## Lecture 1: Introduction to bordism

### Overview

*Bordism* is a notion which can be traced back to Henri Poincaré at the end of the 19th century, but it comes into its own mid-20th century in the hands of Lev Pontrjagin and René Thom [T]. Poincaré originally tried to develop homology theory using smooth manifolds, but eventually simplices were used instead. Recall that a *singular $q$-chain* in a topological space $S$ is a formal sum of continuous maps $\Delta^q \to S$ from the standard $q$-simplex. There is a boundary operation $\partial$ on chains, and a chain $c$ is a *cycle* if $\partial c = 0$; a cycle $c$ is a *boundary* if there exists a $(q+1)$-chain $b$ with $\partial b = c$. If $S$ is a point, then every cycle of positive dimension is a boundary. In other words, abstract chains carry no information. In bordism theory one replaces cycles by *closed*[1] *smooth manifolds* mapping continuously into $S$. A chain is replaced by a compact smooth manifold $X$ and a continuous map $X \to S$; the boundary of this chain is the restriction $\partial X \to S$ to the boundary. Now there is information even if $S = \mathrm{pt}$. For not every closed smooth manifold is the boundary of a compact smooth manifold. For example, $Y = \mathbb{RP}^2$ is *not* the boundary of a compact 3-manifold. (It *is* the boundary of a noncompact 1-manifold with boundary—which? In fact, show that *every* closed smooth manifold $Y$ is the boundary of a noncompact manifold with boundary.)

A variation is to consider smooth manifolds equipped with a *tangential structure* of a fixed type. One type of a tangential structure you already know is an *orientation*, which we review in Lecture 2. We give a general discussion in a few weeks.

One main idea of the course is to extract various algebraic structures of increasing complexity from smooth manifolds and bordism. Today we will use bordism to construct an equivalence relation, and so construct *sets* of bordism classes of manifolds. We will introduce an algebraic structure to obtain *abelian groups* and even a *commutative ring*. These ideas date from the 1950s. The modern results concern more intricate algebraic gadgets extracted from smooth manifolds and bordism: *categories* and their more complicated cousins. Some of the main theorems in the course identify these algebraic structures explicitly. For example, an easy theorem asserts that the bordism group of oriented 0-manifolds is the free abelian group on a single generator, that is, the infinite cyclic group (isomorphic to $\mathbb{Z}$). One of the recent results which is a focal point of the course, the *cobordism hypothesis* [L1, F1], is a vast generalization of this easy classical theorem.

We will also study *bordism invariants*. These are homomorphisms out of a bordism group or category into an abstract group or category. Such homomorphisms, as all homomorphisms, can be used in two ways: to extract information about the domain or to extract information about the codomain. In the classical case the codomain is typically the integers or another simple number system, so we are typically using bordism invariants to learn about manifolds. A classic example of such an invariant is the *signature* of an oriented manifold, and Hirzebruch's signature theorem equates the signature with another bordism invariant constructed from *characteristic numbers*. On the other hand, a typical application of the cobordism hypothesis is to use the structure of manifolds to learn about the codomain of a homomorphism. Incidentally, a homomorphism out of a bordism category is called a *topological quantum field theory* [A1].

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[1] The word 'closed' modifying manifold means 'compact without boundary'.

**(1.1)** *Convention.* All manifolds in this course are smooth, or smooth manifolds with boundary or corners, so we omit the modifier 'smooth' from now on. In bordism theory the manifolds are almost always compact, though we retain that modifier to be clear.

### Review of smooth manifolds

**Definition 1.2.** A *topological manifold* is a paracompact, Hausdorff topological space $X$ such that every point of $X$ has an open neighborhood which is homeomorphic to an open subset of affine space.

