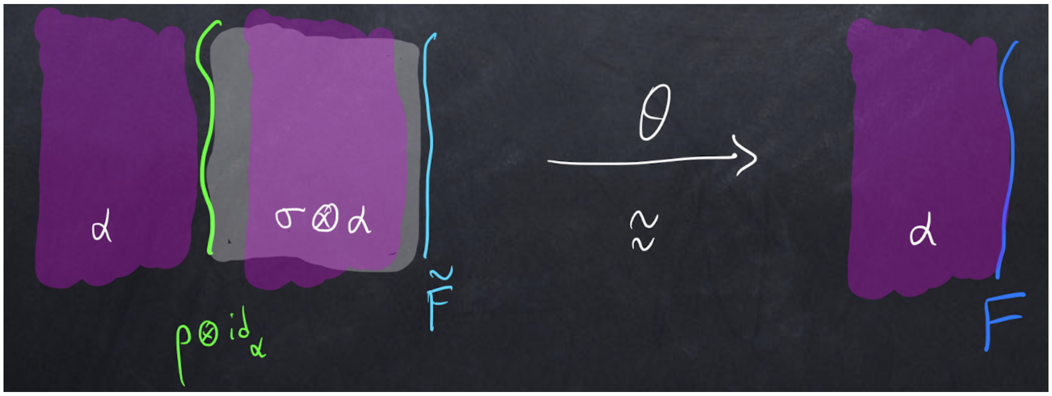


FIGURE 26. The action of (σ, ρ) on the boundary theory F of α

Quotients by a symmetry in field theory

One should think that the quotients in this section are "derived" or "homotopical", though we do not deploy those modifiers. (See Exercise 1.57.)

(3.3) *Augmentations in higher categories.*

Definition 3.4. Let $\mathcal{C}'$ be a symmetric monoidal n-category, and set $\mathcal{C} = \mathrm{Alg}(\mathcal{C}')$. Suppose $A \in \mathcal{C}$ is an algebra object in $\mathcal{C}'$. Then an *augmentation* $\epsilon_A\colon A \to 1$ is an algebra homomorphism from A to the tensor unit $1 \in \mathcal{C}$.

Thus ϵ_A is a 1-morphism in $\mathcal{C}'$ equipped with data that exhibits the structure of an algebra homomorphism. Augmentations may not exist.

Remark 3.5. A general 1-morphism $A \to 1$ in $\mathcal{C}$ is an object of $\mathcal{C}'$ equipped with a right A-module structure. An augmentation is a right A-module structure on the tensor unit $1 \in \mathcal{C}'$.

(3.6) *Augmentations in field theory.*

Definition 3.7. Let $\mathcal{C}'$ be a symmetric monoidal n-category, and set $\mathcal{C} = \mathrm{Alg}(\mathcal{C}')$. Let $\mathcal{F}$ be a collection of $(n+1)$-dimensional fields, and suppose $\sigma\colon \mathrm{Bord}_{n+1}(\mathcal{F}) \to \mathcal{C}$ is a topological field theory. A right boundary theory ϵ for σ is an *augmentation* of σ if $\epsilon(\mathrm{pt})$ is an augmentation of $\sigma(\mathrm{pt})$ in the sense of Definition 3.4.

An augmentation in this sense is often called a *Neumann boundary theory.*

(3.8) *The quotient theory.* We use notations in Definition 1.34 and Definition 1.38 in the following.

Definition 3.9. Suppose given finite symmetry data (σ, ρ) and a (σ, ρ)-module structure $(\widetilde{F}, \theta)$ on a quantum field theory F. Suppose ϵ is an augmentation of σ. Then the *quotient* of F by the symmetry σ is

$$F\underset{\epsilon}{/}\sigma = \epsilon \otimes_\sigma \widetilde{F}. \tag{3.10}$$

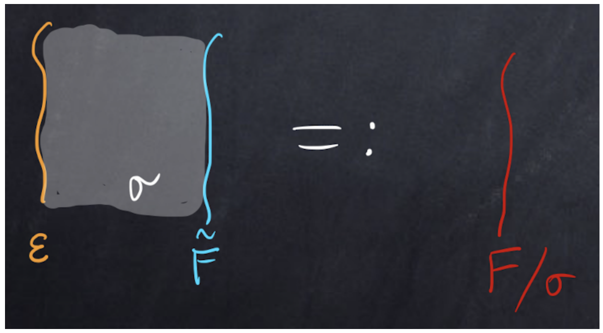


FIGURE 27. The quotient theory

We simply write 'F/σ' if the augmentation ϵ is understood from context. The right hand side of (3.10) is the sandwich in Figure 27.

(3.11) *Quotients in finite homotopy theories.* Recall the definition of a semiclassical boundary theory in **(2.51)**. We now tell what an augmentation is in this context.

Definition 3.12. Let $\mathcal{X}$ be a π-finite space and suppose λ is a cocycle of degree m on $\mathcal{X}$. A *semiclassical right augmentation* of $(\mathcal{X}, \lambda)$ is a trivialization μ of $-\lambda$.

Observe that if $\lambda = 0$, then μ is a cocycle of degree m. Also, there is a canonical choice of μ in this instance: $\mu = 0$.

Remark 3.13. The cocycle λ encodes an *'t Hooft anomaly* in a finite homotopy type theory; it is the projectivity of the symmetry. If $\lambda = 0$, then a cocycle μ encodes a twist of the boundary theory, and it goes by various names: 'discrete torsion', 'θ-angles', etc., depending on the context.

Example 3.14. Let G be a finite group, and let $\sigma = \sigma^{(n+1)}_{BG}$ be the associated finite gauge theory. Use the canonical boundary theory $\mathrm{id}_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{X}$. In the semiclassical picture this corresponds to summing over all principal G-bundles with no additional fields on the ϵ-colored boundaries. This is the usual quotienting operation, oft called 'gauging'.

(3.15) *The Dirichlet-Neumann and Neumann-Dirichlet domain walls.*

Lemma 3.16. *Let σ be a topological field theory with codomain $\mathcal{C} = \mathrm{Alg}(\mathcal{C}')$, and suppose ρ is the right regular boundary theory of σ and ϵ is an augmentation of σ. Then the category of domain walls from σ to ϵ is the trivial theory, as is the category of domain walls from ϵ to σ.*

Roughly: Use the homomorphism $\epsilon(\mathrm{pt}) \colon A \to 1$ to make 1 into a left A-module, where $A = \sigma(\mathrm{pt})$, and so construct a dual *left* boundary theory ϵ^L, the left adjoint. Then the sandwich $\rho \otimes_\sigma \epsilon^L$ is the trivial theory: use the cobordism hypothesis to compute its value on a point as $A \otimes_A 1 \cong 1$. Let

$$\begin{aligned} \delta \colon \rho &\longrightarrow \epsilon \\ \delta^* \colon \epsilon &\longrightarrow \rho \end{aligned} \tag{3.17}$$

be generating domain walls.

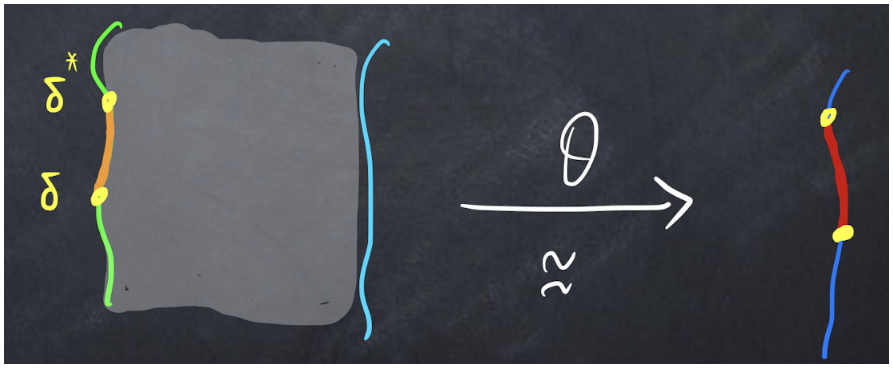


FIGURE 28. The domain walls δ and δ^*

(3.18) *The composition.* Our task is to compute the composition

$$\delta^* \circ \delta \colon \rho \longrightarrow \rho, \tag{3.19}$$

which is a self-domain wall of the boundary theory ρ. (The reverse composition $\delta \circ \delta^*$ is similar.) As always, that computation is done by tracking the links as the points come together, and we obtain the pair of chaps extracted in Figure 29 and isolated in Figure 30. In that figure we have labeled

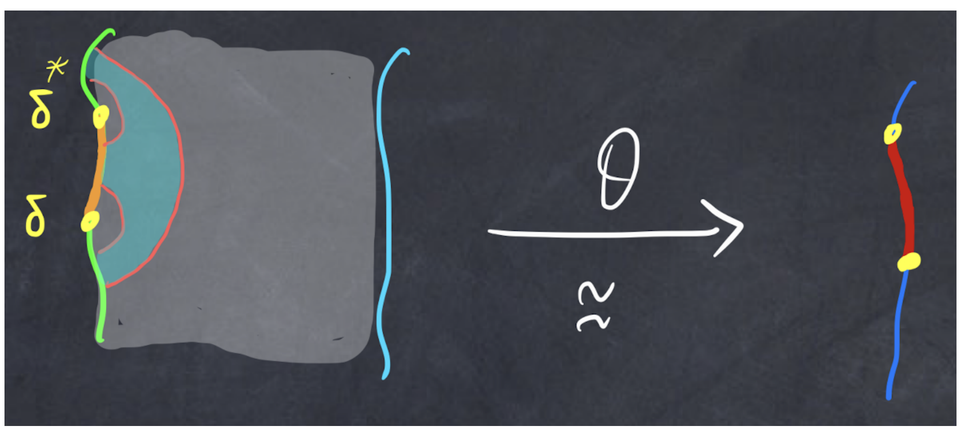


FIGURE 29. Computation of $\delta^* \circ \delta$

the incoming boundary components in accordance with Lemma 3.16. For the outgoing boundary component we are assuming $\sigma(\mathrm{pt}) = A$ is an algebra in $\mathcal{C} = \mathrm{Alg}(\mathcal{C}')$; the label A in the figure is the underlying object of $\mathcal{C}'$. This evaluates to an object in $\mathrm{Hom}_{\mathcal{C}'}(1, A)$. We evaluate it in two cases.

Example 3.20 (Turaev-Viro symmetry). Suppose $n = 2$ and the 3-dimensional theory σ is of Turaev-Viro type with $\sigma(\mathrm{pt}) = \mathcal{A}$ a fusion category. Assume ρ is given by the right regular module $\mathcal{A}_{\mathcal{A}}$ and ϵ is given by a fiber functor $\epsilon_\Phi \colon \mathcal{A} \to \mathrm{Vect}$. Then the codimension 1 quotient defect has local label the object $x_{\mathrm{reg}} \in \mathcal{A}$ defined as

$$x_{\mathrm{reg}} = \sum_x \epsilon_\Phi(x)^* \otimes x, \tag{3.21}$$

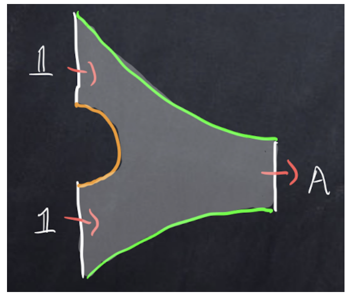


FIGURE 30. The pair of chaps: note the coloring of the boundary components

where the sum is over a representative set of simple objects x. See Proposition 8.9 in [23] for a very similar computation.

