Lecture 15: Quadratic approximation and the second variation formula

Symmetric bilinear forms and inner products

(15.1) The associated self-adjoint operator. Let $V$ be a finite dimensional real vector space and $B: V \times V \to \mathbb{R}$ a symmetric bilinear form. In (14.35) we proved that there is a basis $e_1, \ldots, e_n$ of $V$ in which $B$ is diagonal, in the sense that $B(e_i, e_j) = 0$ if $i \neq j$. By scaling the basis elements we can arrange that the diagonal entries $B(e_i, e_i)$ are either 0, $+1$, or $-1$; see (14.37).

Now suppose $V$ also has an inner product $\langle -, - \rangle$. Define a linear operator $S_B: V \to V$ by

\[ B(\xi_1, \xi_2) = \langle \xi_1, S_B(\xi_2) \rangle, \quad \xi_1, \xi_2 \in V. \]

The symmetry of $B$ implies that $S_B$ is self-adjoint in the sense that

\[ \langle \xi_1, S_B(\xi_2) \rangle = \langle S_B(\xi_1), \xi_2 \rangle, \quad \xi_1, \xi_2 \in V. \]

(15.4) Diagonalization. The operator $S_B$ is diagonalizable, as we now prove.

Theorem 15.5. Let $V$ be a finite dimensional real inner product space and $S: V \to V$ a self-adjoint operator. Then $S$ has a (nonzero) eigenvector.

Of course, not every linear operator has an eigenvector: a nontrivial rotation in the plane does not fix any line. The following proof is essentially a reprise of the second proof of Proposition 9.25.

Proof. Consider the functions $f, g: V \to \mathbb{R}$ defined by

\[ f(\xi) = \frac{1}{2} \langle \xi, S(\xi) \rangle, \]
\[ g(\xi) = \frac{1}{2} \langle \xi, \xi \rangle. \]

Let $S(V) = g^{-1}(1) \subset V$ be the unit sphere. Since $S(V)$ is compact, $f$ has a maximum on $S(V)$, say at $e_1 \in S(V)$. The Lagrange multiplier criterion implies that there exists $\lambda_1 \in \mathbb{R}$ such that $df_{e_1} = \lambda_1 \, dg_{e_1}$, in other words $S(e_1) = \lambda_1 e_1$. □

Corollary 15.7. In the situation of Theorem 15.5 the operator $S$ is diagonalizable.

Proof. Let $V_1$ be the orthogonal complement to the eigenvector $e_1$. The self-adjointness (15.3) implies that if $\xi_2 \in V_1$, then $S(\xi_2) \in V_1$ and the restriction of $S$ to $V_1$ is self-adjoint. Theorem 15.5 produces an eigenvector $e_2$ of this restriction with eigenvalue $\lambda_2 \leq \lambda_1$. Repeat the argument a total of $\dim V$ times to produce an orthonormal basis $e_1, \ldots, e_n$ of eigenvectors with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. □

It follows immediately from (15.2) that $B$ is diagonalized as well:

\[ B(e_i, e_j) = \begin{cases} 0, & i \neq j; \\ \lambda_i, & i = j. \end{cases} \]

The inner product gives meaning to the diagonal entries; compare (14.37).
The second derivative test and quadratic approximation

(15.9) Introduction. Resume our standard setup (5.23) with $B = \mathbb{R}$. Let $p_0 \in U$ be a critical point of $f$, i.e., $df_{p_0} = 0$. Then we would like to say that the function

$$f(p_0 + \xi) = f(p_0) + \frac{1}{2} d^2 f_{p_0}(\xi, \xi)$$

is a good approximation to $f$ near $p_0$. But this is not necessarily true.

Example 15.11. Take $U = A = \mathbb{R}$ and consider the five functions

$$f_1(x) = +x^2$$
$$f_2(x) = -x^2$$
$$f_3(x) = x^3$$
$$f_4(x) = +x^4$$
$$f_5(x) = -x^4$$

Each function has a critical point at $x = 0$. The quadratic approximation (15.10) works—in fact is exact—for $f_1, f_2$. For $f_3, f_4, f_5$ the quadratic approximation is a constant function and that does not predict the local behavior: $x = 0$ is an inflection point of $f_3$, a local minimum of $f_4$, and a local maximum of $f_5$.

(15.13) Second derivative test for local extrema. The quadratic approximation is guaranteed to be good if $d^2 f_{p_0}$ is nondegenerate and if $A$ is finite dimensional. We first prove a special case, when $d^2 f_{p_0}$ is positive definite. The analogous theorem for $d^2 f_{p_0}$ negative definite follows by applying the following to $-f$.

Theorem 15.14. Suppose $A$ is finite dimensional, $f$ is a $C^1$ function, $p_0 \in U$ is a critical point, $f$ is twice differentiable at $p_0$, and $d^2 f_{p_0}$ is positive definite. Then $f$ has a strict local minimum at $p_0$.