Recall that $n$-dimensional affine space is

$$\mathbb{A}^n = \{(x^1, x^2, \dots, x^n) : x^i \in \mathbb{R}\}. \tag{1.3}$$

The vector space $\mathbb{R}^n$ acts transitively on $\mathbb{A}^n$ by translations. The *dimension* $\dim X\colon X \to \mathbb{Z}^{\ge 0}$ assigns to each point the dimension of the affine space in the definition. (It is independent of the choice of neighborhood and homeomorphism, though that is not trivial.) The function $\dim X$ is constant on components of $X$. If $\dim X$ has constant value $n$, we say $X$ is an $n$-dimensional manifold, or $n$-manifold for short.

**(1.4)** *Smooth structures.* For $U \subset X$ an open set, a homeomorphism $x\colon U \to \mathbb{A}^n$ is a *coordinate chart*. We write $x = (x^1, \dots, x^n)$, where each $x^i\colon U \to \mathbb{R}$ is a continuous function. To indicate the domain, we write the chart as the pair $(U, x)$. If $(U, x)$ and $(V, y)$ are charts, then there is a *transition map*

$$y \circ x^{-1}\colon x(U \cap V) \longrightarrow y(U \cap V), \tag{1.5}$$

which is a continuous map between open sets of $\mathbb{A}^n$. We say the charts are $C^\infty$*-compatible* if the transition function (1.5) is *smooth* (=$C^\infty$).

**Definition 1.6.** Let $X$ be a topological manifold. An *atlas* or *smooth structure* on $X$ is a collection of charts such that

(i) the union of the charts is $X$;
(ii) any two charts are $C^\infty$-compatible; and
(iii) the atlas is maximal with respect to (ii).

A topological manifold equipped with an atlas is called a *smooth manifold*.

We usually omit the atlas from the notation and simply notate the smooth manifold as '$X$'.

**(1.7)** *Empty set.* The empty set $\emptyset$ is trivially a manifold of any dimension $n \in \mathbb{Z}^{\ge 0}$. We use '$\emptyset^n$' to denote the empty manifold of dimension $n$.

**(1.8)** *Manifolds with boundary.* A simple modification of Definition 1.2 and Definition 1.6 allow for manifolds to have boundaries. Namely, we replace affine space with a closed half-space in affine space. So define

$$\mathbb{A}^n_- = \{(x^1, x^2, \dots, x^n) \in \mathbb{A}^n : x^1 \le 0\} \tag{1.9}$$

and ask that coordinate charts take values in open sets of $\mathbb{A}^n_-$. Then if $p \in X$ satisfies $x^1(p) = 0$ in some coordinate system $(x^1, \dots, x^n)$, that will be true in all coordinate systems. In this way $X$ is partitioned into two disjoint subsets, each of which is a manifold: the *interior* (consisting of points with $x^1 < 0$ in every coordinate system) and the *boundary* $\partial X$ (consisting of points with $x^1 = 0$ in every coordinate system).

*Remark* 1.10. I remember the convention on charts by the mnemonic 'ONF', which stands for 'Outward Normal First'. The fact that it also stands for 'One Never Forgets' helps me remember! An outward normal in a coordinate system is represented by the first coordinate vector field $\partial/\partial x^1$, and it points out of the manifold at the boundary.

**(1.11)** *Tangent bundle at the boundary.* At any point $p \in \partial X$ of the boundary there is a canonical subspace $T_p(\partial X) \subset T_pX$; the quotient space is a real line $\nu_p$. So over the boundary $\partial X$ there is a short exact sequence

$$0 \longrightarrow T(\partial X) \longrightarrow TX \longrightarrow \nu \longrightarrow 0 \tag{1.12}$$

of vector bundles. In any boundary coordinate system the vector $\partial/\partial x^1(p)$ projects to a nonzero element of $\nu_p$, but there is no canonical basis independent of the coordinate system. However, any two such vectors are in the same component of $\nu_p \setminus \{0\}$, which means that $\nu$ carries a canonical *orientation*. (We review orientations in Lecture 2.)

**Definition 1.13.** Let $X$ be a manifold with boundary. A *collar* of the boundary is an open set $U \subset X$ which contains $\partial X$ and a diffeomorphism $(-\epsilon, 0] \times \partial X \to U$ for some $\epsilon > 0$.

