What are topological symmetries in QFT?

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November 17, 2022

Joint work with Greg Moore and Constantin Teleman

arXiv:2209.07471

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Crucially, the implementation of symmetries and anomalies is *quantum*, not *classical*

Outline

- Motivation: representation theory of Lie groups and Lie algebras

- Definitions and examples

- Quotients (gauging)

- Line defects in 4-dimensional gauge theory

Computations in... $\mathfrak{sl}_2(\mathbb{R})$

Set

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

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Namely, both sides equal

$$\begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}$$

In the 3-dimensional representation of $\mathfrak{sl}_2(\mathbb{R})$ we have

$$h' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad e' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \qquad f' = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

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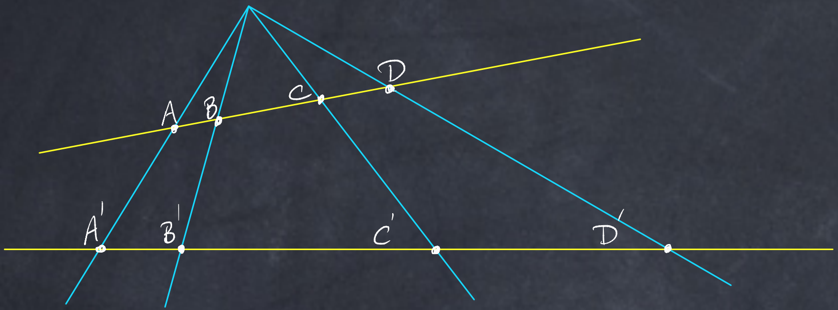
Now slightly less simple matrix manipulations verify the identity

$$\boxed{\frac{1}{2}(h')^2 + e'f' + f'e' = \frac{1}{2}(h')^2 + h' + 2f'e'}$$

Namely, both sides equal

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

The Lie group $\mathrm{SL}_2(\mathbb{R})$ acts on the projective line $\mathbb{RP}^1$ as fractional linear transformations



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There is an induced action on differentials $\phi(x)(dx)^\lambda$ for each $\lambda \in \mathbb{C}$

The infinitesimal action of $\mathfrak{sl}_2(\mathbb{R})$ is:

$$\begin{aligned} \tilde{h} &: \phi \longmapsto -2x\phi' - 2\lambda\phi \\ \tilde{e} &: \phi \longmapsto -\phi' \\ \tilde{f} &: \phi \longmapsto \quad x^2\phi' + 2\lambda x\phi \end{aligned}$$

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Some calculus manipulations verify the identity

$$\boxed{\frac{1}{2}\tilde{h}^2 + \tilde{e}\tilde{f} + \tilde{f}\tilde{e} = \frac{1}{2}\tilde{h}^2 + \tilde{h} + 2\tilde{f}\tilde{e}}$$

Both sides act as multiplication by $4\lambda^2 - 2\lambda$

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Each representation defines a module over the *universal enveloping algebra* $A = U\big(\mathfrak{sl}_2(\mathbb{R})\big)$

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Many recent results about extended notions of symmetry in QFT: Apruzzi, Bah, Benini, Bhardwaj, Bonetti, Bullimore, Córdova, Choi, Cvetič, Del Zotto, Dumitrescu, Frölich, Fuchs, Gaiotto, García Etxebarria, Gould, Gukov, Heckman, Heidenreich, Hopkins, Hosseini, Hsin, Hübner, Intriligator, Ji, Jian, Johnson-Freyd, Jordan, Kaidi, Kapustin, Komargodski, Lake, Lam, McNamara, Minasian, Montero, Ohmari, Pantev, Pei, Plavnik, Reece, Robbins, Roumpedakis, Rudelius, Runkel, Schäfer-Nameki, Scheimbauer, Schweigert, Seiberg, Seifnashri, Shao, Sharpe, Tachikawa, Thorngren, Torres, Vandermeulen, Wang, Wen, Willett, . . . , . . . , . . .