Example 3.22 (finite homotopy theories)**.** Let $\mathcal{X}$ be a π-finite space. Then the composition is the homotopy fiber product

$$\begin{tikzcd}
& & \Omega\mathcal{X} \arrow[dl, dashed] \arrow[dr, dashed] & & \\
& \ast \arrow[dl] \arrow[dr] & & \ast \arrow[dl] \arrow[dr] & \\
\ast \arrow[drr] & & \mathcal{X} \arrow[d] & & \ast \arrow[dll] \\
& & \mathcal{X} & &
\end{tikzcd} \tag{3.23}$$

which is then the domain wall

$$\begin{tikzcd}
& \Omega\mathcal{X} \arrow[dl] \arrow[dr] & \\
\ast \arrow[dr] & & \ast \arrow[dl] \\
& \mathcal{X} &
\end{tikzcd} \tag{3.24}$$

Note that the points $*$ in the second row of (3.23) are obtained as fiber products from the parts of the diagram which lie below it, and the fact that they are single points (contractible spaces) proves Lemma 3.16 in the finite homotopy theory case.

Example 3.25 (special case)**.** Now take $\mathcal{X} = BG$ for G a finite group. Then $\Omega\mathcal{X} \simeq G$, so the composition $\delta^* \circ \delta$ sums over maps to G. Suppose M is a bordism on which we evaluate F, and suppose $Z \subset M$ is a cooriented codimension submanifold on which we place the defect $\delta^* \circ \delta$. (As usual, we do not make background fields explicit.) Form the sandwich $[0,1) \times M$ with $\{0\} \times (M \setminus Z)$ colored with ρ and $\{0\} \times Z$ colored with $\delta^* \circ \delta$. Then the theory sums over principal G-bundles

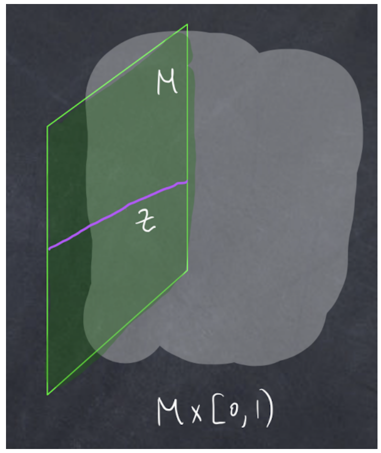


FIGURE 31. The defect $\delta^* \circ \delta$ on Z

together with a trivialization on the ρ-colored boundary $\{0\} \times (M \setminus Z)$. For each $g \in G$ there is a defect $\eta(g)$ (of "'t Hooft type") which constrains the jump in the trivializations across Z to be g. Then quantization using (3.24) shows

$$\delta^* \circ \delta = \sum_{g \in G} \eta(g). \tag{3.26}$$

This equation appears in several recent physics papers. A similar equation holds for $\mathcal{X}$ in general, and in particular for $\mathcal{X} = B^q A$ for a finite abelian group A, only now automorphisms appear in the quantization.

Quotient defects: quotienting on a submanifold

The following discussion is inspired by the paper [28]. The basic idea is to execute the quotient construction on a submanifold, not necessarily to take the quotient of the entire theory.

Fix a positive integer n and finite n-dimensional symmetry data (σ, ρ). Suppose ϵ is an augmentation of σ, as in Definition 3.7. As explained in Definition 3.9, if $(\widetilde{F}, \theta)$ is a (σ, ρ)-module structure on an n-dimensional quantum field theory F, then dimensional reduction of σ depicted in Figure 27, which is the sandwich $\epsilon \otimes_\sigma \widetilde{F}$, is the quotient F/σ of F by the symmetry. This can be interpreted as placing the topological defect ϵ on the entire theory.

There is a generalization which places the defect on a submanifold. Suppose M is a bordism on which we evaluate F, and suppose $Z \subset M$ is a submanifold on which we place the defect. (As usual, we do not make background fields explicit.) Form the sandwich $[0, 1) \times M$ with $\{0\} \times M$ colored with ρ. Let $\nu \subset M$ be an open tubular neighborhood of $Z \subset M$ with projection $\pi \colon \nu \to Z$, and arrange that the closure $\bar{\nu}$ of ν is the total space of a disk bundle $\bar{\nu} \to Z$.

Definition 3.27. The *quotient defect* $\epsilon(Z)$ is the ρ-defect supported on $\{0\} \times \bar{\nu}$ with $\{0\} \times \nu$ colored with ϵ and $\{0\} \times \partial\bar{\nu}$ colored with δ.

[28] Konstantinos Roumpedakis, Sahand Seifnashri, and Shu-Heng Shao, *Higher Gauging and Non-invertible Condensation Defects*, arXiv:2204.02407.

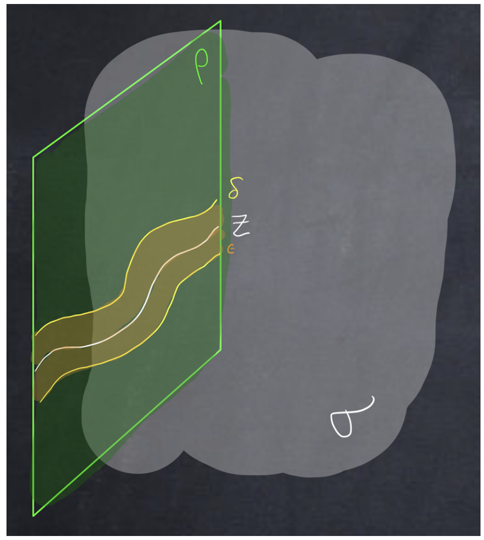


FIGURE 32. The quotient defect $\epsilon(Z)$

This defect is depicted in Figure 32.

Next we compute the local label of the quotient defect $\epsilon(Z)$, as in Definition 2.65(1), and so express $\epsilon(Z)$ as a defect supported on Z. Consider a somewhat larger tubular neighborhood, now of $\{0\} \times Z \subset [0,1) \times M$. Let $\ell = \operatorname{codim}_M Z$. The tubular neighborhood for $\ell = 1$ is depicted in Figure 33. It is a pair of chaps, two of whose incoming boundary components are δ-colored. Its value in the topological theory σ—with boundaries and defects ρ, ϵ, δ—is an object in $\operatorname{Hom}\bigl(1, \sigma(D^1, S^0_\delta)\bigr)$. (If $\mathcal{C} = \operatorname{Alg}(\mathcal{C}')$ is the codomain of σ, and $\sigma(\mathrm{pt}) = A$ is an algebra object in $\mathcal{C}'$, then $\sigma(D^1, S^0_\delta) = A$ as an object of $\mathcal{C}'$.)

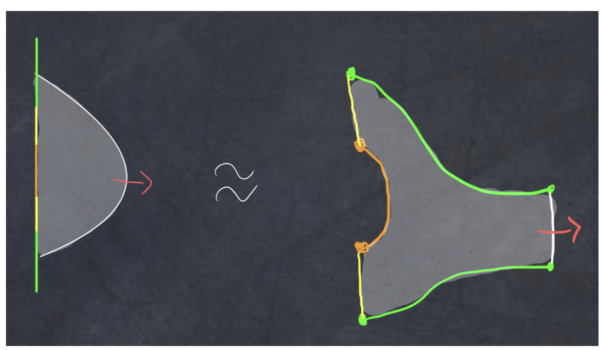


FIGURE 33. The local label of $\epsilon(Z)$ in codimension 1

Remark 3.28. The pair of chaps picture makes clear that the defect $\epsilon(Z)$ for $\ell = 1$ can be interpreted as follows, assuming $Z \subset M$ has trivialized normal bundle. Let Z_1, Z_2 be parallel normal translates of Z, color the region in between $\{0\} \times Z_1$ and $\{0\} \times Z_2$ with ϵ, color the remainder of $\{0\} \times M$ with ρ, and use the domain wall δ at $\{0\} \times Z_1$ and δ^* at $\{0\} \times Z_2$. Then $\epsilon(Z)$ is the composition $\delta^*(Z_2) * \delta(Z_1)$. If a quantum field theory F has a (σ, ρ)-module structure, then $\delta(Z_1)$ is a domain wall from F to F/σ and $\delta^*(Z_2)$ is a domain wall from F/σ to F; the composition $\epsilon(Z)$ is a self domain wall of F, precisely the one computed in (**3.18**).

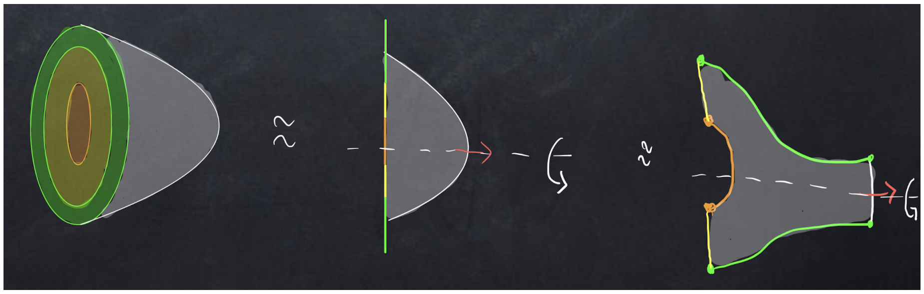


FIGURE 34. The local label of $\epsilon(Z)$ in codimension 2

The tubular neighborhood of $\{0\} \times Z \subset [0,1) \times M$ for codimension $\ell = 2$ is the 3-dimensional bordism obtained from the pair of chaps by revolution in 3-space, as illustrated in Figure 34. For general $\ell > 1$, the bordism is the $(\ell+1)$-disk $D^{\ell+1}$ with boundary S^ℓ partitioned as

$$\partial D^{\ell+1} = D^\ell_\epsilon \cup A^\ell_\rho \cup D^\ell \tag{3.29}$$

into disks D^ℓ and an annulus A^ℓ with the domain wall δ at the intersection of the ϵ and ρ-colored regions. (In Figure 34 that domain wall is thickened from a sphere $S^{\ell-1}$ to an annulus A^ℓ.)

Remark 3.30. These are the local defects. The global defects are a section of a bundle (local system) of local defects over the submanifold $Z \subset M$.

Example 3.31 (finite homotopy theory)**.** Let $\sigma = \sigma^{(n+1)}_{\mathcal{X}}$ be the finite homotopy theory built from a π-finite space $\mathcal{X}$. Then we can use the semiclassical calculus for π-finite spaces to compute semiclassical spaces of defects. Suppose ρ is specified by a basepoint $* \to \mathcal{X}$ and ϵ is specified by the identity map $\mathcal{X} \xrightarrow{\text{id}} \mathcal{X}$. Then δ is specified by the homotopy fiber product

$$\begin{tikzcd} & * \arrow[dl, dashed] \arrow[dr, dashed] & \\ * \arrow[dr] & & \mathcal{X} \arrow[dl, "\text{id}"] \\ & \mathcal{X} & \end{tikzcd} \tag{3.32}$$

which is a point. This is the manifestation of the uniqueness of δ (Lemma 3.16), as already remarked after (3.24).

The semiclassical space of local ρ-defects of codimension ℓ is

$$\text{Map}\big((D^\ell, S^{\ell-1}), (\mathcal{X}, *)\big) = \Omega^\ell \mathcal{X}. \tag{3.33}$$

Set

$$N^\ell = \big(D^{\ell+1}, D^\ell \cup A^\ell \cup D^\ell\big), \tag{3.34}$$

with boundary as in (3.29); see Figure 34. The semiclassical local label of the defect $\epsilon(Z)$ is

$$\operatorname{Map}(N^\ell, \mathcal{X}) \longrightarrow \Omega^\ell\mathcal{X}, \tag{3.35}$$

the map induced by restriction to $\Omega^\ell\mathcal{X}$.