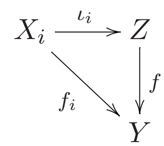
**Theorem 1.14.** *The boundary $\partial X$ of a manifold $X$ with boundary has a collar.*

This is not a trivial theorem. We only need it when $X$, hence also $\partial X$, is compact, in which case it is somewhat simpler.

**Exercise 1.15.** Prove Theorem 1.14 assuming $X$ is compact. (Hint: Cover the boundary with a finite number of coordinate charts; use a partition of unity to glue the vector fields $-\partial/\partial x^1$ in each coordinate chart into a smooth vector field; and use the fundamental existence theorem for ODEs, including smooth dependence on initial conditions.)

**(1.16)** *Disjoint union.* Let $\{X_1, X_2, \dots\}$ be a countable collection of manifolds. We can form a new manifold, the *disjoint union* of $X_1, X_2, \dots$, which we denote $X_1 \amalg X_2 \amalg \cdots$. As a set it is the disjoint union of the sets underlying the manifolds $X_1, X_2, \dots$. One may wonder how to define the disjoint union. For example, what is $X \amalg X$? This is ultimately a question of set theory, and we will meet such problems again. One solution is to fix an infinite dimensional affine space $\mathbb{A}^\infty$ and regard all manifolds as embedded in it. (This is no loss of generality by the Whitney Embedding Theorem.) Then we can replace $X_i$ (embedded in $\mathbb{A}^\infty$) by $\{i\} \times X_i$ (embedded in $\mathbb{A}^\infty = \mathbb{A}^1 \times \mathbb{A}^\infty$) and define the disjoint union to be the ordinary union of subsets of $\mathbb{A}^\infty$. Another way out is to characterize the disjoint union by a universal property: a disjoint union of $X_1, X_2, \dots$ is a manifold $Z$ and a collection of smooth maps $\iota_i \colon X_i \to Z$ such that for any manifold $Y$ and any collection $f_i \colon X_i \to Y$ of smooth maps, there exists a unique map $f \colon Z \to Y$ such that for each $i$ the diagram

(1.17)




commutes. (The last statement means $f \circ \iota_i = f_i$.) If you have not seen universal properties before, you might prove that $\iota_i$ is an embedding and that any two choices of $\bigl(Z, \{\iota_i\}\bigr)$ are canonically isomorphic. (You should also spell out what 'canonically isomorphic' means.) We will encounter such categorical notions more later in the course.

**(1.18)** *Terminology.* A manifold is *closed* if it is compact without boundary. By contrast, many use the term 'open manifold' to mean a noncompact manifold without boundary, but I am not particularly fond of that usage.

## Bordism

We now come to the fundamental definition. Fix an integer $n \geq 0$.

**Definition 1.19.** Let $Y_0, Y_1$ be closed $n$-manifolds. A *bordism* $\bigl(X\,,\,(\partial X)_0 \amalg (\partial X)_1\,,\,\theta_0\,,\,\theta_1\bigr)$ *from* $Y_0$ *to* $Y_1$ consists of a compact $(n{+}1)$-manifold $X$ with boundary, a decomposition $\partial X = (\partial X)_0 \amalg (\partial X)_1$ of its boundary, and embeddings

$$\theta_0 \colon [0, +1) \times Y_0 \longrightarrow X \tag{1.20}$$

$$\theta_1 \colon (-1, 0] \times Y_1 \longrightarrow X \tag{1.21}$$

such that $\theta_i(Y_i) = (\partial X)_i$, $i = 0, 1$.