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Main idea: Make analogous universal computations with symmetries in QFT

Warning

The word 'symmetry' in mathematics usually refers to *groups* ("invertible symmetries") rather than algebras ("noninvertible symmetries"), but in modern QFT-speak the term 'symmetry' is also used for the latter. Algebras of operators, including those that commute with a Hamiltonian, date from the earliest days of quantum mechanics

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Motivation: algebras

Abstract symmetry data (for algebras) is a pair (A, R):

A algebra

R right regular module

Definition: Let V be a vector space. An (A, R)-*action on* V is a pair (L, θ) consisting of a left A-module L together with an isomorphism of vector spaces

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Analogy:

algebra $\rightsquigarrow$ topological field theory

element of algebra $\rightsquigarrow$ defect in TFT

Field theory

Analogy: field theory $\sim$ module over an algebra OR $\sim$ representation of a Lie group

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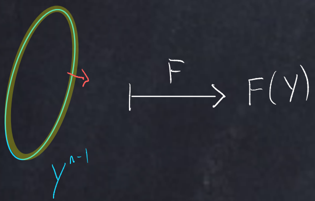
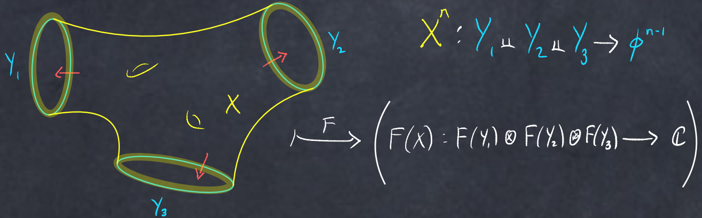
Segal Axiom System: A (Wick-rotated) field theory F is a linear representation of a bordism (multi)category $\mathrm{Bord}_n(\mathcal{F})$

n	dimension of spacetime
$\mathcal{F}$	background fields (orientation, Riemannian metric, . . .)





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Kontsevich-Segal: Axioms for 2-tier nontopological theory $F\colon\ \mathrm{Bord}_{\langle n-1,n\rangle}(\mathcal{F}) \to t\,\mathrm{Vect}$

Domain walls, boundary theories, defects

$\sigma, \sigma_1, \sigma_2$	$(n+1)$-dimensional theories
$\delta: \ \sigma_1 \to \sigma_2$	domain wall
$\rho: \ \sigma \to \mathbb{1}$	right boundary theory
$\widetilde{F}: \mathbb{1} \to \sigma$	left boundary theory

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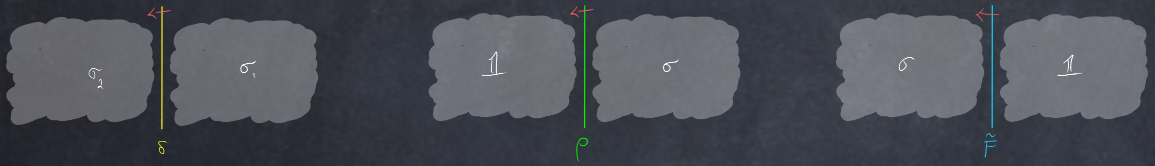

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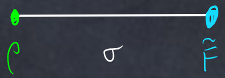

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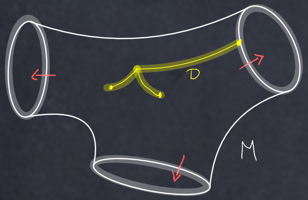
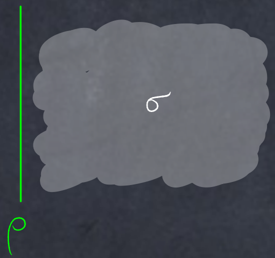
More generally, one can have *defects* supported on any (stratified) manifold $D \subset M$



Main definition: abstract symmetry data

Fix a dimension n and background fields $\mathcal{F}$ (which we keep implicit)

Definition: A *quiche* is a pair (σ, ρ) in which $\sigma\colon \mathrm{Bord}_{n+1}(\mathcal{F}) \to \mathcal{C}$ is an $(n+1)$-dimensional topological field theory and ρ is a right topological σ-module.