The strictness means that there exists a neighborhood $U' \subset U$ of $p_0$ such that $f(p) > f(p_0)$ for all $p \in U' \setminus \{p_0\}$.

Remark 15.15. The following proof works for $A$ infinite dimensional if the given norm on $V$ is equivalent to the norm (15.16) defined by the second differential. In finite dimensions all norms are equivalent, and so we can and do use (15.16) as the norm in the definition of differentiability.

Proof. Since $d^2 f$ is positive definite,

$$\|\xi\| = \sqrt{d^2 f_{p_0}(\xi, \xi)}, \quad \xi \in V,$$

is a norm on $V$. The twice differentiability of $f$ at $p_0$ is the assertion: given $\epsilon > 0$ there exists $\delta > 0$ such that if $\xi \in V$ satisfies $\|\xi\| < \delta$, then $p_0 + \xi \in U$ and

$$| df_{p_0 + \xi} - df_{p_0} - d^2 f_{p_0}(\xi) | \leq \epsilon \|\xi\|.$$
Fix $0 < \epsilon < 1$ and choose $\delta > 0$ so that (15.17) holds if $\|\xi\| < \delta$. Now fix $\xi_0 \in B_\delta(0)$ and set $g(t) = f(p_0 + t\xi_0)$, $t \in [0,1]$. Then $g'(t) = df_{p_0 + t\xi_0}(\xi_0)$. Evaluate the linear functionals in (15.17) on $\xi_0$ and use the fact that $df_{p_0} = 0$ to conclude

\[(15.18) \quad (1 - \epsilon) t \|\xi_0\|^2 \leq g'(t) \leq (1 + \epsilon) t \|\xi_0\|^2.\]

This is an inequality of real-valued functions of $t$. Integrating we conclude that

\[(15.19) \quad \frac{1 - \epsilon}{2} \|\xi_0\|^2 \leq g(1) - g(0) \leq \frac{1 + \epsilon}{2} \|\xi_0\|^2,
\]

which is

\[(15.20) \quad f(p_0) + \frac{1 - \epsilon}{2} \|\xi_0\|^2 \leq f(p_0 + \xi_0) \leq f(p_0) + \frac{1 + \epsilon}{2} \|\xi_0\|^2.
\]

In particular, $f(p_0 + \xi_0) \geq f(p_0)$ and $f(p_0 + \xi_0) > f(p_0)$ if $\xi_0 \neq 0$. Therefore, $f$ has a strict local minimum at $p_0$. \qed

(15.21) More general quadratic approximations. The inequalities (15.20) sandwich the function $f$ between two quadratic functions, an approximation valid in a neighborhood of the critical point $p_0$ with positive definite second differential. More generally, suppose $p_0$ is a critical point of $f$ with nondegenerate second differential. Choose a decomposition $V = P \oplus N$ such that $d^2f_{p_0}$ is positive definite on $P$ and negative definite on $N$. Define the norm

\[(15.22) \quad \|\xi' + \xi''\| = \sqrt{d^2f_{p_0}(\xi', \xi')} + \sqrt{-d^2f_{p_0}(\xi'', \xi'')}, \quad \xi', \xi'' \in P \oplus N,
\]

on $V$.

**Theorem 15.23.** Suppose $A$ is finite dimensional, $f$ is a $C^1$ function, $p_0 \in U$ is a critical point, $f$ is twice differentiable at $p_0$, and $d^2f_{p_0}$ is nondegenerate. Then in a neighborhood $U' \subset U$ of $p_0$ we have

\[(15.24) \quad f(p_0) + \frac{1 - \epsilon}{2} \|\xi'\|^2 - \frac{1 + \epsilon}{2} \|\xi''\|^2 \leq f(p_0 + \xi' + \xi'') \leq f(p_0) + \frac{1 + \epsilon}{2} \|\xi'\|^2 - \frac{1 - \epsilon}{2} \|\xi''\|^2,
\]

in terms of the norm (15.22), for $\xi' + \xi'' \in V = P \oplus N$.

**Proof.** We only need a small modification of the proof of Theorem 15.14. Namely, (15.17) implies that for $\xi = \xi' + \xi''$ of norm less than $\delta$ and $0 \leq t \leq 1$ we have

\[(15.25) \quad | df_{p_0 + t\xi' + t\xi''}(\xi' + \xi'') - t \|\xi'\|^2 + t \|\xi''\|^2 | \leq \epsilon t (\|\xi'\|^2 + \|\xi''\|^2),
\]

and so writing $\xi_0 = \xi'_0 + \xi''_0$ we replace (15.18) with the inequalities

\[(15.26) \quad (1 - \epsilon) t \|\xi'_0\|^2 - (1 + \epsilon) t \|\xi''_0\|^2 \leq g'(t) \leq (1 + \epsilon) t \|\xi'_0\|^2 - (1 - \epsilon) t \|\xi''_0\|^2.
\]

The inequalities (15.24) follow by integrating (15.26). \qed
Second variation formula