Each of $(\partial X)_0, (\partial X)_1$ is a union of components of $\partial X$; note that there is a finite number of components since $X$, and so too $\partial X$, is compact. The map $\theta_i$ is a diffeomorphism onto its image, which is a collar neighborhood of $(\partial X)_i$. The collar neighborhoods are included in the definition to make it easy to glue bordisms. Without them we could as well omit the diffeomorphisms and give a simpler informal definition: a bordism $X$ from $Y_0$ to $Y_1$ is a compact $(n+1)$-manifold with boundary $Y_0 \amalg Y_1$. But we will keep the slightly more elaborate Definition 1.19. The words 'from' and 'to' in the definition distinguish the roles of $Y_0$ and $Y_1$, and indeed the intervals in (1.20) and (1.21) are different. But not that different—for the moment that distinction is only one of semantics and not any mathematics of import. For example, in the informal definition just given

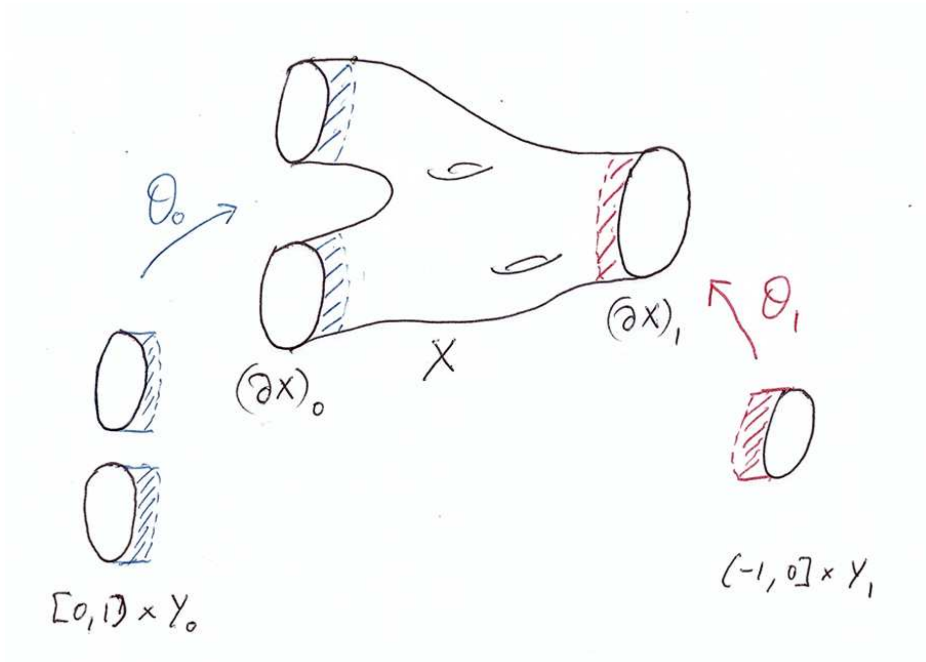



Figure 1. $X$ is a bordism from $Y_0$ to $Y_1$

the manifolds $Y_0, Y_1$ play symmetric roles. We picture a bordism in Figure 1. In the older literature a bordism is called a "cobordism". If the context is clear, we notate a bordism $\bigl(X\,,\,(\partial X)_0 \amalg (\partial X)_1\,,\,\theta_0\,,\,\theta_1\bigr)$ as '$X$'.

**Definition 1.22.** Let $\bigl(X\,,\,(\partial X)_0 \amalg (\partial X)_1\,,\,\theta_0\,,\,\theta_1\bigr)$ be a bordism from $Y_0$ to $Y_1$. The *dual bordism* from $Y_1$ to $Y_0$ is $\bigl(X^\vee\,,\,(\partial X^\vee)_0 \amalg (\partial X^\vee)_1\,,\,\theta_0^\vee\,,\,\theta_1^\vee\bigr)$, where: $X^\vee = X$; the decomposition of the boundary is swapped, so $(\partial X^\vee)_0 = (\partial X)_1$ and $(\partial X^\vee)_1 = (\partial X)_0$; and

$$\begin{aligned} \theta_0^\vee(t,y) &= \theta_1(-t,y), \qquad & t \in [0,+1), \quad & y \in Y_1, \\ \theta_1^\vee(t,y) &= \theta_0(-t,y), \qquad & t \in (-1,0], \quad & y \in Y_0. \end{aligned} \tag{1.23}$$

More informally, we picture the dual bordism $X^\vee$ as the original bordism $X$ "turned around".