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Regular ρ: Suppose $\mathcal{C}'$ is a symmetric monoidal n-category and σ is an $(n+1)$-dimensional topological field theory with codomain $\mathcal{C} = \mathrm{Alg}(\mathcal{C}')$. Let $A = \sigma(\mathrm{pt})$. Then A is an algebra in $\mathcal{C}'$ which, as an object in $\mathcal{C}$, is $(n+1)$-dualizable. Assume that the right regular module A_A is n-dualizable as a 1-morphism in $\mathcal{C}$. Then the boundary theory ρ determined by A_A is the *right regular boundary theory* of σ, or the *right regular σ-module.*

An important generalization

The bulk topological theory σ need not be defined on $(n+1)$-manifolds; it can be a *once-categorified n-dimensional theory*

Analog of boundary theories: *relative field theories* (Stolz-Teichner called them *twisted field theories*)

Defects are also defined in once-categorified theories; the link is a *raviolo* or *UFO*

In this talk we do not pursue these ideas further

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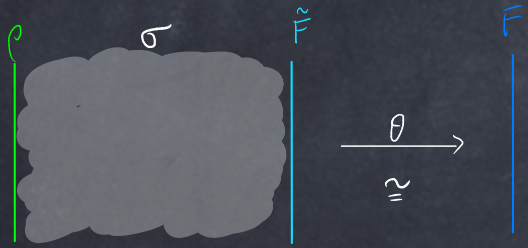
of absolute n-dimensional theories.



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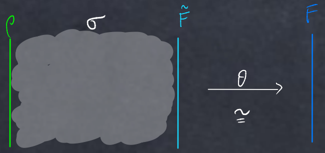

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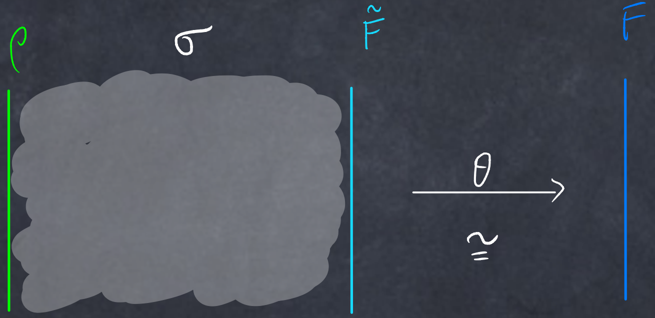
- The sandwich picture of F as $\rho \otimes_\sigma \widetilde{F}$ separates out the topological part (σ, ρ) of the theory from the potentially nontopological part $\widetilde{F}$ of the theory.
- Symmetry persists under renormalization group flow, hence a low energy approximation to F should also be an (σ, ρ)-module. If F is gapped, then we can bring to bear powerful methods and theorems in topological field theory to investigate *topological* left σ-modules. This leads to dynamical predictions

Example: quantum mechanics with G-symmetry

$n = 1$	
$\mathcal{F}$	{orientation, Riemannian metric} for F and $\widetilde{F}$
$\mathcal{H}$	Hilbert space
H	Hamiltonian
$G \circlearrowright \mathcal{H}$	finite group
$\sigma(\mathrm{pt})$	$\mathbb{C}[G]$
$F(\mathrm{pt})$	$\mathcal{H}$
$\widetilde{F}(\mathrm{pt})$	${}_{\mathbb{C}[G]}\mathcal{H}$ (left module)



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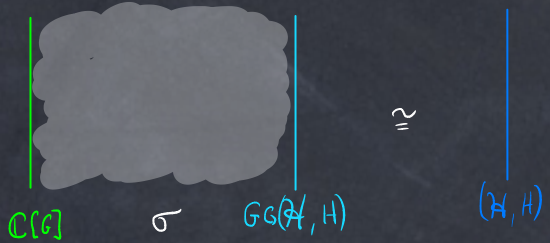

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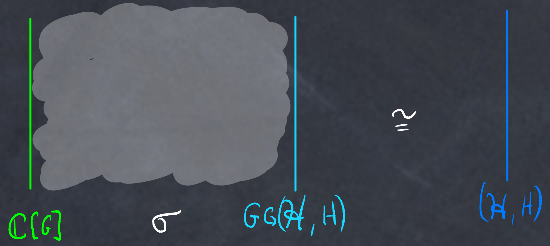

Label the defect beginning with the highest dimensional strata and work down in dimension:

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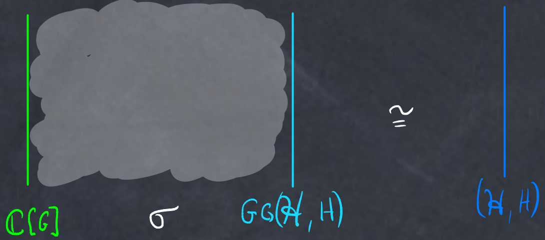

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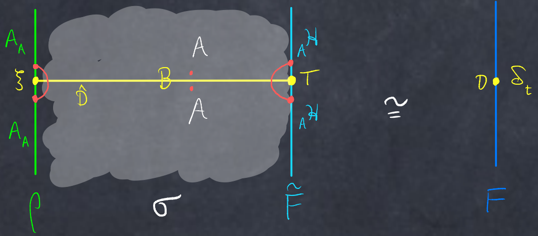
B	(A, A)-bimodule
ξ	vector in B
T	(A, A)-bimodule map $B \longrightarrow \mathrm{End}(\mathcal{H})$



Example: gauge theory with BA-symmetry

n	any dimension
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BA	a homotopical/shifted A ("1-form A-symmetry")
H	Lie group with $A \subset Z(H)$ $H = \mathrm{SU}_2$
$\overline{H} = H/A$	$\overline{H} = \mathrm{SO}_3$
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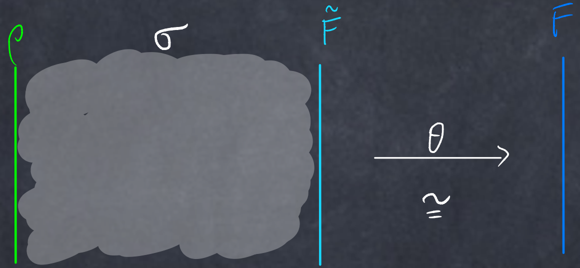

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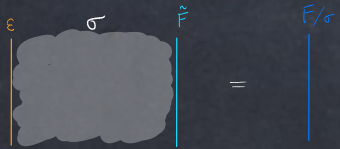
A quotient construction allows to recover absolute $\overline{H}$-gauge theory as a sandwich (later)



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- Definitions and examples
- Quotients (gauging)
- Line defects in 4-dimensional gauge theory

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Example: $A = \mathbb{C}[G]$:

$$\epsilon\colon \mathbb{C}[G] \longrightarrow \mathbb{C}$$
$$\sum_{g\in G} \lambda_g g \longmapsto \sum_{g\in G} \lambda_g$$

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Example: $A = \mathbb{C}[G]$:

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The "quotient" of a left A-module L is the vector space

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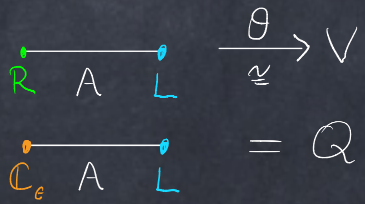

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Augmentations for higher algebras: Φ tensor category $\epsilon\colon \Phi \to \mathrm{Vect}$ *fiber functor*

Projective symmetries

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An augmentation $\epsilon\colon A^\tau \to \mathbb{C}$ splits the extension, so does not exist if $[\tau] \neq 0$

Quotients and quotient defects

We use the yoga of fully local topological field theory: let $\mathcal{C}'$ be a symmetric monoidal n-category and set $\mathcal{C} = \mathrm{Alg}(\mathcal{C}')$, the $(n+1)$-category whose objects are algebras in $\mathcal{C}'$

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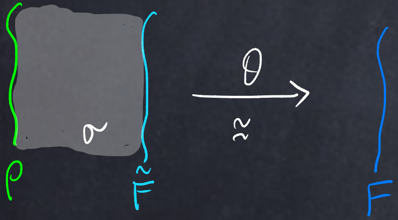
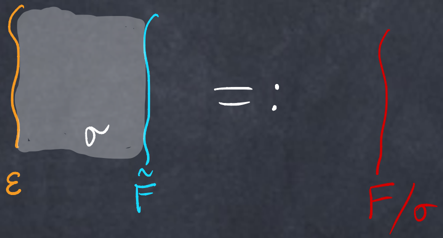
Augmentations do not always exist