(15.27) Recalling the setup. In Lecture 1 we computed the first variation formula, that is, the differential of the length function. The setup is that $V$ is a finite dimensional real inner product space, $E$ an affine space over $V$, and $p, q \in E$ fixed points. Parametrized paths from $p$ to $q$ form an affine space

(15.28) \[ A = \{ \gamma: [0, 1] \rightarrow E \text{ such that } \gamma(0) = p, \gamma(1) = q, \gamma \in C^2([0, 1], E) \} \]

whose tangent space is the vector space

(15.29) \[ X = \{ \xi: [0, 1] \rightarrow V \text{ such that } \xi(0) = \xi(1) = 0, \xi \in C^2([0, 1], V) \} \]

with norm

(15.30) \[ \| \xi \| = \max_{s \in [0, 1]} \| \dot{\xi}(s) \|_V, \]

where the dot denotes $d/ds$. Set

(15.31) \[ U = \{ \gamma \in A \text{ such that } \dot{\gamma}(s) \neq 0 \text{ for all } s \in [0, 1] \} \]

and define the length function

(15.32) \[ f: U \rightarrow \mathbb{R}, \quad \gamma \mapsto \int_0^1 ds \| \dot{\gamma}(s) \| \]

In Theorem 10.28 we proved that $f$ is differentiable with differential

(15.33) \[ df(\gamma)(\xi) = \xi f(\gamma) = \int_0^1 ds \frac{\langle \dot{\gamma}, \dot{\xi} \rangle}{\langle \dot{\gamma}, \dot{\gamma} \rangle^{1/2}}. \]

Furthermore, $\gamma$ is a critical point if it is a constant velocity motion, or a reparametrization of a constant velocity motion.

(15.34) The second directional derivative. Assume $\gamma$ is a unit speed, so $\langle \dot{\gamma}, \dot{\gamma} \rangle = 1$ and $\ddot{\gamma} = 0$. We do not prove that $f$ is twice differentiable at $\gamma$, but content ourselves with computing the iterated second directional derivative. Fix $\xi_1, \xi_2 \in X$. Then commuting differentiation and integration, as
in (10.12) and with the same justification, we have

\[
\xi_1\xi_2f(\gamma) = \frac{d}{dt} \bigg|_{t=0} \xi_2f(\gamma + t\xi_1)
\]

\[
= \frac{d}{dt} \bigg|_{t=0} \int_0^1 ds \frac{\langle \dot{\gamma} + t\xi_1, \dot{\xi}_2 \rangle}{\langle \dot{\gamma} + t\xi_1, \dot{\gamma} + t\xi_1 \rangle^{1/2}}
\]

\[
= \int_0^1 ds \left\{ \langle \dot{\xi}_1, \dot{\xi}_2 \rangle - \frac{1}{2} \langle \dot{\gamma}, \dot{\gamma} \rangle^{-3/2} 2 \langle \dot{\gamma}, \dot{\xi}_1 \rangle \langle \dot{\gamma}, \dot{\xi}_2 \rangle \right\}
\]

\[
= \int_0^1 \left\{ \langle \dot{\xi}_1, \dot{\xi}_2 \rangle - \langle \dot{\gamma}, \dot{\xi}_1 \rangle \langle \dot{\gamma}, \dot{\xi}_2 \rangle \right\}.
\]

This is a symmetric bilinear form in the variables \(\xi_1, \xi_2\), as it should be. It has an infinite dimensional kernel due to reparametrization invariance. Namely, if \(\rho_1 : [0, 1] \rightarrow \mathbb{R}\) is a \(C^2\) function with \(\rho_1(0) = \rho_1(1) = 0\), then for \(\xi_1(s) = \rho_1(s)\dot{\gamma}\) we have \(\dot{\xi}_1 = \dot{\rho}_1\dot{\gamma}\) and (15.35) vanishes for all \(\xi_2 \in X\). We claim that (15.35) is positive semidefinite. Namely, in general we write \(\xi \in X\) as

\[
(15.36) \quad \xi(s) = \rho(s)\dot{\gamma} + \eta, \quad \langle \eta(s), \dot{\gamma} \rangle = 0,
\]

for \(\eta : [0, 1] \rightarrow V\) with \(\eta(0) = \eta(1) = 0\). Differentiating the constraint we find \(\langle \dot{\eta}, \dot{\gamma} \rangle = 0\). Setting \(\xi_1 = \xi_2 = \rho\dot{\gamma} + \eta\) we compute

\[
(15.37) \quad \xi\xi f(\gamma) = \int_0^1 \|\dot{\eta}\|^2
\]

which is nonnegative.

Remark 15.38. Theorem 15.14 does not apply since the second differential is only semidefinite, not definite, and the domain is infinite dimensional. We can work modulo the kernel to obtain a positive definite form, but it is not equivalent to (15.30) (on the quotient), so we would need further argument to prove that—spoiler alert!—the shortest distance between two points in Euclidean space is a straight line segment.