*Remark* 1.24. We should view the dual bordism as a bordism from $Y_1^\vee$ to $Y_0^\vee$ where for naked manifolds we set $Y_i^\vee = Y_i$. When we come to manifolds with tangential structure, such as an orientation, we will not necessarily have $Y_i^\vee = Y_i$.

We use Definition 1.19 to extract our first algebraic gadget from compact manifolds: a *set*. Namely, define closed $n$-manifolds $Y_0, Y_1$ to be equivalent if there exists a bordism from $Y_0$ to $Y_1$.

**Lemma 1.25.** *Bordism defines an equivalence relation.*

*Proof.* For any closed manifold $Y$, the manifold $X = [0,1] \times Y$ determines a bordism from $Y$ to $Y$: set $(\partial X)_0 = \{0\} \times Y$, $(\partial X)_1 = \{1\} \times Y$, and use simple diffeomorphisms $[0,1) \to [0,1/3)$ and $(-1,0] \to (2/3,1]$ to construct (1.20) and (1.21). So bordism is a reflexive relation. Definition 1.22 shows that the relation is symmetric: if $X$ is a bordism from $Y_0$ to $Y_1$, then $X^\vee$ is a bordism from $Y_1$ to $Y_0$. For transitivity, suppose $\big(X\,,\,(\partial X)_0 \amalg (\partial X)_1\,,\,\theta_0\,,\,\theta_1\big)$ is a bordism from $Y_0$ to $Y_1$ and $\big(X'\,,\,(\partial X')_0 \amalg (\partial X')_1\,,\,\theta_0'\,,\,\theta_1'\big)$ a bordism from $Y_1$ to $Y_2$. Then Figure 2 illustrates how to glue $X$ and $X'$ together along $Y_1$ using $\theta_1$ and $\theta_0'$ to obtain a bordism from $Y_0$ to $Y_2$. □

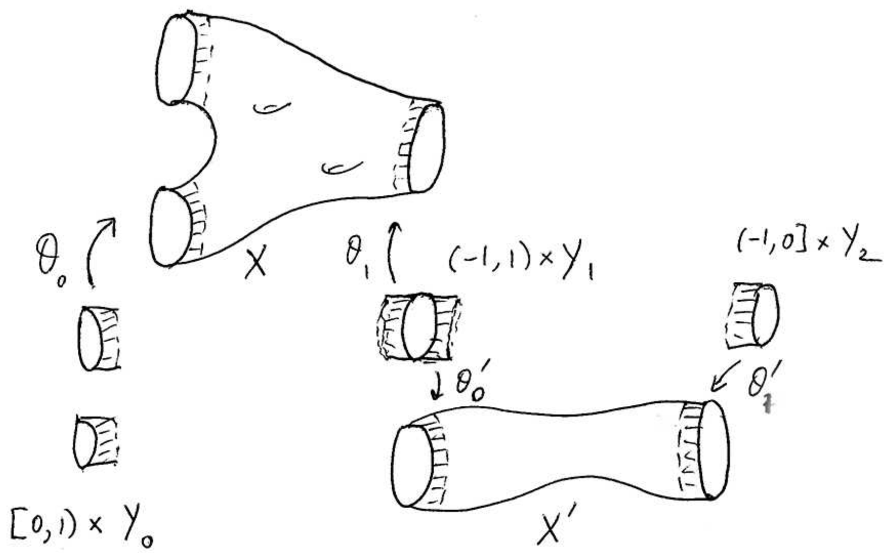



Figure 2. Gluing bordisms

**Exercise 1.26.** Write out the details of the gluing argument. Show carefully that the glued space is a manifold with boundary. Note that $(\partial X)_1 \approx (\partial X')_0$ is a submanifold of the glued manifold, and the maps $\theta_1$ and $\theta_0'$ combine to give a diffeomorphism $(-1,1) \times Y_1$ onto an open tubular neighborhood. This is sometimes called a *bi-collaring*.