Definition: Suppose given finite symmetry data (σ, ρ) and a (σ, ρ)-module structure $(\widetilde{F}, \theta)$ on a quantum field theory F. Suppose ϵ is an augmentation of σ. Then the *quotient* of F by the symmetry σ is

$$F\underset{\epsilon}{/}\sigma = \epsilon \otimes_\sigma \widetilde{F}$$





Outline

- Motivation: representation theory of Lie groups and Lie algebras
- Definitions and examples
- Quotients (gauging)
- Line defects in 4-dimensional gauge theory

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Our goal is to explain the selection rule for line operators which appears in the papers *Framed BPS States* (arXiv:1006.0146) by Gaiotto-Moore-Neitzke and *Reading between the lines of four-dimensional gauge theories* (arXiv:1305.0318) by Aharony-Seiberg-Tachikawa

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Definition: A topological space $\mathcal{X}$ is *π-finite* if (i) $\pi_0\mathcal{X}$ is a finite set, (ii) for all $x \in \mathcal{X}$, the homotopy group $\pi_q(\mathcal{X}, x)$, $q \geqslant 1$, is finite, and (iii) there exists $Q \in \mathbb{Z}^{>0}$ such that $\pi_q(\mathcal{X}, x) = 0$ for all $q > Q$, $x \in \mathcal{X}$.

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Examples: (1) An Eilenberg-MacLane space $K(\pi, q)$ is π-finite if π is a finite group. Denote $K(G, 1)$ by BG for G a finite group, and if $q \geqslant 1$ and A is a finite abelian group, we denote $K(A, q)$ by B^qA.

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(2) Let G be a finite group, let A be a finite abelian group, and fix a cocycle k for a cohomology class $[k] \in H^3(G; A)$. (One can also include an action of G on A.) Realize k as a map $k\colon BG \to B^3A$, and form the π-finite space $\mathcal{X}$ as a pullback:

$$\begin{array}{ccccc} B^2A & \longrightarrow & \mathcal{X} & \longrightarrow & BG \\ \| & & \downarrow & & \downarrow{\scriptstyle k} \\ B^2A & \longrightarrow & * & \longrightarrow & B^3A \end{array}$$

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Remark: If we drop the π-finiteness assumption, then we can construct a once-categorified theory from *any* topological space

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m	(spacetime) dimension
$\mathfrak{X}$	π-finite space
λ	cocycle of degree m on $\mathfrak{X}$ $[\lambda] \in H^m(\mathfrak{X}; \mathbb{C}^\times)$
M	closed manifold
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$$m = 5: \qquad \sigma(M) = \sum_{[\phi]\in\pi_0(\mathcal{X}^M)} \frac{\#\pi_2(\mathcal{X}^M,\phi)}{\#\pi_1(\mathcal{X}^M,\phi)} = \frac{\#H^0(M;A)}{\#H^1(M;A)}\,\#H^2(M;A)$$

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codim 1—the vector space of locally constant complex-valued functions on $\mathcal{X}^M$:

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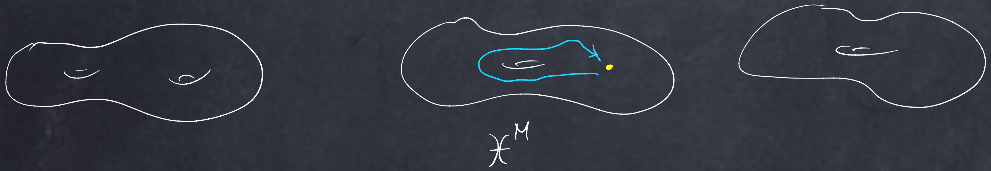

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The quantization of a bordism $M\colon N_0 \to N_1$ uses the correspondence of mapping spaces:

$$\mathcal{X}^{N_0} \xleftarrow{\;p_0\;} \mathcal{X}^M \xrightarrow{\;p_1\;} \mathcal{X}^{N_1}$$

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Compositions of defects are computed using homotopy fiber products