Lemma 3.36. *There is a homotopy equivalence* $\operatorname{Map}(N^\ell, \mathcal{X}) \simeq \Omega^\ell\mathcal{X}$ *under which* (3.35) *is the identity map.*

Proof. Use the technique in Example 0.8 of [29]. First, deformation retract A^ℓ to $S^{\ell-1}$, and so define

$$\overline{N}^\ell = \big(D^{\ell+1}, D^\ell \cup S^{\ell-1} \cup D^\ell\big). \tag{3.37}$$

Choose a basepoint for $\overline{N}^\ell$ on $S^{\ell-1}$. Form the correspondence of pointed spaces

$$\begin{tikzcd}[column sep=small] & \overline{N}^\ell \cup D^\ell \arrow[dl] \arrow[dr] & \\ \overline{N}^\ell \,/\, S^{\ell-1} & & D^\ell \,/\, S^{\ell-1} \end{tikzcd} \tag{3.38}$$

in which D^ℓ is attached to $S^{\ell-1} \subset \overline{N}^\ell$, the left map collapses this new D^ℓ, and the right map collapses $D^{\ell+1}$. Since $D^\ell, D^{\ell+1}$ are contractible, each of these arrows is a homotopy equivalence. Now take the pointed mapping spaces into $\mathcal{X}$. $\square$

The quantization of $\mathrm{id}\colon \Omega^\ell\mathcal{X} \to \Omega^\ell\mathcal{X}$ is typically a noninvertible object. For example, if the quantization is a vector space, then the vector space is[30] $\operatorname{Fun}(\pi_0\Omega^\ell\mathcal{X}) = \operatorname{Fun}(\pi_\ell\mathcal{X})$; the local label is the constant function 1. If the quantization is a linear category, then it is the category $\operatorname{Vect}(\Omega^\ell\mathcal{X})$ of flat vector bundles over $\Omega^\ell\mathcal{X}$, i.e., vector bundles on the fundamental groupoid $\pi_{\le 1}\Omega^\ell\mathcal{X}$; the local label is the trivial bundle with fiber $\mathbb{C}$.

$$\mathrm{id}\colon \operatorname{Map}(Z^\nu, \mathcal{X}) \longrightarrow \operatorname{Map}(Z^\nu, \mathcal{X}), \tag{3.39}$$

where Z^ν is the Thom space of the normal bundle. As an example, suppose $\ell = 1$ and assume that the normal bundle $\nu \to Z$ has been trivialized. (This amounts to a coorientation of the codimension 1 submanifold $Z \subset M$—a direction for the domain wall.) Then

$$\operatorname{Map}(Z^\nu, \mathcal{X}) \simeq \operatorname{Map}(Z, \Omega\mathcal{X}). \tag{3.40}$$

For example, if A is a finite abelian group and $\mathcal{X} = B^2A$—so σ encodes a BA-symmetry—then $\operatorname{Map}(Z^\nu, B^2A) \simeq \operatorname{Map}(Z, BA)$ is the "space" of principal A-bundles $P \to Z$. One should, rather,

[29] Allen Hatcher, *Algebraic topology.* available at https://pi.math.cornell.edu/~hatcher/AT/ATpage.html.
[30] The homotopy group $\pi_\ell\mathcal{X} = \pi_\ell(\mathcal{X}, *)$ uses the basepoint $* \in \mathcal{X}$.

treat it as a groupoid, the groupoid $\mathrm{Bun}_A(Z)$ of principal A-bundles over Z and isomorphisms between them. A point $* \to \mathrm{Bun}_A(Z)$ is a principal A-bundle $P \to Z$, and this map quantizes to a defect $\eta(P)$ on Z. The quantization of $\mathrm{id}\colon \mathrm{Bun}_A(Z) \to \mathrm{Bun}_A(Z)$ is a sum of the quantizations of $*\big/\mathrm{Aut}\,P \to \mathrm{Bun}_A(Z)$ over isomorphism classes of principal A-bundles $P \to Z$. Informally, we might write this as a sum of

$$\frac{1}{\mathrm{Aut}\,P}\,\eta(P) = \frac{1}{H^0(Z;A)}\,\eta(P). \tag{3.41}$$

This sort of expression appears in [31], for example; compare Example 3.25.

The ρ-defect $\eta(P)$ has a geometric semiclassical interpretation. Without the defect one is summing over A-gerbes on $[0,1)\times M$ which are trivialized on $\{0\}\times M$. The defect $\eta(P)$ on $\{0\}\times Z$ tells to only trivialize the A-gerbe on $\bigl(\{0\}\times M\bigr) \ \setminus\ \bigl(\{0\}\times Z\bigr)$ and to demand—relative to the coorientation of Z—that the trivialization jump by the A-bundle $P\to Z$.

Remark 3.42. If the π-finite space $\mathcal{X}$ is equipped with a cocycle λ which represents a cohomology class $[\lambda]\in h^n(\mathcal{X})$ for some cohomology theory h, then a codimension ℓ quotient defect has semiclassical label space $\Omega^\ell\mathcal{X}$ with transgressed cocycle and its cohomology class $[\tau^\ell\lambda]\in h^{n-\ell}(\Omega^\ell\mathcal{X})$. A nonzero cohomology class obstructs the existence of the quotient. However, as observed in [28] it is possible that $[\lambda]\neq 0$ but $[\tau^\ell\lambda]=0$ for some ℓ, which means that the quotient of the entire theory by σ does not exist, but quotient defects of sufficiently high codimension do exist.

Projectivity

We begin with some ruminations on projective symmetry in quantum theory, in part to make contact with Clay's lecture series. A review of **(1.61)** at this point is warranted.

(3.43) *Linear and projective geometry.* Let V be a *linear* space, say finite dimensional and complex. The automorphism group $\mathrm{Aut}\,V$ consists of invertible linear maps $T\colon V\to V$; after a choice of basis it is isomorphic to the group of invertible square complex matrices of size equal to $\dim V$. The *projective* space $\mathbb{P}V$ is the space of lines (1-dimensional subspaces) of V. A linear automorphism $T\in\mathrm{Aut}\,V$ induces an automorphism $\overline{T}$ of $\mathbb{P}V$; the linear map T takes lines to lines. A homothety (scalar multiplication) of V induces the identity map of $\mathbb{P}V$. More precisely, there is a group extension

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \mathrm{Aut}\,V \longrightarrow \mathrm{Aut}\,\mathbb{P}V \longrightarrow 1 \tag{3.44}$$

which serves to define $\mathrm{Aut}\,\mathbb{P}V$. This extension is *central*; the kernel $\mathbb{C}^\times$ lies in the center of $\mathrm{Aut}\,V$ (and equals the center). Each projective transformation in $\mathrm{Aut}\,\mathbb{P}V$ has a $\mathbb{C}^\times$-torsor of linear lifts.

[31] Yichul Choi, Clay Cordova, Po-Shen Hsin, Ho Tat Lam, and Shu-Heng Shao, *Non-Invertible Duality Defects in 3+1 Dimensions*, arXiv:2111.01139.

The projective action of a group G on $\mathbb{P}V$ is a group homomorphism $G \to \operatorname{Aut}\mathbb{P}V$. One can use it to pull back the central extension (3.44) to a central extension

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow G^\tau \longrightarrow G \longrightarrow 1 \tag{3.45}$$

of G. The group G^τ acts linearly on V through a group homomorphism $G^\tau \to \operatorname{Aut} V$ and, as our starting point, the group G acts projectively on $\mathbb{P}V$. The central extension (3.45) is a measure of the projectivity of this projective action. If G is a discrete group, say a finite group, then the equivalence class of the central extension is an element of the cohomology group $H^2(G;\mathbb{C}^\times)$. If this class is zero, then there exist splittings of (3.45). Furthermore, the splittings form a torsor over the group of characters of G, i.e., over the cohomology group $H^1(G;\mathbb{C}^\times)$: given a splitting s any other splitting is the product of s with a character $\chi\colon G \to \mathbb{C}^\times$. In summary:

$$\begin{aligned} \text{existence: }\ & H^2(G;\mathbb{C}^\times) \\ \text{uniqueness: }\ & H^1(G;\mathbb{C}^\times) \end{aligned} \tag{3.46}$$

(3.47) *Quantum theory is projective, not linear.* The space of pure states of a quantum system is a projective space $\mathcal{P}$, at first with no topology. Instead there is a function

$$\mathcal{P} \times \mathcal{P} \longrightarrow [0,1] \tag{3.48}$$

which maps an ordered pair of states to the probability of transitioning from one state to the other.[32] Symmetries of the quantum system are automorphisms of $\mathcal{P}$ which preserve the function (3.48).

A fundamental theory due to Wigner (and von Neumann) states that for $\mathcal{P} = \mathbb{P}\mathcal{H}$ any symmetry of the quantum system lifts either to a unitary automorphism of $\mathcal{H}$ or an antiunitary automorphism of $\mathcal{H}$. (See [33] for geometric proofs.) In fact, up to a simple transformation, the transition function (3.48) equals the distance function for the Fubini-Study metric on $\mathbb{P}\mathcal{H}$, and so Wigner's theorem becomes a theorem about its isometries. It follows that the projectivity of a group G of symmetries of quantum theory is measured by a $\mathbb{Z}/2\mathbb{Z}$-graded central extension of G, where the $\mathbb{Z}/2\mathbb{Z}$-grading keeps track of the unitary vs. antiunitary dichotomy. (See [34] for a fuller discussion of symmetry in quantum mechanics.)

[32] If we write $\mathcal{P} = \mathbb{P}\mathcal{H}$ for a Hilbert space $\mathcal{H}$, and $\psi_1, \psi_2 \in \mathcal{H}$ are nonzero vectors, then the value of (3.48) on the pair of lines generated is

$$\frac{|\langle \psi_1, \psi_2\rangle|^2}{\|\psi_1\|^2\, \|\psi_2\|^2}. \tag{3.49}$$

[33] Daniel S. Freed, *On Wigner's theorem*, Proceedings of the Freedman Fest (Vyacheslav Krushkal Rob Kirby and Zhenghan Wang, eds.), Geometry & Topology Monographs, vol. 18, Mathematical Sciences Publishers, 2012, pp. 83–89. arXiv:1211.2133.

[34] Daniel S. Freed and Gregory W. Moore, *Twisted equivariant matter*, Ann. Henri Poincaré **14** (2013), no. 8, 1927–2023, arXiv:1208.5055. I apologize for the self-referential tendencies I learned from Gilderoy Lockhart. The paper on which these summer school notes are based will not be so disgustingly provincial.

(3.50) *Projectivity in quantum field theory: anomalies.* Here the metaphor (2.2) comes in handy. A field theory $F\colon \operatorname{Bord}_n(\mathcal{F}) \to \mathcal{C}$ is a linear representation of $\operatorname{Bord}_n(\mathcal{F})$, and since quantum theory is projective we expect this representation to be projective, and furthermore its projectivity should be measured by some kind of "cocycle" on $\operatorname{Bord}_n(\mathcal{F})$. Indeed, the right kind of cocycle in this context is an *invertible field theory*. In this context the projectivity is called an *anomaly (theory)*;[35] it is an invertible field theory over $\mathcal{F}$ of dimension $n+1$. Observe that in the case of a finite group G acting on a quantum mechanical system ($n=1$), this matches the cocycle of a group extension (3.45), which has degree 2.