**Exercise 1.27.** Show that diffeomorphic manifolds are bordant.

Let $\Omega_n$ denote the set of equivalence classes of closed $n$-manifolds under the equivalence relation of bordism. We use the term *bordism class* for an element of $\Omega_n$. Note that the empty manifold $\emptyset^0$ is a special element of $\Omega_n$, so we may consider $\Omega_n$ as a *pointed set*.

*Remark* 1.28. Again there is a set-theoretic worry: is the collection of closed $n$-manifolds a set? One way to make it so is to consider all manifolds as embedded in $\mathbb{A}^\infty$, as in (**1.16**). We will not make such considerations explicit.

## Disjoint union and the abelian group structure

Simple operations on manifolds—disjoint union and Cartesian product—give $\Omega_n$ more structure.

**Definition 1.29.**

(i) A *commutative monoid* is a set with a commutative, associative composition law and identity element.

(ii) An *abelian group* is a commutative monoid in which every element has an inverse.

Typical examples: $\mathbb{Z}^{\geq 0}$ is a commutative monoid; $\mathbb{Z}$ and $\mathbb{R}/\mathbb{Z}$ are abelian groups.

Disjoint union is an operation on manifolds which passes to bordism classes: if $Y_0$ is bordant to $Y_0'$ and $Y_1$ is bordant to $Y_1'$, then $Y_0 \amalg Y_1$ is bordant to $Y_0' \amalg Y_1'$. So $(\Omega_n, \amalg)$ is a commutative monoid.

**Lemma 1.30.** $(\Omega_n, \amalg)$ *is an abelian group. In fact,* $Y \amalg Y$ *is null-bordant.*

The identity element is represented by $\emptyset^n$. A null bordant manifold is one which is bordant to $\emptyset^n$.

*Proof.* The manifold $X = [0,1] \times Y$ provides a null bordism: let $(\partial X)_0 = \partial X = Y \amalg Y$ and $(\partial X)_1 = \emptyset^n$ and define $\theta_0, \theta_1$ appropriately. □

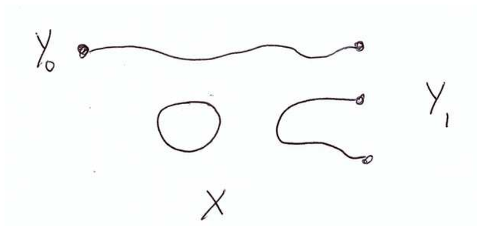



FIGURE 3. 1 point is bordant to 3 points

It is also true that the abelian group $(\Omega_n, \amalg)$ is finitely generated, though we do not prove that here. It follows that it is isomorphic to a product of cyclic groups of order 2. We denote this abelian group simply by '$\Omega_n$'.

**Proposition 1.31.** $\Omega_0 \cong \mathbb{Z}/2\mathbb{Z}$ *with generator* pt.

*Proof.* Any 0-manifold has no boundary, and a compact 0-manifold is a finite disjoint union of points. Lemma 1.30 implies that the disjoint union of two points is a boundary, so is zero in $\Omega_0$. It remains to prove that pt is not the boundary of a compact 1-manifold with boundary. That follows from the classification theorem for compact 1-manifolds with boundary: any such is a finite disjoint union of circles and closed intervals, so its boundary has an even number of points. □

The bordism group in dimensions 1,2 can also be computed from elementary theorems.

**Proposition 1.32.** $\Omega_1 = 0$ *and* $\Omega_2 \cong \mathbb{Z}/2\mathbb{Z}$ *with generator the real projective plane* $\mathbb{RP}^2$.