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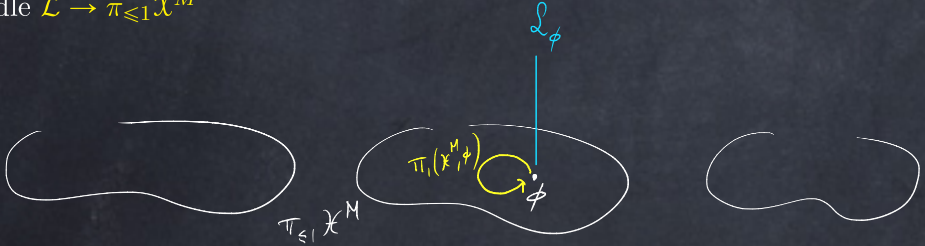

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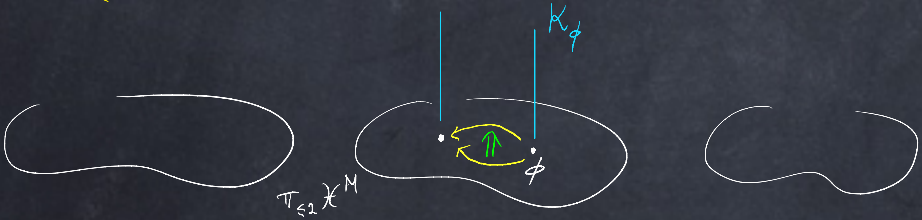

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Higher Gauss law: At a point $\phi \in \mathfrak{X}^M$, if $\pi_1(\mathfrak{X}^M, \phi) = 0$ then $\pi_2(\mathfrak{X}^M, \phi)$ acts on $\mathcal{K}_\phi$ by automorphisms of the identity functor via a character, and sections of $\mathcal{K} \to \mathfrak{X}^M$ vanish on the component which contains ϕ if that character is not the identity

Remark

It is because we treat symmetry as *quantum*—not simply as a *classical* background gauge field—that we can formulate this higher Gauss law and so deduce the selection rule

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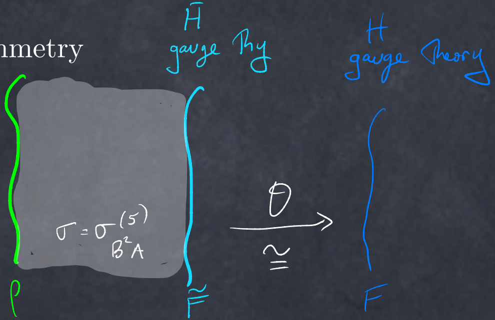
BA **symmetry**

H	compact Lie group
A	finite subgroup of center(H)
$\overline{H}$	H/A
σ	5-dimensional finite homotopy with $\mathcal{X} = B^2A$
ρ	right topological boundary theory $* \to B^2A$
F	a 4-dimensional H-gauge theory with BA symmetry
$\widetilde{F}$	the corresponding $\overline{H}$-gauge theory



Topological right σ-modules

Recall the semiclassical description: $f\colon \mathcal{Y} \to B^2A$ and a trivialization μ of $-f^*\lambda = 0$, so a 4-cocycle μ on $\mathcal{Y}$ with coefficients in $\mathbb{C}^\times$

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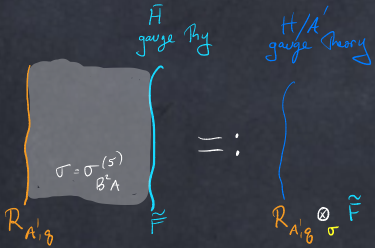

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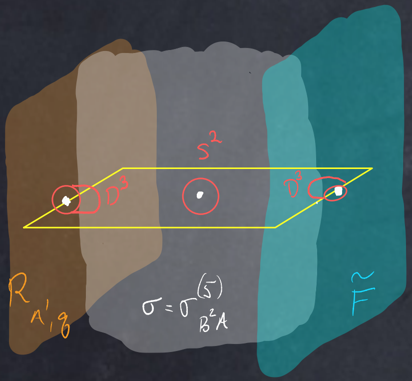
The quadratic form q gives rise to the *Pontrjagin square* cohomology operation

$$\mathcal{P}_q\colon H^2(X;A') \longrightarrow H^4(X;\mathbb{C}^\times)$$

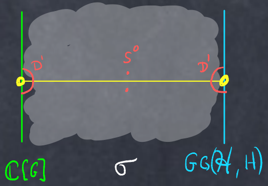
which enters the formula for the partition function in the theory $R_{A',q} \otimes_\sigma \widetilde{F}$, which is an H/A'-gauge theory