Anomalies are not a sickness of a theory; to the contrary, they are useful tools for investigating its behavior. They are only potentially a sickness when we want to "integrate out" some fields, i.e., turn some background fields into fluctuating fields. Roughly speaking, that is because we can integrate functions valued in a *linear space*, not functions valued in a *projective space*. So to carry out the integration, we must lift the projective geometry (field theory) to linear geometry. In this situation we have a fiber bundle of fields, such as

$$\begin{aligned} &\mathcal{F} = \{\text{Riemannian metric, orientation, } H\text{-connection}\}\\ &\Big\downarrow{\scriptstyle\pi} \\ &\overline{\mathcal{F}} = \{\text{Riemannian metric, orientation}\}\end{aligned} \tag{3.51}$$

The total space is the space of background fields in the starting theory, the base is the space of background fields in the pushforward theory, and the fibers are the background fields in the original theory that we have promoted to fluctuating fields. Suppose the original theory over $\mathcal{F}$ has an anomaly α: an invertible $(n+1)$-dimensional theory over $\mathcal{F}$. Then to integrate over the fibers of π—to push forward under π—we need to provide *descent data* for α. In other words, we need to provide an $(n+1)$-dimensional field theory $\bar{\alpha}$ over $\overline{\mathcal{F}}$ and an isomorphism $\alpha \xrightarrow{\cong} \pi^*\bar{\alpha}$. This is the formal part—the main work is the analysis required to integrate over an infinite dimensional space—but if this can be done, we obtain a pushforward theory over $\overline{\mathcal{F}}$ with anomaly $\bar{\alpha}$. This descent problem has existence and uniqueness aspects, analogous to (3.46), only now with invertible field theories.

Remark 3.52. Changing descent data by tensoring with an n-dimensional invertible theory is sometimes called "changing the scheme", and pushforwards which differ in this way share many physical properties.

(3.53) *'t Hooft anomalies in finite homotopy theories.* Recall **(2.23)**, which I recommend you review at this point. In terms of the discussion above, a "cocycles" on a π-finite space $\mathcal{X}$ defines an invertible field theory in which the map to $\mathcal{X}$ remains a background field. Once we sum—the finite path integral—over these maps, with the cocycle as a weight, then we obtain a typically noninvertible theory σ. When equipped with a semiclassical regular boundary theory ρ, the quantization

[35]Note that the anomaly is a theory over the background fields $\mathcal{F}$, which may or may not have to do directly with symmetry.

of data in (**2.51**), then (σ,ρ) represents a symmetry with an *'t Hooft* anomaly represented by the cocycle.

(3.54) *Twisted boundary theories.* At this point recall Definition 2.52. Note that if $\mathcal{X}$ is a π-finite space, and we take the zero cocycle, then a (right or left) semiclassical boundary theory is a map $\mathcal{Y}\to\mathcal{X}$ of π-finite spaces together with a *cocycle* μ on $\mathcal{Y}$. If we are working with an $m=(n+1)$-dimensional theory, then μ has degree m. The cocycle μ is used to weight/twist the quantization of the boundary.

Example: BA-symmetry in 4 dimensions and line defects

The following discussion is inspired by [36].

(3.55) *Symmetry data.* Let A be a finite abelian group, set $\mathcal{X}=B^2A$, and fix a basepoint $*\to B^2A$. This defines the semiclassical data of a BA-symmetry, what is often referred to as a "1-form A-symmetry". We set $\lambda=0$: there is no 't Hooft anomaly. For definiteness set $n=4$, so we use the 5-dimensional finite homotopy theory $\sigma=\sigma^{(5)}_{B^2A}$. The basepoint gives a regular right boundary theory, and we study the pair (σ,ρ) as abstract 4-dimensional symmetry data.

(3.56) *The left (σ,ρ)-module: 4-dimensional gauge theory.* Let H be a Lie group, and suppose $A\subset H$ is a subgroup its center. Set $\overline{H}=H/A$. From the exact sequence

$$A\longrightarrow H\longrightarrow \overline{H} \tag{3.57}$$

of Lie groups we obtain a sequence of fiberings

$$BA\longrightarrow BH\longrightarrow B\overline{H}\longrightarrow B^2A \tag{3.58}$$

In fact, we can promote (3.58) to a fibering of classifying spaces[37] of connections:

$$B_\nabla H\longrightarrow B_\nabla\overline{H}\longrightarrow B^2A \tag{3.59}$$

The fibering of $B_\nabla H$ over B^2A is the structure of the action of BA on it. (Given a principal H-bundle with connection and a principal A-bundle, use the *homomorphism* $A\times H\to H$ to construct a new principal H-bundle with connection.) If H is a finite group, and therefore $\overline{H}$ too is a finite group, then the map $B_\nabla\overline{H}=B\overline{H}\to B^2A$ would be a semiclassical left boundary theory. In general, of course, there is no finiteness nor is $\overline{H}$-gauge theory a topological field theory. Nonetheless, we set $\widetilde{F}$ to be 4-dimensional $\overline{H}$-gauge theory, and so obtain the (σ,ρ)-module exhibited in Figure 35.

[36] Ofer Aharony, Nathan Seiberg, and Yuji Tachikawa, *Reading between the lines of four-dimensional gauge theories*, JHEP **08** (2013), 115, `arXiv:1305.0318 [hep-th]`; see also Section 2.3 of Davide Gaiotto, Gregory W. Moore, and Andrew Neitzke, *Framed BPS States*, Adv. Theor. Math. Phys. **17** (2013), no. 2, 241–397, `arXiv:1006.0146 [hep-th]`.

[37] These are simplicial sheaves on Man; see [11].

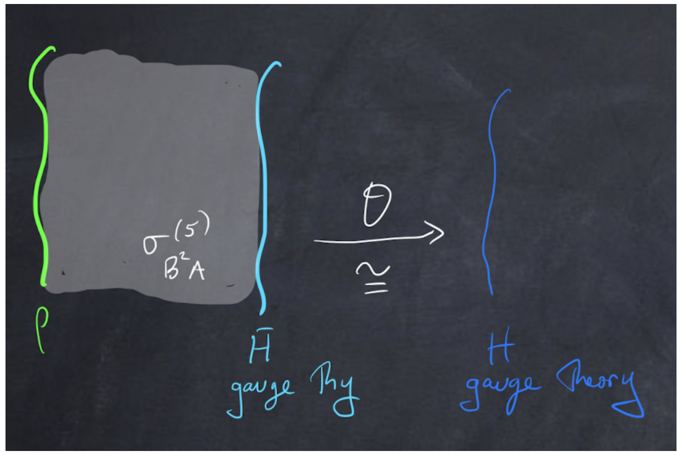


FIGURE 35. The BA-action on H-gauge theory

Remark 3.60. We have used the phrase "H-gauge theory" without specifying which one. All we need here that it is a 4-dimensional gauge theory with BA-symmetry. Other details do not enter this discussion.

(3.61) *Topological right* (σ, ρ)*-modules.* For any subgroup $A' \subset A$ there is a map of π-finite spaces

$$B^2A' \longrightarrow B^2A \tag{3.62}$$

We use this as semiclassical right boundary data, but we allow a twisting as in **(3.54)**, i.e., a degree 4 cocycle μ on B^2A'. In this case we use ordinary cohomology (so assume an orientation is among the background fields), and then the class of μ lives in the cohomology group $H^4(B^2A'; \mathbb{C}^\times)$. A theorem of Eilenberg-MacLane computes

$$H^4(B^2A'; \mathbb{C}^\times) \cong \{\text{quadratic functions } q\colon A' \longrightarrow \mathbb{C}^\times\} \tag{3.63}$$

Thus pairs (A', q) determine a right topological boundary theory $R_{A',q}$. The sandwich $R_{A',q} \otimes_\sigma \widetilde{F}$, depicted in Figure 36, is a twisted form of H/A'-gauge theory.

Remark 3.64. Under the isomorphism (3.63), the quadratic form q gives rise to the *Pontrjagin square* cohomology operation

$$\mathcal{P}_q\colon H^2(X; A') \longrightarrow H^4(X; \mathbb{C}^\times) \tag{3.65}$$

on any space X. It enters the formula for the partition function in the theory $R_{A',q} \otimes_\sigma \widetilde{F}$.

(3.66) *Local line defects: interior label.* We study local line defects in the twisted H/A'-gauge theory using the sandwich picture in Figure 37. So we have the surface $[0,1] \times C$ built over a curve C (in some manifold M), with topological $R_{A',q}$-boundary at $\{0\} \times C$ and with nontopological

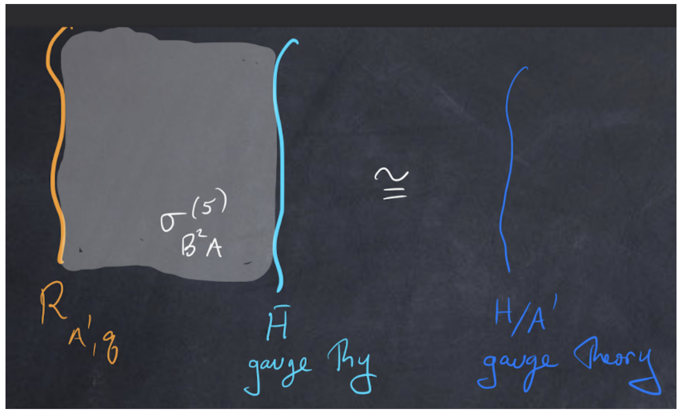


FIGURE 36. A quotient of H-gauge theory

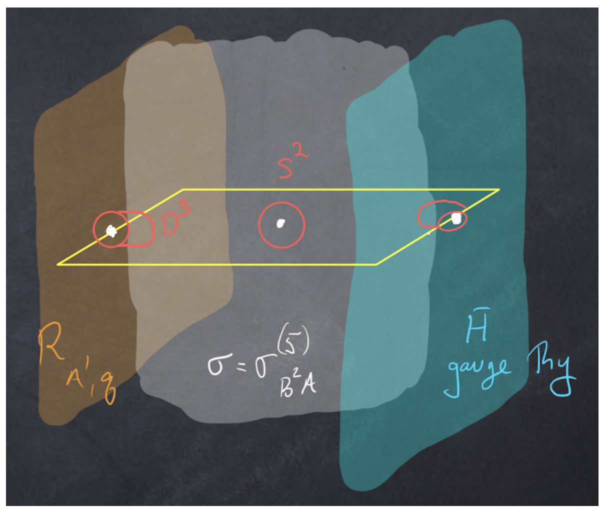


FIGURE 37. Line defects in twisted H/A'-gauge theory

$\widetilde{F}$-boundary at $\{1\} \times C$. See Figure 12 for an analogous figure in quantum mechanics for point defects. We find the relevant labels by working down in dimension of strata, starting at the top. Thus we begin with the link of an interior point, which is S^2. Its quantization $\sigma(S^2)$ is, say, a linear 2-category—one step beyond our illustration in Example 2.27. Observe that for the mapping space $\mathrm{Map}(S^2, B^2A)$ we have

$$\begin{aligned} \pi_0\big(\mathrm{Map}(S^2, B^2A)\big) &= H^2(S^2;A) \cong A \\ \pi_1\big(\mathrm{Map}(S^2, B^2A)\big) &= H^1(S^2;A) = 0 \\ \pi_2\big(\mathrm{Map}(S^2, B^2A)\big) &= H^0(S^2;A) \cong A \end{aligned} \tag{3.67}$$

The quantization consists of flat bundles (local systems) of linear categories over the 2-groupoid with these homotopy groups, depicted in Figure 38. In other words, for each

$$m \in H^2(S^2;A) \cong A \tag{3.68}$$

there is a linear category $\mathcal{L}_m$. Furthermore, π_2 based at m acts on $\mathcal{L}_m$ as automorphisms of the identity functor. Under suitable assumptions, then, we can decompose according to the characters

$$e \in H^0(S^2; A)^\vee \cong A^\vee \tag{3.69}$$

to obtain

$$\mathcal{L}_m = \bigoplus_e \mathcal{L}_{m,e} \cdot e \tag{3.70}$$

In summary, then, an object of $\sigma(S^2)$ is a collection of linear categories $\mathcal{L}_{m,e}$ labeled by $m \in A$ and $e \in A^\vee$.