*Proof.* The first statement follows from the classification theorem in the previous proof: any closed 1-manifold is a finite disjoint union of circles, and a circle is the boundary of a 2-disk, so is null bordant. The second statement follows from the classification theorem for closed 2-manifolds. Recall that there are two connected families. The oriented surfaces are boundaries (of 3-dimensional handlebodies, for example). Any unoriented surface is a *connected sum*[2] of $\mathbb{RP}^2$'s, so it suffices to prove that $\mathbb{RP}^2$ does not bound and $\mathbb{RP}^2 \# \mathbb{RP}^2$ does bound. A nice argument emerged in lecture for the former. Namely, if $X$ is a compact 3-manifold with boundary $\partial X = \mathbb{RP}^2$, then the *double* $D = X \cup_{\mathbb{RP}^2} X$ has Euler characteristic $2\chi(X) - 1$, which is odd. But $D$ is a closed odd dimensional manifold, so has vanishing Euler characteristic. This contradiction shows $X$ does not exist. We

[2]The connected sum is denoted '#'. We do not pause here to define it carefully. The definition depends on choices, but the diffeomorphism class, hence bordism class, does not depend on the choices.

give a different argument in the next lecture. For the latter, recall that $\mathbb{RP}^2\#\mathbb{RP}^2$ is diffeomorphic to a Klein bottle $K$, which has a map $K \to S^1$ which is a *fiber bundle* with fiber $S^1$. There is an associated fiber bundle with fiber the disk $D^2$ which is a compact 3-manifold with boundary $K$. □

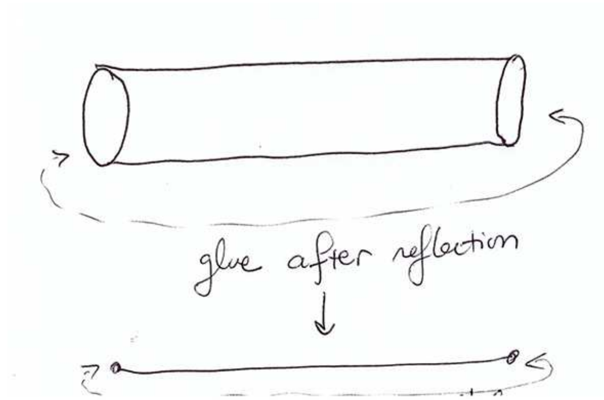



FIGURE 4. Constructing the Klein bottle by gluing

There is a separate set of notes on fiber bundles. For now recall that we can construct $K$ by gluing together the ends of a cylinder $[0,1] \times S^1$ using a reflection on $S^1$. Then projection onto the first factor, after gluing, is the map $K \to S^1$. The disk bundle is formed analogously starting with $[0,1] \times D^2$. This is depicted in Figure 2.

## Cartesian product and the ring structure

Now we bring in another operation, Cartesian product, which takes an $n_1$-manifold and an $n_2$-manifold and produces an $(n_1+n_2)$-manifold.

**Definition 1.33.**

(i) A *commutative ring* $R$ is an abelian group $(+,0)$ with a second commutative, associative composition law $(\cdot)$ with identity $(1)$ which distributes over the first: $r_1 \cdot (r_2 + r_3) = r_1 \cdot r_2 + r_1 \cdot r_3$ for all $r_1, r_2, r_3 \in R$.

(ii) A $\mathbb{Z}$*-graded commutative ring* is a commutative ring $S$ which as an abelian group is a direct sum

$$S = \bigoplus_{n \in \mathbb{Z}} S_n \tag{1.34}$$

of abelian subgroups such that $S_{n_1} \cdot S_{n_2} \subset S_{n_1+n_2}$.

Elements in $S_n \subset S$ are called *homogeneous of degree* $n$; the general element of $S$ is a finite sum of homogeneous elements.

