Final Exam
M367K: Topology
May 9, 2018

T/F.  _____/30  1. _____/20  2. _____/10  3. _____/10  4. _____/15
5. _____/15  Total Score. _____/100
6 (EC). _____/10  7 (EC). _____/10  8 (EC). _____/10

Please put your name at the top of the exam. Read over the entire exam before you begin. Continue your work on the backs of pages or on extra sheets. If your solution runs over onto these pages, please indicate that clearly. If you use extra sheets, be sure to staple them to the exam.

Only attempt the extra credit problem after completing the rest of the exam. Extra credit problems are scored separately and are not included in the Total Score.

Write neatly! The more neat work you show, the more (partial) credit you will receive.

Have fun!

Answer the following true/false questions. If you find a question ambiguous, write a brief explanation. Each true/false question is worth 2 points.

True   False

1. Suppose topological spaces $X$ and $Y$ are homotopy equivalent. Then there is a bijection $\pi_0 X \cong \pi_0 Y$ of the sets of path components.

2. Suppose $X$ is a topological space which is not connected. Then there exists a nonempty subset $A \subseteq X$ which is both open and closed.

3. Let $X$ be a set endowed with the discrete topology. Then $X$ is metrizable.

4. Suppose $X, Y$ are topological spaces, $f, g: X \to Y$ homotopic continuous maps, and $f$ is a homeomorphism. Then $g$ is also a homeomorphism.

5. Let $X$ be a topological space and $x_0 \in X$. Suppose $f, g, h: [0, 1] \to X$ are paths which begin and end at $x_0$. Then $f \ast (g \ast h) = (f \ast g) \ast h$ as paths $[0, 1] \to X$.

6. Let $X$ be a metric space and $C \subset X$ a compact subset. Then $C$ is closed and bounded.

7. There exists a continuous map $r: D^2 \to S^1$, where $D^2$ is the closed disk in $\mathbb{R}^2$ and $S^1$ is its boundary circle.
8. A connected space is path connected.

9. Suppose $X$ is contractible. Then for any $x_0 \in X$, the fundamental group $\pi_1(X, x_0)$ is the group with one element.

10. The polynomial $f(z) = z^3 - 6z + 8$ has a real root: $z \in \mathbb{R}$ such that $f(z) = 0$.

11. Let $X$ be a topological space, $A, B \subset X$ connected subspaces. Then $A \cap B$ is connected.

12. Let $X$ be a compact space and $C_1 \supset C_2 \supset C_3 \cdots$ a nested sequence of compact subsets. Then there exists $x \in X$ such that $x \in C_n$ for all $n$.

13. Suppose $A \subset \mathbb{R}^2$ is connected and $p, q \in A$. Then the midpoint $(p + q)/2$ of the line segment connecting $p$ and $q$ also lies in $A$.

14. Let $D^2$ be the closed unit disk in the plane. Then any continuous map $f : D^2 \to D^2$ has a fixed point, i.e., a point $x \in D^2$ such that $f(x) = x$.

15. Suppose $f : X \to Y$ is a bijective continuous map of topological spaces. Assume $X$ is compact and $Y$ is Hausdorff. Then $f$ is a homeomorphism.
1. (20 points) Here are descriptions of five topological spaces $X$ and subsets $A \subset X$:

A : $X = \mathbb{Q}$  
A = $\{x \in \mathbb{Q} : x^2 < \sqrt{2}\}$

B : $X = \{x, y, z\}$, $\mathcal{T} = \{\emptyset, \{x\}, \{x, y\}, X\}$  
A = $\{x, z\}$

C : $X =$ topologists’ sine curve,  
A = $\{0\} \times [-1, 1]$

D : $X = S^2$  
A = $\{(x, y, z) \in S^2 : x, y \in \mathbb{Q}\}$

E : $X = \{f : \{1, 2, \ldots, 10\} \rightarrow [-100, 100]\}$  
A = $\{f : |f(n)| \leq 1 \text{ for all } n\}$

Recall that $\mathbb{Q} \subset \mathbb{R}$ is the subset of rational numbers with the subspace topology, $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ with the subspace topology induced from $\mathbb{R}^3$, and the topologists’ sine curve is the subspace of $\mathbb{R}^2$ formed as the union of the graph of

$$f : \mathbb{R}^+ \rightarrow \mathbb{R}, \quad x \mapsto \sin(1/x)$$

(viewed as a subset of $\mathbb{R}^2$) and $\{0\} \times [-1, 1] \subset \mathbb{R}^2$, topologized with the subspace topology. The function space in E can be identified with $[-100, 100]^{\times 10}$ and is endowed with the product topology.