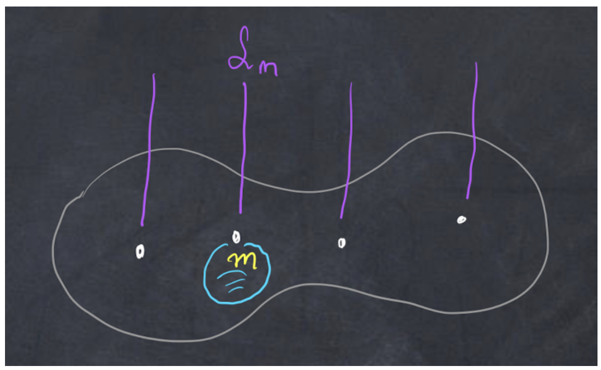


FIGURE 38. The 2-category $\mathrm{Cat}\big((B^2A)^{S^2}\big)$

(3.71) *Local line defects: collating the labels.* Now we proceed to the lower dimensional strata, so determine the labels on the boundary strata of $[0,1]\times C$, depicted with their links in Figure 37. A label on the $R_{A',q}$-colored boundary is an object $\mathcal{L}_0$ in this category—part of the topological field theory (σ,ρ)—and a label of the $\widetilde{F}$-colored boundary is another object $\mathcal{L}_1$. The image under θ of this configuration is the sum over m,e of $\mathrm{Hom}\big((\mathcal{L}_0)_{m,e},(\mathcal{L}_1)_{m,e}\big)$, which is the category of defects in the twisted H/A'-gauge theory. What we will now compute is that $\mathcal{L}_0$ is supported at a subset of pairs (m,e) determined by the subgroup $A'\subset A$ and the quadratic function $q\colon A'\to\mathbb{C}^\times$. This is the information gained from the symmetry; it matches the examples in [36]. Note that the information is from the topological part of the sandwich; the particulars of the analytic part (the $\overline{H}$-gauge theory) do not enter.

(3.72) *Higher Gauss law.* We begin with the lower Gauss law. Suppose $\mathcal{G}$ is a finite 1-groupoid, and $L\to\mathcal{G}$ is a complex line bundle over $\mathcal{G}$. We want to compute the global sections (which is a limit; the colimit is equivalent). At a point $m\in\mathcal{G}$ the group $\pi_1(\mathcal{G},m)$ acts by a character on the fiber L_m. The value of a global section at m must be a fixed point of this action, so it lies in the invariant subspace of L_m, that is, the section vanishes unless the character vanishes.

Now suppose $\mathcal{K} \to \mathcal{G}$ is a bundle of invertible complex linear categories over a 2-groupoid $\mathcal{G}$, the higher analog of a complex line bundle. (We call the fibers $\mathbb{V}$-*lines*, where $\mathbb{V} = \mathrm{Vect}$ is the category of complex vector spaces, regarded as the categorification of a ring in this context.) Then at each $m \in \mathcal{G}$ there are two layers to consider, and we need fixed point data for both. If $f \in \pi_1(\mathcal{G}, m)$ and $x \in \mathcal{K}_m$, then the first piece of fixed point data is an isomorphism

$$\eta_f(x)\colon x \longrightarrow f(x) \tag{3.73}$$

Then if $a\colon f \to g$ is a 2-morphism in $\mathcal{G}$ at m, we also need an isomorphism $\lambda_a(x)$ in

$$\begin{tikzcd} & f(x) \arrow[dd, "\lambda_a(x)"] \\ x \arrow[ur, "\eta_f(x)"] \arrow[dr, "\eta_g(x)"'] & \\ & g(x) \end{tikzcd} \tag{3.74}$$

that makes the diagram commute. Now specialize to $f = g = \mathrm{id}_x$. Then (3.74) becomes

$$\begin{tikzcd} & x \arrow[dd, "\lambda_a(x)"] \\ x \arrow[ur, "\eta"] \arrow[dr, "\eta"'] & \\ & x \end{tikzcd} \tag{3.75}$$

for some automorphism η. This commutes only if $\lambda_a(x) = \mathrm{id}$. Hence, if $\pi_1\mathcal{G} = 0$ then the global sections are only nonzero on the components on which $\pi_2\mathcal{G}$ acts trivially. This is the higher Gauss law.

Exercise 3.76. Make this paragraph precise using the theory of semiadditive categories.

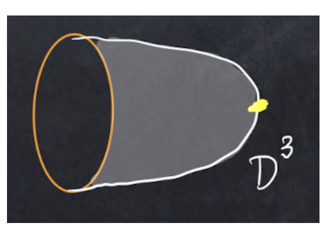


FIGURE 39. The link in Figure 37 at the $R_{A',q}$-colored boundary

(3.77) *Local line defects: the missing link.* First, associated to the quadratic function $q\colon A' \to \mathbb{C}^\times$ is a bihomomorphism

$$b\colon A' \times A' \longrightarrow \mathbb{C}^\times \tag{3.78}$$

It induces a Pontrjagin-Poincare duality

$$H^2(S^2;A') \times H^0(S^2;A') \longrightarrow \mathbb{C}^\times \tag{3.79}$$

and so an isomorphism

$$e'\colon H^2(S^2;A') \longrightarrow H^0(S^2;A')^\vee \tag{3.80}$$

We proceed to quantize the link depicted in Figure 39, which is a 3-disk D^3 with boundary colored by $R_{A',q}$. Label the center point (m,e), which means the tensor unit category Vect sitting over m with π_2 acting via the character e; see (3.70). (Thus $\mathcal{L}_{m,e} = \mathrm{Vect}$ and the linear categories labeled by other pairs (m',e') are zero.) There is a semiclassical description of this defect (m,e) in terms of Definition 2.76: the space $\mathcal{Y} = B^2A$ is equipped with a 2-cocycle that represents the class in $H^2(B^2A;\mathbb{C}^\times) \cong A^\vee$ given by the character e; the map to $\mathrm{Map}(S^2,B^2A)$ is the identity onto the component indexed by m. At the $R_{A',q}$-colored boundary, in semiclassical terms the boundary theory is given as the map

$$\bigl(\mathrm{Map}(S^2,B^2A'),\tau^2(\mu_q)\bigr) \longrightarrow \mathrm{Map}(S^2,B^2A), \tag{3.81}$$

where $\tau^2\mu_q$ is the transgression of a cocycle which represents the class of the quadratic function in (3.63). Altogether we have a diagram

$$\begin{tikzcd}[column sep=tiny] \bigl(\mathrm{Map}(S^2,B^2A'),\tau^2(\mu_q)\bigr) \arrow[dr] & & \mathrm{Map}(D^3\setminus B^3,B^2A) \arrow[dl] \arrow[dr] & & (B^2A,e) \arrow[dl, "m"] \\ & \mathrm{Map}(S^2,B^2A) & & \mathrm{Map}(S^2,B^2A) & \end{tikzcd} \tag{3.82}$$

Interpret in terms of the bordism obtained by cutting out a ball around the yellow defect; in the bottom row the first entry is the orange boundary and the second the link of the yellow defect. Take the homotopy limit to conclude that the quantization is supported on pairs (m,e) which satisfy

$$\begin{aligned} m &\in A' \\ e &= e'(m)^{-1} \end{aligned} \tag{3.83}$$

This selection rule is the one in [36].

Exercise 3.84. Check this last statement in some examples from [36].

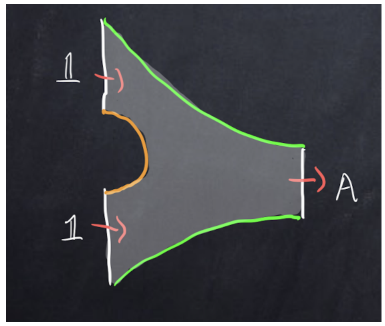


FIGURE 40. An element in $\operatorname{Hom}(1, A)$

Some additional problems

Problem 3.85. Let G be a finite group, let $\chi\colon G \to \mathbb{C}^\times$ be a character (it could be the trivial character that sends each $g \in G$ to $1 \in \mathbb{C}^\times$), and let $\sigma = \sigma_{BG}^{(2)}$ be the 2-dimensional finite G-gauge theory with values in $\mathcal{C} = \mathrm{Alg}(\mathrm{Vect})$ the Morita 2-category of algebras. Then $\sigma(\mathrm{pt}) = A = \mathbb{C}[G]$. The regular boundary theory (constructed from the regular module A_A) is indicated in Figure 40 in green, and the augmentation boundary theory in orange. The depicted bordism maps to an element of $\operatorname{Hom}(1, A)$, where this is Hom in $\Omega\mathcal{C} = \operatorname{Hom}_{\mathcal{C}}(1, 1) = \mathrm{Vect}$. (Why?) So it can be identified with an element of $A = \mathbb{C}[G]$. Compute that element.

Problem 3.86. Let $(\mathcal{H}, H)$ define a quantum mechanical theory F which is invariant under the action of a finite group G. Let σ be the 2-dimensional G-gauge theory and consider Figure 41, which shows F in the "sandwich" picture. There a general point defect is depicted. Point defects in F form a vector space. What extra structure is encoded in this sandwich picture? Now replace the right regular boundary ρ with a quotient $\epsilon_{G'}$ for a subgroup $G' \subset G$: it is associated to the right module $\mathbb{C}\langle G' \backslash G\rangle$ for the group algebra $A = \mathbb{C}[G]$. What theory is obtained? What are the point defects in terms of the point defects for F? What happens if you twist by a character of G'?

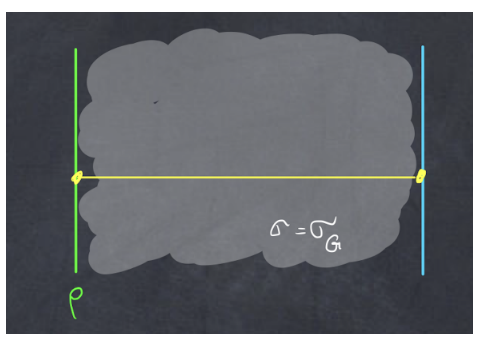


FIGURE 41. Point defects in quantum mechanics

Lecture 4: Duality defects and applications

So far we have emphasized the structure of symmetry in field theory at the expense of concrete examples. In many ways this lecture is no different: its focus is on abelian duality in many forms and then on a particular defect related to duality. But at least we have a few definite applications to honest quantum field theories! They come from two papers,[38] whose authors include two of the other lecturers at this school. This lecture is to a large extent an exposition of these papers.

We begin with a broad outline of the dynamical question addressed and how symmetry plays a role in solving it. It emerges that the question addressed is whether a topological field theory has an augmentation in the sense of **(3.6)**. Then we turn to *abelian duality*: for finite abelian groups; for spectra; and finally for special finite homotopy theories, where it is a form of *electromagnetic duality*. This duality induces a map on boundary theories, and one interest here is that the transform of a theory under electromagnetic duality is the quotient theory by the symmetry (Corollary 4.42). We also discuss *self-duality*: topological theories which are self-electromagnetic dual. In the last part of the lecture we consider situations in which the quotient F/σ of a theory by a symmetry is isomorphic to the original theory F. Composition with the topological defect δ in **(3.15)** is a self-domain wall of the theory F, the so-called *duality defect*. At the end of the lecture we return to the applications in [38]. The main argument is in **(4.63)**.