Line defects in the H/A'-gauge theory $R_{A',q} \otimes_\sigma \widetilde{F}$

M	4-manifold
$C \subset M$	1-dimensional submanifold
$[0,1] \times C$	2-dimensional submanifold of $[0,1] \times M$



Recall + compare:



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$$\pi_0\big(\mathrm{Map}(S^2, B^2A)\big) = H^2(S^2; A) \cong A$$
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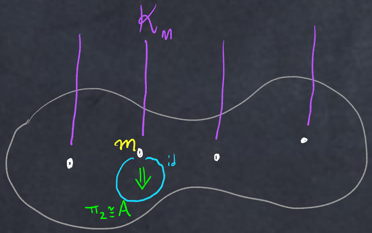

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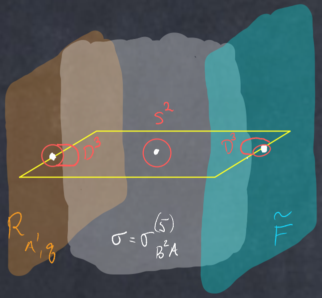
2-category of local systems of linear categories over the indicated 2-groupoid, so for $m \in H^2(S^2; A) \cong A$ we have a linear category $\mathcal{K}_m$ equipped with an action of $\pi_2 \cong A$ by automorphisms of the identity functor, hence $\mathcal{K}_m$ decomposes as

$$\mathcal{K}_m = \bigoplus_e \mathcal{K}_{m,e} \cdot e, \qquad e \in H^0(S^2; A)^\vee \cong A^\vee$$

The line defect $[0,1) \times C$ in $(\sigma, R_{A',q})$

First, fix a pair $(m_0, e_0) \in A \times A^\vee$ and choose the interior label $\mathcal{K}$ to be the "δ-function" supported at (m_0, e_0):

$$\mathcal{K}_{m,e} = \begin{cases} \text{Vect}, & (m,e) = (m_0, e_0); \\ 0, & (m,e) \neq (m_0, e_0). \end{cases}$$



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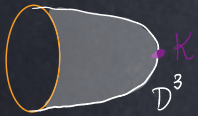
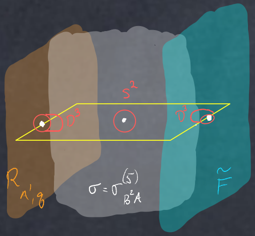
The quantization of the link D^3 at the $R_{A',q}$ boundary is a 1-category

Claim: This 1-category vanishes unless (m_0, e_0) obeys a selection rule





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The selection rule is an assertion in the *topological* field theory $(\sigma, R_{A',q})$

The selection rule

From the quadratic function $q\colon A' \to \mathbb{C}^\times$ we obtain a bicharacter

$$b\colon A' \times A' \longrightarrow \mathbb{C}^\times$$

which induces a pairing

$$H^2(S^2; A') \times H^0(S^2; A') \longrightarrow \mathbb{C}^\times$$

and so too a homomorphism

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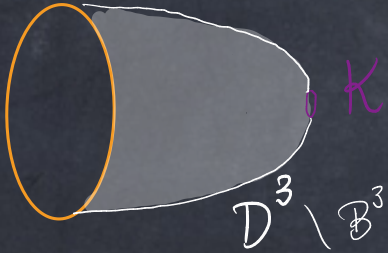
Selection rule for (m_0, e_0):

$$\boxed{\begin{aligned} m_0 &\in A' \\ e_0\big|_{A'} &= e'(m_0)^{-1} \end{aligned}}$$

Sketch proof of the selection rule

Compute the homotopy limit of the diagram:

$$
\begin{array}{ccccccc}
(\mathrm{Map}(S^2, B^2A'), \tau^2(\mu_q)) & & & \mathrm{Map}(D^3 \backslash B^3, B^2A) & & & (B^2A, e_0) \\
& \searrow & \swarrow & & \searrow & \swarrow{\scriptstyle m_0} & \\
& & \mathrm{Map}(S^2, B^2A) & & \mathrm{Map}(S^2, B^2A) & &
\end{array}
$$



$m_0 \in A'$

$e_0\big|_{A'} = e'(m_0)^{-1}$