For each of the five cases tell if $X$ is path connected, if $X$ is metrizable, if $A$ is compact, and specify the closure $\overline{A}$. Each of the first three answers is either ‘Y’ or ‘N’.

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<tr>
<th>X Path Connected?</th>
<th>X Metrizable?</th>
<th>A Compact?</th>
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2. (10 points) Write a careful proof of the following theorem.

**Theorem:** Let $f : X \to Y$ be a continuous map. Assume $X$ is path connected. Then $f(X)$ is path connected.

Let $y_0, y_1 \in f(X)$. Choose $x_0, x_1 \in X$ such that $f(x_0) = y_0$ and $f(x_1) = y_1$. Since $X$ is path connected, we can find a continuous path $p : [0, 1] \to X$ such that $p(0) = x_0$ and $p(1) = x_1$. Then $f \circ p : [0, 1] \to f(X)$ is continuous and $(f \circ p)(0) = y_0$, $(f \circ p)(1) = y_1$. Since arbitrary points of $f(X)$ can be connected by a path, $f(X)$ is path connected.

**Remark:** It is better to replace $f$ by the map $\hat{f} : X \to f(X)$ which satisfies $f = c \circ \hat{f}$, where $c : f(X) \to Y$ is the inclusion. A theorem we proved in class shows $\hat{f}$ is continuous.
3. (10 points) The letters \(PHY\) are not homeomorphic, but they are homotopy equivalent. Sketch arguments for both assertions.

If \( f: H \to Y \) is a homeomorphism, then so too is \( f |_{H \setminus \{p, q\}} : H \setminus \{p, q\} \to Y \setminus \{f(p), f(q)\} \), where \( p, q \in H \) are the indicated points. This induces an isomorphism on \( \pi_0 \), the set of path components. Now \( \# \pi_0 (H \setminus \{p, q\}) = 5 \) and, for any two points \( p, q \in Y \), \( \# \pi_0 (Y \setminus \{p, q\}) \leq 4 \). (If \( y \notin \{p, q\} \) then there are 3 path components; if \( y \in \{p, q\} \) there are 2.) This shows \( f \) cannot exist.

Each of \( H, Y \) is contractible, so they are homotopy equivalent:

\[
\begin{align*}
H & \to H \to 1 \to 1 \\
Y & \to V \to 1 \to 1
\end{align*}
\]
4. (15 points) Tell whether each of the following statements is true or false, and then provide a proof if true or a counterexample if false.

(a) Let $X$ be a topological space and $B_1, B_2$ two bases for the topology. Then $B_1 \cap B_2$ is also a basis for the topology.

(b) Let $X$ be a finite set and $\mathcal{T}$ a topology on $X$. Then any subset of $X$ is compact.

(c) Let $f : X \to Y$ be a map of topological spaces. Suppose $X = U_1 \cup U_2$ is a union of two open sets such that each restriction $f|_{U_i} : U_i \to Y$, $i = 1, 2$, is continuous. Then $f$ is continuous.

(a) False. Note $B_1, B_2 \subset \mathcal{P}(X)$ so $B_1 \cap B_2 \subset \mathcal{P}(X)$; the intersection take place inside $\mathcal{P}(X)$, not $X$. For a counterexample take $X = \mathbb{R}$ with the (usual) order topology. Let $B_1$ be the set of nonempty open intervals of rational length and $B_2$ the set of nonempty open intervals of irrational length. Then $B_1 \cap B_2 = \emptyset$.

(b) True. Any finite set $F$ with any topology is compact: If $U : I \to \mathcal{P}(F)$ is a cover (open or not), then choose $s : F \to I$ by the condition $f(U(s))$ for $f \in F$. The restriction $U|_{S(F)}$ is a finite subcover.

(c) True. Let $V \subset Y$ be open. For $i = 1, 2$ we have $(f|_{U_i})^{-1}(V) \subset U_i$ is open. Since $U_i \subset X$ is open, $(f|_{U_i})^{-1}(V) \subset X$ is open. Then $f^{-1}(V) = (f|_{U_1})^{-1}(V) \cup (f|_{U_2})^{-1}(V) \subset X$ is also open, so $f$ is continuous. (This equality of subsets of $X$ were $X = U_1 \cup U_2$.)
5. (15 points) Give an example of each of the following.