Trivially gapped theories with finite symmetry

In quantum field theory one is often interested in the low energy, or quantum relativistically equivalent[39] long distance, behavior of a system. As with classical physics, posed in terms of differential equations, the a quantum field theory is formulated at short distance/time and the interesting dynamics—the "answers" to the short range "questions"—happen at long distance/time. The term *ultraviolet* (UV) is used for short range, and the term *infrared* (IR) is used for long range.[40] The *renormalization group flow* is meant to be[41] a flow on a space of theories which has trajectories that limit at negative infinite time to an ultraviolet theory and limit at positive infinite time to a corresponding infrared theory.

(4.1) *Possible infrared behaviors.* Clay discussed this more in his lectures. First, there is a basic dichotomy in quantum systems: *gapped* vs. *gapless.* A quantum mechanical system is *gapped* if its minimum energy is an eigenvalue of finite multiplicity of the Hamiltonian, assumed bounded below, and is an isolated point of the spectrum. This notion generalizes to a relativistic quantum field theory if we understand 'spectrum' to mean the spectrum of representations of the translation group of Minkowski spacetime. In the infrared, a gapless system is typically thought to be well-approximated

[38] Yichul Choi, Clay Cordova, Po-Shen Hsin, Ho Tat Lam, and Shu-Heng Shao, *Non-Invertible Duality Defects in 3+1 Dimensions*, arXiv:2111.01139

Justin Kaidi, Kantaro Ohmori, and Yunqin Zheng, *Kramers-Wannier-like Duality Defects in (3+1)D Gauge Theories*, Phys. Rev. Lett. **128** (2022), no. 11, 111601, arXiv:2111.01141 [hep-th].

[39] In terms of L=length, T=time, and M=mass, energy has units $\mathrm{ML}^2/\mathrm{T}^2$. The fundamental constants have units $[c] = \mathrm{L/T}$ and $[\hbar] = \mathrm{ML}^2/\mathrm{T}$. Use $\hbar$ to relate energy and inverse time, and then use c to get inverse length.

[40] These derive from wavelength λ, which related to energy in a relativistic quantum theory by $E = 2\pi\hbar c/\lambda$.

[41] The cautious language reflects the lack of rigorous constructions in many situations.

by a conformal field theory. A gapped system is typically thought to be well-approximated by a topological field theory, roughly up to tensoring by an invertible field theory (which need not be topological).[42] In the gapped case there is a further dichotomy: the low energy approximation can be invertible or not. In case it is invertible, the term 'trivially gapped' is sometimes used. One can view this invertibility condition as a nondegeneracy condition on the vacuum, namely that there be a unique vacuum state on each space. That condition typically holds for quantum mechanical systems, but is often violated in supersymmetric field theories; noninvertible low energy topological field theories also occur for nonsupersymmetric systems.

(4.2) *Persistence of symmetry.* Let (σ,ρ) be symmetry data that acts on an ultraviolet theory F_{UV}. Under renormalization group flow, one imagines that the symmetry persists:

$$\begin{tikzcd} (\sigma,\rho) \circlearrowright F_{\mathrm{UV}} \arrow[d, rightsquigarrow, "\text{RG flow}"] \\ (\sigma,\rho) \circlearrowright F_{\mathrm{IR}} \end{tikzcd} \tag{4.3}$$

In other words, if there is (σ,ρ)-symmetry in the ultraviolet, then there should be (σ,ρ)-symmetry in the infrared as well.

Remark 4.4. In general, the symmetry in the infrared could be bigger: an *emergent symmetry* may occur. As well, there may be symmetries in the UV which act trivially in the IR. The assertion is that there is a homomorphism from UV-symmetry to IR-symmetry. (We have not discussed "homomorphisms" of symmetry in these lectures.)

(4.5) *Trivially gapped theories with symmetry.* The question, then, of whether a quantum field theory with symmetry (σ,ρ) can be trivially gapped is whether there exists a left σ-module $\tilde\lambda$ such that

$$\lambda := \rho \otimes_\sigma \tilde\lambda \tag{4.6}$$

is an *invertible* field theory. If we are in the situation of Definition 1.35 and $A = \sigma(\mathrm{pt})$ is an algebra, then this is equivalent[43] to asking for an augmentation $A \to 1$ of A; see Definition 3.4. One can envision obstructions to the existence of an augmentation without knowing anything specific about F_{UV} or the left (σ,ρ)-module structure on it. Indeed, that is precisely how we argue at the end of the lecture. (See §**(4.62)** to be sure which theory does not have an augmentation.)

[42] These statements are hardly universal. For example, fracton models in condensed matter physics do not have long range field theory approximations in the traditional sense.

[43] Evaluate (4.6) on a point to see that $\lambda(\mathrm{pt}) = A \otimes_A \tilde\lambda(\mathrm{pt})$, and so $\tilde\lambda(\mathrm{pt})$ is an invertible A-module.

Abelian duality

(4.7) *Finite abelian groups.* This is the classical version of *Pontrjagin duality.*

Definition 4.8. Let A, A' be finite abelian groups. A bihomomorphism

$$b\colon A \times A' \longrightarrow \mathbb{C}^\times \tag{4.9}$$

is a *duality pairing* if b is nondegenerate in the sense that (1) if $a \in A$ satisfies $b(a,a') = 1$ for all $a' \in A'$, then $a = 1$; and (1) if $a' \in A'$ satisfies $b(a,a') = 1$ for all $a \in A$, then $a' = 1$.

The bihomomorphism b induces homomorphisms $A \to (A')^\vee$ and $A' \to A^\vee$; the nondegeneracy condition states that these are isomorphisms. Here $A^\vee$ is the Pontrjagin dual abelian group of homomorphisms $A \to \mathbb{C}^\times$. The duality relation is symmetric: $(A^\vee)^\vee \cong A$ (canonically).

Remark 4.10. For any finite abelian group A there is a canonical duality pairing (evaluation)

$$A^\vee \times A \to \mathbb{C}^\times \tag{4.11}$$

If A, A' are in Pontrjagin duality, then there is a Fourier transform isomorphism

$$\mathrm{Fun}(A) \xrightarrow{\;\cong\;} \mathrm{Fun}(A') \tag{4.12}$$

on the vector spaces of complex functions.

(4.13) *Example.* For $\ell \in \mathbb{Z}^{>0}$, the group μ_ℓ is the subgroup of $\mathbb{C}^\times$ consisting of ℓ^{th} roots of unity. It is a cyclic group of order ℓ; there are several generators but none canonically picked out (Galois symmetry). The group $\mathbb{Z}/\ell\mathbb{Z}$ of classes of integers under the mod ℓ equivalence is also cyclic of order ℓ, again not canonically so. These two groups are not canonically isomorphic, but they are canonically dual via the pairing

$$\begin{aligned} b\colon\ \mu_\ell \times \mathbb{Z}/\ell\mathbb{Z} &\longrightarrow \mathbb{C}^\times \\ \lambda \quad , \quad k \quad &\longmapsto \lambda^k \end{aligned} \tag{4.14}$$

Remark 4.15. For $\ell = 2$ these are cyclic groups of order 2, and any two cyclic groups of order 2 are canonically isomorphic. Perhaps it is worth pointing out that the center of SU_N is canonically μ_N: think of the scalar matrices that comprise the center.

Remark 4.16. Classical Pontrjagin duality extends to topological abelian groups that are locally compact. In some sense the $\ell \to \infty$ limit of (4.14) is the duality pairing

$$\begin{aligned} \mathbb{T} \times \mathbb{Z} &\longrightarrow \mathbb{C}^\times \\ \lambda \ , \ k &\longmapsto \lambda^k \end{aligned} \tag{4.17}$$

The corresponding isomorphism (4.12) of infinite dimensional vector spaces (of L^2 functions) is the usual Fourier transform.

(4.18) *Pontrjagin self-duality.* If $A = A'$ in Definition 4.8, then we say that b is a *Pontrjagin self-duality pairing*; it identifies A as its own Pontrjagin dual. We have already said that μ_2 has a canonical such Pontrjagin self-duality. We emphasize that Pontrjagin self-duality is *data*, not merely a condition.

(4.19) *π-finite spectra.* Recall Definition 1.12(3) of a π-finite spectrum $\boldsymbol{T}$. There is an extension of duality to such spectra; the role of $\mathbb{C}^\times$ is played by the *Brown-Comenetz* dual[44] is played by the spectrum $I\mathbb{C}^\times$, a "character dual" to the sphere characterized by the universal property

$$[\boldsymbol{T}, I\mathbb{C}^\times] \cong (\pi_0\boldsymbol{T})^\vee \tag{4.20}$$

for any spectrum $\boldsymbol{T}$. (Here $[\boldsymbol{T}, \boldsymbol{T}']$ denotes the abelian group of homotopy classes of spectrum maps $\boldsymbol{T} \to \boldsymbol{T}$.) The *spectrum* of maps $\boldsymbol{T} \to I\mathbb{C}^\times$ is the character dual $\boldsymbol{T}^\vee$ to the π-finite spectrum $\boldsymbol{T}$; it is also π-finite.

Example 4.21. Let A be a finite abelian group and HA the corresponding Eilenberg-MacLane spectrum. There is a duality pairing

$$HA^\vee \wedge HA \longrightarrow H\mathbb{C}^\times \longrightarrow I\mathbb{C}^\times \tag{4.22}$$

which identifies $HA^\vee$ as the character dual to HA.

Finite electromagnetic duality

(4.23) *The electromagnetic dual theory.* Let $\boldsymbol{T}$ be a π-finite spectrum, and denote by $\mathfrak{X}_{\boldsymbol{T}}$ its 0-space, which is an infinite loop space. By the definition of a spectrum, there is a basepoint $* \to \mathfrak{X}_{\boldsymbol{T}}$. For any dimension $n \in \mathbb{Z}^{>0}$, the semiclassical data $(\mathfrak{X}_{\boldsymbol{T}}, *)$ quantizes to symmetry data (σ, ρ) in which $\sigma = \sigma^{(n+1)}_{\boldsymbol{T}}$ is an $(n+1)$-dimensional topological field theory. The duality pairing

$$\Sigma^n\boldsymbol{T}^\vee \wedge \boldsymbol{T} \longrightarrow \Sigma^n I\mathbb{C}^\times \tag{4.24}$$

determines a cohomology class on $\mathfrak{X}_{\Sigma^n\boldsymbol{T}^\vee} \wedge \mathfrak{X}_{\boldsymbol{T}}$; let μ be a cocycle representative.[45]

Definition 4.25.

(1) The *dual symmetry data* $(\sigma^\vee, \rho^\vee)$ to (σ, ρ) is the finite homotopy theory $\sigma^\vee = \sigma^{(n+1)}_{\mathfrak{X}_{\Sigma^n\boldsymbol{T}^\vee}}$ with Dirichlet boundary theory $\rho^\vee$ from the basepoint $* \to \mathfrak{X}_{\Sigma^n\boldsymbol{T}^\vee}$.

[44] They use $\mathbb{Q}/\mathbb{Z}$ in place of $\mathbb{C}^\times$, as could we.

[45] In many cases of interest the pairing (4.24) factors through a simpler cohomology theory than $I\mathbb{C}^\times$. For example, if $\boldsymbol{T} = \Sigma^s HA$ is a shifted Eilenberg-MacLane spectrum of a finite abelian group, then (4.24) factors through $\Sigma^n H\mathbb{C}^\times$ and we can choose μ to be a singular cocycle with coefficients in $\boldsymbol{T}$. Recall that we use the word 'cocycle' for any geometric representative of a generalized cohomology class.