The integers $\mathbb{Z}$ form a commutative ring, and for any commutative ring $R$ there is a polynomial ring $S = R[x]$ in a single variable which is $\mathbb{Z}$-graded. To define the $\mathbb{Z}$-grading we must assign an integer degree to the indeterminate $x$. Typically we posit $\deg x = 1$, in which case $S_n$ is the abelian group of homogeneous polynomials of degree $n$ in $x$. More generally, there is a $\mathbb{Z}$-graded polynomial ring $R[x_1, \dots, x_k]$ in any number of indeterminates with any assigned integer degrees $\deg x_k \in \mathbb{Z}$.

Define

$$\Omega = \bigoplus_{n \in \mathbb{Z}^{\geq 0}} \Omega_n. \tag{1.35}$$

We formally define $\Omega_{-m} = 0$ for $m > 0$. The Cartesian product of manifolds is compatible with bordism, so passes to a commutative, associative binary composition law on $\Omega$.

**Proposition 1.36.** *$(\Omega, \amalg, \times)$ is a $\mathbb{Z}$-graded ring. A homogeneous element of degree $n \in \mathbb{Z}$ is represented by a closed manifold of dimension $n$.*

We leave the proof to the reader. The ring $\Omega$ is called the *unoriented bordism ring.*

In his Ph.D. thesis Thom [T] proved the following theorem (among many other foundational results).

**Theorem 1.37** ([T])**.** *There is an isomorphism of $\mathbb{Z}$-graded rings*

$$\Omega \cong \mathbb{Z}/2\mathbb{Z}[x_2, x_4, x_5, x_6, x_8, \dots] \tag{1.38}$$

*where there is a polynomial generator of degree $k$ for each positive integer $k$ not of the form $2^i - 1$.*

Furthermore, Thom proved that if $k$ is even, then $x_k$ is represented by the real projective manifold $\mathbb{RP}^k$. Dold later constructed manifolds representing the odd degree generators: they are fiber bundles[3] over $\mathbb{RP}^m$ with fiber $\mathbb{CP}^\ell$.

**Exercise 1.39.** Work out $\Omega_{10}$. Find manifolds which represent each bordism class.

Thom proved that the *Stiefel-Whitney numbers* determine the bordism class of a closed manifold. The *Stiefel-Whitney classes* $w_i(Y) \in H^i(Y; \mathbb{Z}/2\mathbb{Z})$ are examples of *characteristic classes* of the tangent bundle; we will discuss them later, and we will also give a quick review of cohomology later as well. Any closed $n$-manifold $Y$ has a *fundamental class* $[Y] \in H_n(Y; \mathbb{Z}/2\mathbb{Z})$. If $x \in H^\bullet(Y; \mathbb{Z}/2\mathbb{Z})$, then the pairing $\langle x, [Y] \rangle$ produces a number in $\mathbb{Z}/2\mathbb{Z}$.

**Theorem 1.40** ([T])**.** *The Stiefel-Whitney numbers*

$$\langle w_{i_1}(Y) \smile w_{i_2}(Y) \smile \cdots \smile w_{i_k}(Y)\,,\, [Y] \rangle \in \mathbb{Z}/2\mathbb{Z}, \tag{1.41}$$

*determine the bordism class of a closed $n$-manifold $Y$.*

That is, if closed $n$-manifolds $Y_0, Y_1$ have the same Stiefel-Whitney numbers, then they are bordant. Notice that not all naively possible nonzero Stiefel-Whitney numbers can be nonzero. For example, $\langle w_1(Y), [Y] \rangle$ vanishes for any closed 1-manifold $Y$. Also, the theorem implies that a closed $n$-manifold is the boundary of a compact $(n+1)$-manifold iff all of the Stiefel-Whitney numbers of $Y$ vanish. If it is a boundary, it is immediate that the Stiefel-Whitney numbers vanish; the converse is hardly obvious.

[3] They are the quotient of $S^m \times \mathbb{CP}^\ell$ by the free involution which acts as the antipodal map on the sphere and complex conjugation on the complex projective space.

*Remark* 1.42. The modern developments in bordism use disjoint union heavily, so generalize the study of classical abelian bordism groups. However, they do not use Cartesian product in the same way.

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