(a) A surjective continuous map \( f : X \to Y \) of topological spaces and elements \( y_1, y_2 \in Y \) so that \( f^{-1}(y_1) \) is compact but \( f^{-1}(y_2) \) is not compact.

(b) A metric space \( X \) and a subset \( A \subset X \) which is compact but not connected.

(c) A continuous map \( f : S^1 \to S^1 \) so that the cardinality of \( f^{-1}(y) \) is 2 for all \( y \in S^1 \). (You can take \( S^1 = \mathbb{R}/\mathbb{Z} \), as in lecture.)

\[
\begin{align*}
(a) \quad X &= (0, 1) \cup \{2, 3\} \subset \mathbb{R} \\
&\quad Y = \{y_1, y_2\} \\
&\quad f|_{(0, 1)} = y_2, \quad f|_{\{2, 3\}} = y_1
\end{align*}
\]

\[
(b) A = X = \{a, b\} \text{ with the discrete topology}
\]

\[
(c) \quad f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \\
&\quad x \mapsto 2x
\]

For \( y \in \mathbb{R}/\mathbb{Z} \) the equation

\[2x \equiv y \pmod{1}\]

has two solutions: if \( x_0 \) is one solution, then \( x_0 + \frac{1}{2} \pmod{1} \) is another.
6. (10 points) (Extra Credit) Prove the following.

**Theorem:** Let \( f : X \to Y \) be a homotopy equivalence. Fix \( x_0 \in X \). Prove that the induced map \( f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0)) \) on fundamental groups is an isomorphism.

Let \( g : Y \to X \) be an inverse homotopy equivalence; so \( g \circ f \) \& \( f \circ g \) are identities. Let \( f_x, g_x \) be the induced maps on \( \pi_0 \). Then \( g_x \circ f_x = \text{id}_{\pi_0 X} \) and \( f_x \circ g_x = \text{id}_{\pi_0 Y} \), since \( f_x, g_x, (g \circ f)_x = g_x \circ f_x \), and \((f \circ g)_x = f_x \circ g_x \). Only depend on homotopy class of the given map. Hence \( f_x, g_x \) are inverse isomorphisms (bijective on sets).
7. (10 points) (Extra Credit) Prove the following.

**Theorem:** Suppose $X$ and $Y$ are connected topological spaces. Then the Cartesian product $X \times Y$ is connected.

If $X$ or $Y$ is empty, so is $X \times Y$, which is connected.

If not, choose $x_0 \in X$ and $y_0 \in Y$. Let

$$A_x = X \times \{y_0\} \cup \{x_0\} \times Y$$

Since each term in the union is connected (being homeomorphic to $X$, $Y$, respectively), and their intersection is nonempty, $A_x$ is connected. Then so is $X \times Y = \bigcup_x A_x$. Since for any $x, x' \in X$ we have $(x_0, y_0) \in A_x \cap A_{x'}$.
8. (10 points) (Extra Credit) Let $X$ be a topological space, and let $D^1 = [-1, 1] \subset \mathbb{R}$ be the 1-disk with boundary the 0-sphere $S^0 = \{ -1, 1 \} \subset D^1$. Suppose $f: S^0 \rightarrow X$ is a continuous map. Then the space $X \cup_f D^1$ obtained by attaching $D^1$ to $X$ along $f$ is the quotient of the disjoint union $X \sqcup D^1$ by the equivalence relation which identifies $+1 \in D^1$ with $f(+1) \in X$ and $-1 \in D^1$ with $f(-1) \in X$. Find spaces $X_1, X_2$ and attaching maps $f_1, f_2$ so that $X_1$ is not homeomorphic to $X_2$ yet there exists a homeomorphism

$$X_1 \cup_{f_1} D^1 \cong X_2 \cup_{f_2} D^1.$$ 

Set $X_1 = \text{pt}$

$$X_2 = [0, 1]$$

There is a unique attaching map $f_1$. Define $f_2(-1) = 0$, $f_2(1) = 1$.

The space with a cell attached at each homotopy to $S^1$, whereas $X_1$ is not even in bijection with $X_2$, much less homeomorphic.