(2) The *canonical domain wall* $\zeta\colon \sigma \to \sigma^\vee$—i.e., a $(\sigma^\vee, \sigma)$-bimodule—is the finite homotopy theory constructed from the correspondence of π-finite spaces

$$\begin{tikzcd} & (\mathcal{X}_{\Sigma^n \boldsymbol{T}^\vee} \times \mathcal{X}_{\boldsymbol{T}}, \mu) \arrow[dl] \arrow[dr] & \\ \mathcal{X}_{\Sigma^n \boldsymbol{T}^\vee} & & \mathcal{X}_{\boldsymbol{T}} \end{tikzcd} \tag{4.26}$$

in which the maps are projections onto the factors in the Cartesian product. There is a similar canonical domain wall $\zeta^\vee\colon \sigma^\vee \to \sigma$.

(3) The *canonical Neumann boundary theories* $\epsilon, \epsilon^\vee$ are the finite homotopy theories induced from the identity maps on $\mathcal{X}_{\boldsymbol{T}}, \mathcal{X}_{\Sigma^n \boldsymbol{T}^\vee}$, respectively.

Our formulation emphasizes the role of σ as a symmetry for another quantum field theory. But σ is a perfectly good $(n+1)$-dimensional field theory in its own right. From that perspective $\sigma^\vee$ is the $(n+1)$-dimensional *electromagnetic dual* theory. See [46] for more about electromagnetic duality in this context.

Remark 4.27.

(1) In this finite version of electromagnetic duality there is a shift from what one expects based on usual electromagnetic duality of U_1-gauge fields. So, for example, if A is a finite abelian group, then in 3 dimensions the finite gauge theory which counts principal A-bundles has as its electromagnetic dual the gauge theory which counts principal $A^\vee$-bundles.

(2) As usual, we have not made explicit the background fields for σ, $\sigma^\vee$, and ζ, $\zeta^\vee$. In fact, the theories σ and $\sigma^\vee$ are defined on bordisms unadorned by background fields: they are "unoriented theories". For ζ we need a set of (topological) background fields which orient manifolds sufficiently to integrate μ. For example, if μ is a singular cocycle with coefficients in $\mathbb{T}$, then we need a usual orientation.

(4.28) *Electromagnetic self-duality.* Analogous to Pontrjagin self-duality of abelian groups (4.18), there is the possibility of an isomorphism $\sigma \xrightarrow{\cong} \sigma^\vee$, which is a self-duality of finite abelian gauge theories. It is based on a self-duality pairing

$$\boldsymbol{T} \wedge \boldsymbol{T} \longrightarrow \Sigma^n I\mathbb{C}^\times \tag{4.29}$$

(Keep in mind that the theory σ is in dimension $n+1$.)

Example 4.30 (Finite version of "p-form gauge fields")**.** Suppose that A is a finite abelian group equipped with a Pontrjagin self-duality, and for $p \in \mathbb{Z}^{>0}$ set $\boldsymbol{T} = \Sigma^p HA$. Then in odd dimension $2p+1$ there is an electromagnetic self-duality of $\sigma = \sigma_{\boldsymbol{T}}^{(2p+1)}$. The cases $p=1$ and $p=2$ are of particular interest later in this lecture.

[46] Yu Leon Liu, *Abelian duality in topological field theory*, arXiv:2112.02199.

Example 4.31 (A non-Eilenberg-MacLane spectrum)**.** Consider the spectrum $\boldsymbol{T}$ which is an extension (Postnikov tower)

$$\Sigma^3 H\mathbb{Z}/2\mathbb{Z} \longrightarrow \boldsymbol{T} \longrightarrow \Sigma^2 H\mathbb{Z}/2\mathbb{Z} \tag{4.32}$$

with extension class (k-invariant or Postnikov invariant)

$$\mathrm{Sq}^2\colon \Sigma^2 H\mathbb{Z}/2\mathbb{Z} \longrightarrow \Sigma^4 H\mathbb{Z}/2\mathbb{Z} \tag{4.33}$$

Then there is an electromagnetic self-duality of $\sigma^{(6)}_{\boldsymbol{T}}$.

Remark 4.34. There is a field theory built on a "self-dual" field, which is very interesting; it is a kind of square root of $\sigma^{(n+1)}_{\boldsymbol{T}}$, and it requires a quadratic refinement of (4.29) to define.

Duality and quotients

(4.35) *Duality swaps Dirichlet and Neumann.* We keep the notation of **(4.23)**, including Definition 4.25. The topological field theories σ and $\sigma^\vee$ have dimension $n+1$.

Proposition 4.36. *There is an isomorphism of right σ-modules*

$$\psi\colon \rho^\vee \otimes_{\sigma^\vee} \zeta \xrightarrow{\;\cong\;} \epsilon \tag{4.37}$$

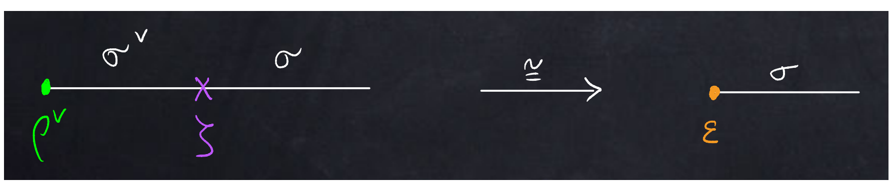


Figure 42. An isomorphism of right σ-modules

This isomorphism is depicted in Figure 42.

Proof. We use the calculus of π-finite spectra, as described in **(2.46)**—see especially the composition law (2.50). The theory σ is induced from $\mathfrak{X}_{\boldsymbol{T}}$, the theory $\sigma^\vee$ from $\mathfrak{X}_{\Sigma^n\boldsymbol{T}^\vee}$, the boundary theory $\rho^\vee$ from $* \to \mathfrak{X}_{\Sigma^n\boldsymbol{T}^\vee}$, and the domain wall ζ from the correspondence diagram (4.26). Hence $\rho^\vee \otimes_{\sigma^\vee} \zeta$ is induced from the homotopy fiber product:

$$\begin{tikzcd} * \arrow[dr] & (\mathfrak{X}_{\boldsymbol{T}}, 0) \arrow[l, dashed] \arrow[r, dashed] & (\mathfrak{X}_{\Sigma^n\boldsymbol{T}^\vee} \times \mathfrak{X}_{\boldsymbol{T}}, \mu) \arrow[dl] \arrow[dr] & \\ & \mathfrak{X}_{\Sigma^n\boldsymbol{T}^\vee} & & \mathfrak{X}_{\boldsymbol{T}} \end{tikzcd} \tag{4.38}$$

Here we use that the restriction of μ to $*\times\mathcal{X}_{\boldsymbol{T}}$ is zero. So the sandwich is the right σ-module induced from the composition

$$\begin{tikzcd} (\mathcal{X}_{\boldsymbol{T}},0) \arrow[r, dashed] & (\mathcal{X}_{\Sigma^n\boldsymbol{T}^\vee}\times\mathcal{X}_{\boldsymbol{T}},\mu) \arrow[dr] & \\ & & \mathcal{X}_{\boldsymbol{T}} \end{tikzcd} \tag{4.39}$$

which is $\mathrm{id}_{\mathcal{X}_{\boldsymbol{T}}}$. That theory is the augmentation boundary theory ϵ. ☐

FIGURE 43. The dual symmetry on the quotient F/σ

(4.40) *The effect on left σ-modules.* The duality domain wall ζ maps left σ-modules to left $\sigma^\vee$-modules; the domain wall $\zeta^\vee$ induces a map in the opposite direction. The following theorem, which follows immediately from Proposition 4.36, illuminates this involutive correspondence.

Corollary 4.41. *Let F be a quantum field theory equipped with a (σ,ρ)-module structure. Then the quotient F/σ carries a canonical $(\sigma^\vee,\rho^\vee)$-module structure.*

Corollary 4.42. *The transform of F under electromagnetic duality is the quotient F/σ.* ☐

Proof of Corollary 4.41. The proof is contained in Figure 43. In words: Let $(\widetilde{F},\theta)$ be the (σ,ρ)-module data, as in Definition 1.38. Define the left $\sigma^\vee$-module

$$\widehat{F}=\zeta\otimes_\sigma\widetilde{F} \tag{4.43}$$

and the isomorphism

$$\widehat{\theta}\colon \rho^\vee\otimes_{\sigma^\vee}\widehat{F}=\rho^\vee\otimes_{\sigma^\vee}\zeta\otimes_\sigma\widetilde{F}\xrightarrow{\psi\otimes\theta}\epsilon\otimes_\sigma\widetilde{F}=F/\sigma \tag{4.44}$$

Then $(\widehat{F},\widehat{\theta})$ is the desired $(\sigma^\vee,\rho^\vee)$-module structure. ☐

The 2-dimensional Ising model

Here we briefly indicate an important example of duality: the Ising model. We take it as a *stat mech model*, which is a discrete quantum system, rather than a continuous quantum field theory. (There is another basic class of discrete quantum models—*lattice systems*—which are discussed in Mike Hopkins' lectures.) The same basic structure applies. The paper [23] contains much more information. The story there is for an arbitrary finite group; here for simplicity we take the finite abelian group $A=\mu_2$.

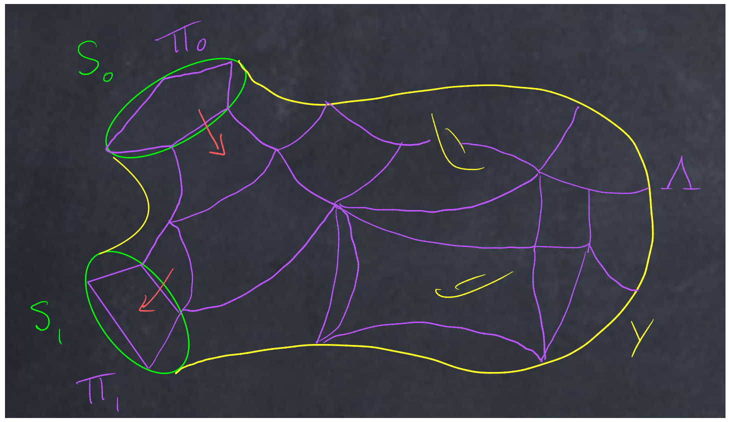


FIGURE 44. A bordism $(Y, \Lambda)\colon (S_0, \Pi_0) \to (S_1, \Pi_1)$ with lattices

(4.45) *The basic model.* Fix a positive real number $\beta \in \mathbb{R}^{>0}$. There is a 2-dimensional field theory F_β of "latticed" 1- and 2-manifolds; a typical bordism in the bordism category of such is depicted in Figure 44. (We defer to [23] for details, such as the definition of the lattices.) The "fluctuating field" over which we sum—the "spin"—is a function from the vertices of the lattice to the group A, and as usual a bordism gives rise to a correspondence of fluctuating fields:

$$\begin{tikzcd} & A^{\mathrm{Vert}(\Lambda)} \arrow[dl, "r_0"'] \arrow[dr, "r_1"] & \\ A^{\mathrm{Vert}(\Pi_0)} & & A^{\mathrm{Vert}(\Pi_1)} \end{tikzcd} \tag{4.46}$$

Linearize by push-pull

$$F_\beta(Y, \Lambda) = (r_1)_* \circ K_\beta \circ (r_0)^*\colon F_\beta(S_0, \Pi_0) \longrightarrow F_\beta(S_1, \Pi_1)$$

where we include an "integral kernel" K_β. The vector space of functions $F_\beta(S_i, \Pi_i) = \mathrm{Fun}\big(A^{\mathrm{Vert}(\Lambda)}\big)$ is independent of β. The function K_β is defined in terms of a weight on the group:

$$\begin{aligned} \theta_\beta\colon \mu_2 &\longrightarrow \mathbb{R}^{>0} \\ +1 &\longmapsto 1 \\ -1 &\longmapsto e^{-2\beta} \end{aligned} \tag{4.47}$$

Physically, β is the inverse temperature. Then for $s\colon \mathrm{Vert}(\Lambda) \to A = \mu_2$ we define

$$K_\beta(s) = \prod_e \theta\big(g(s;e)\big), \qquad e \text{ incoming or interior},$$

where for an edge $e \in \mathrm{Edge}(\Lambda)$ we let $g(s;e) \in A$ be the ratio of the two spins $s\big|_{\partial e}$ on the endpoints of e: it is $+1$ if they agree and -1 if they disagree.

Exercise 4.48. How does this model behave as $\beta \to 0$? As $\beta \to \infty$?

(4.49) *Symmetry.* The group A acts as a symmetry that exchanges spin $+1$ with spin -1, vertex by vertex. It is straightforward to construct a model $\widetilde{F}_\beta$ which couples the theory to a background principal A-bundle $Q \to Y$: the spin s becomes a section of this bundle over the vertices, and the ratio function $g(s;e)$ is still well-defined using parallel transport along the edge e. Then, summing over principal A-bundles, we see that $\widetilde{F}_\beta$ is a left σ-module, where $\sigma = \sigma^{(3)}_{BA}$ is the 3-dimensional finite A-gauge theory. Let ρ be the usual regular boundary theory for σ.

(4.50) *Kramers-Wannier duality.* Under electromagnetic duality $(\sigma,\rho) \to (\sigma^\vee,\rho^\vee)$, we obtain a left $\sigma^\vee$-module $\widehat{F}_\beta$ and a corresponding dual theory $F^{\mathrm{dual}}_\beta = \rho^\vee \otimes_{\sigma^\vee} \widehat{F}_\beta$. The isomorphism (4.44) is $F^{\mathrm{dual}}_\beta = F_\beta \big/ \sigma$. Now we need an input which, of course, depends on the details of the Ising model and which we do not elaborate here. Namely, there is a *Kramers-Wannier* isomorphism

$$\phi_\beta \colon F_\beta \big/ \sigma \xrightarrow{\;\cong\;} F_{\beta^\vee}, \tag{4.51}$$

where the dual inverse temperature $\beta^\vee$ is defined by the condition

$$\sinh(2\beta)\sinh(2\beta^\vee) = 1. \tag{4.52}$$

Furthermore, this isomorphism is equivariant for the symmetry action: it is induced from an isomorphism of left $\sigma^\vee$-modules. The map $\beta \leftrightarrow \beta^\vee$ is an involution on $\mathbb{R}^{>0}$, and it has a unique fixed point β_c. At this self-dual point there is an isomorphism

$$\phi = \phi_{\beta_c} \colon F/\sigma \xrightarrow{\;\cong\;} F, \tag{4.53}$$

where $F = F_{\beta_c}$. Use the electromagnetic self-duality to conclude that (4.53) is an isomorphism of left (σ,ρ)-modules.

The duality defect and an obstruction to trivial gappedness

(4.54) *The defect.* We resume our general setup: (σ,ρ) is symmetry data in n dimensions, with ρ a regular right σ-module, and F is a quantum field theory equipped with a (σ,ρ)-module structure $(\sigma,\rho,\widetilde{F})$. Assume further that F is equipped with an isomorphism

$$\phi \colon F/\sigma \xrightarrow{\;\cong\;} F. \tag{4.55}$$

Remark 4.56. One could ask for some compatibility of (4.55) with symmetry. By Corollary 4.41 the quotient F/σ is a $(\sigma^\vee,\rho^\vee)$-module, so what we would need is self-duality data for (σ,ρ); then we could ask that (4.55) be compatible with the (σ,ρ)-module structure. I do not see where that enters the arguments below, but it would be a more satisfactory story and would make the duality defect part of *self*-duality. More likely, is that there is a more specialized story with self-duality, and it would be nice to track down the extra information one can glean.

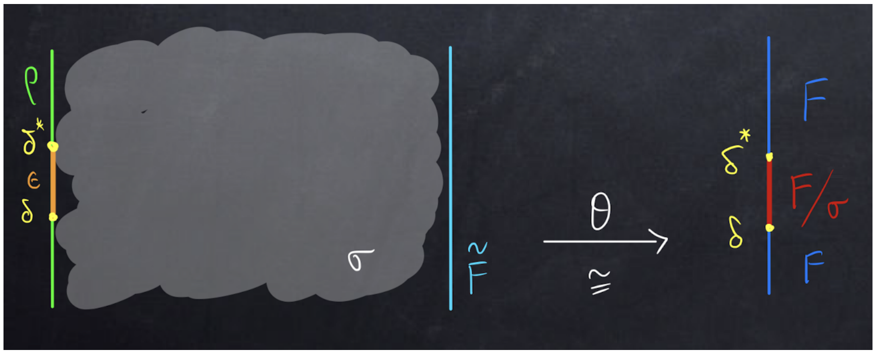


FIGURE 45. Figure 28 redux

Figure 45 recalls the domain walls $\delta\colon F \to F/\sigma$ and $\delta^*\colon F/\sigma \to F$ which are defined in (**3.15**).

Definition 4.57. The *duality defect* is the self-domain wall

$$\Delta = \phi \circ \delta\colon F \longrightarrow F. \tag{4.58}$$

Since δ is a topological defect, and ϕ is simply an isomorphism of theories, the composition Δ is also a topological defect.

(4.59) *The composition $\Delta^* \circ \Delta$.* As a preliminary we observe the following. View ϕ as a domain wall from F to F/σ, and furthermore imagine that there is a 2-category of theories, domain walls, and domain walls between domain walls. Then we can contemplate the adjoint ϕ^*. It is a general fact that if ϕ is invertible, then its adjoint equals its inverse: $\phi^* = \phi^{-1}$.

Exercise 4.60. Formulate and prove this result in 2-categories.

Using this we compute

$$\Delta^* \circ \Delta = (\phi\delta)^*(\phi\delta) = \delta^*\phi^*\phi\delta = \delta^*\phi^{-1}\phi\delta = \delta^* \circ \delta \tag{4.61}$$

The composition $\delta^* \circ \delta$ is discussed in (**3.18**). We use those computations below.

(4.62) *A larger symmetry.* The theory F has a $(\hat\sigma, \hat\rho)$-module structure for a topological field theory $\hat\sigma$ which is σ with Δ adjoined. I have not thought through in detail how to define $(\hat\sigma, \hat\rho)$. The equation (4.61) holds in $\hat\sigma$. One can also work out compositions of Δ with other defects in (σ, ρ).

It is this larger theory which we will prove in some instances does not admit an augmentation.

(4.63) *The obstruction to infrared triviality.* Here 'triviality' means 'trivially gapped' in the sense of **(4.1)**: the low energy approximation to a gapped theory is invertible.[47] Continuing in this situation with the duality defect, suppose as in **(4.5)** that there exists a left σ-module $\tilde{\lambda}$ such that Δ also acts, Δ satisfies (4.61), and the sandwich

$$\lambda := \rho \otimes_{\sigma} \tilde{\lambda} \tag{4.64}$$

is invertible. (In terms of **(4.62)**, $\tilde{\lambda}$ is a left $\hat{\sigma}$-module.) Because λ is invertible, self-domain walls act as multiplication by an n-dimensional field theory; they do not couple to λ. (Compare: an endomorphism of a line is multiplication by a complex number, an endomorphism of a $\mathbb{V}$-line is tensoring by a vector space, etc.) So $\delta^* \circ \delta$ acts as multiplication by an $(n-1)$-dimensional *topological* field theory, and so too does Δ act as multiplication by a topological field theory. Those theories satisfy (4.61): Δ is a kind of square root of $\delta^* \circ \delta$. But in some situations no such square root exists, as we can prove using the well-developed principles of topological field theory. The conclusion in that case is that no sandwich (4.64) exists with λ invertible.

We emphasize that the argument does not use any left (σ, ρ)-module; it just uses (σ, ρ) and the additional defect Δ. So it applies to any quantum field theory with an (σ, ρ)-module structure.

Examples

(4.65) *A 2-dimensional example.* Let $\sigma = \sigma^{(3)}_{B\mu_2}$ be 3-dimensional μ_2-gauge theory with its usual regular right boundary theory. Adjoin a codimension one defect Δ which satisfies (4.61). This acts on 2-dimensional theories F equipped with an isomorphism $\phi\colon F/\sigma \xrightarrow{\cong} F$. Now the composition $\delta^* \circ \delta$ was computed in Example 3.25, and it acts on an invertible 2-dimensional theory as multiplication by the 1-dimensional topological field theory which is the σ-model into μ_2. In particular, the vector space attached to a point has dimension 2. Hence there is no square root: Δ acts as multiplication by a 1-dimensional topological field theory, as does Δ^*, and the vector space attached to a point has the same dimension in both. Since $\sqrt{2}$ is not an integer, this cannot happen.

We can an alternative proof in this case which directly shows that $\hat{\sigma}$ in **(4.62)** does not have an augmentation. Namely, $\hat{\sigma}(\mathrm{pt})$ is the fusion category $\mathrm{Vect}[\mu_2]$ with an additional simple object x adjoined. Let $\mu_2 = \{1, g\}$, so that the set of simple objects in $\hat{\sigma}(\mathrm{pt})$ is $\{1, g, x\}$. Then the relations are

$$\begin{aligned} g^2 &= 1 \\ gx &= x \\ x^2 &= 1 + g \end{aligned} \tag{4.66}$$

One can see directly that no fiber functor exists, for example by the same dimension argument applied to the last relation. The fusion category $\hat{\sigma}(\mathrm{pt})$ was introduced by Tambara-Yamagami.

[47] It is not necessarily topological, just invertible.

An example of a theory with this symmetry, including Δ, is the Ising model at the critical temperature; see (4.53). But that theory is not gapped, so learning that it is not trivially gapped is not a particular victory.

(4.67) *A 4-dimensional example.* Now take $\sigma = \sigma^{(5)}_{B^2\mu_2}$ to be the μ_2-gerbe theory in 5 dimensions; it acts on 4-dimensional theories with $B\mu_2$-symmetry. We give examples below. Assume that Δ is adjoined and that (4.61) is satisfied. The composition $\delta^*\circ\delta$ is computed in Example 3.22; from (3.24) we see that it acts on an invertible 4-dimensional theory as multiplication by 3-dimensional μ_2-gauge theory $\Gamma = \sigma^{(3)}_{B\mu_2}$. We claim there is no 3-dimensional topological field theory T such that $T^*\circ T = \Gamma$. If so, evaluate on a point to obtain fusion categories $T(\mathrm{pt})$ and $\Gamma(\mathrm{pt}) = \mathrm{Vect}[\mu_2]$. The number of simple objects in $\mathrm{Vect}[\mu_2]$ is 2, which is not a perfect square. The number of simple objects in $T^*(\mathrm{pt})\otimes T(\mathrm{pt})$ is a perfect square. This contradiction proves that there is no invertible left σ-module on which Δ acts.

Remark 4.68. We have not identified the 5-dimensional topological field theory $\hat{\sigma}$. It would be interesting to do so and to argue directly that $\hat{\sigma}(\mathrm{pt})$ has no augmentation. (Perhaps the argument is a rewording of the one given.)

We conclude with two 4-dimensional gauge theories to which the argument applies to rule out trivial gappedness; see [38] for details.

The first is U_1 Yang-Mills theory with coupling constant τ. The map ϕ is S-duality, which maps $\tau\mapsto -1/\tau$. The quotient by the $B\mu_2$-symmetry maps $\tau\mapsto\tau/4$, so the fixed value is $\tau = 2\sqrt{-1}$. This is the theory to which the argument applies.

The second example is SO_3 Yang Mills theory with θ-angle $\theta = \pi$.

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