Lecture 10: Thom spectra and $\mathcal{X}$-bordism

We begin with the definition of a spectrum and its antecedents: prespectra and Ω-prespectra. Spectra are the basic objects of *stable homotopy theory*. We construct a prespectrum—then a spectrum—for each unstable or stable tangential structure. They are built using the Thom complex of vector bundles, so they are known as *Thom spectra*. For stable tangential structures there is a version of the Pontrjagin-Thom construction and then the main theorem identifies $\mathcal{X}$-bordism groups with the homotopy groups of an appropriate Thom spectrum. We then focus on oriented bordism and summarize the computation of its rational homotopy groups.

Prespectra and spectra

This definition is basic to stable homotopy theory. A good reference is [Ma]. All spaces in this section are pointed.

Let X, Y be pointed spaces. Recall from Exercise 4.29 that there is an isomorphism of spaces

$$\operatorname{Map}_*(\Sigma X, Y) \xrightarrow{\cong} \operatorname{Map}_*(X, \Omega Y) \tag{10.1}$$

if we use the correct topologies. In the following definition we only need (10.1) as an isomorphism of sets.

Definition 10.2.

(i) A *prespectrum* $T_\bullet$ is a sequence $\{T_q\}_{q\in\mathbb{Z}^{>0}}$ of pointed spaces and maps $s_q\colon \Sigma T_q \to T_{q+1}$.

(ii) An Ω*-prespectrum* is a prespectrum $T_\bullet$ such that the adjoints $t_q\colon T_q \to \Omega T_{q+1}$ of the structure maps are weak homotopy equivalences.

(iii) A *spectrum* is a prespectrum $T_\bullet$ such that the adjoints $t_q\colon T_q \to \Omega T_{q+1}$ of the structure maps are homeomorphisms.

Obviously a spectrum is an Ω-prespectrum is a prespectrum. We can take the sequence of pointed spaces $T_{q_0}, T_{q_0+1}, T_{q_0+2}, \dots$ to begin at any integer $q_0 \in \mathbb{Z}$. If $T_\bullet$ is a *spectrum* which begins at q_0, then we can extend to a sequence of pointed spaces T_q defined for *all* integers q by setting

$$T_q = \Omega^{q_0-q}T_{q_0}, \qquad q < q_0. \tag{10.3}$$

Note that each T_q, in particular T_0, is an *infinite* loop space:

$$T_0 \simeq \Omega T_1 \simeq \Omega^2 T_2 \simeq \cdots \tag{10.4}$$

There are shift maps on prespectra, Ω-prespectra, and spectra: simply shift the indexing.

Example 10.5. Let X be a pointed space. The *suspension prespectrum* of X is defined by setting $T_q = \Sigma^q X$ for $q \geq 0$ and letting the structure maps s_q be the identity maps. In particular, for $X = S^0$ we obtain the sphere prespectrum with $T_q = S^q$.

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(10.6) *Spectra from prespectra.* Associated to each prespectrum $T_\bullet$ is a spectrum[1] $LT_\bullet$ called its *spectrification.* It is easiest to construct in case the adjoint structure maps $t_q\colon T_q \to \Omega T_{q+1}$ are inclusions. Then set $(LT)_q$ to be the colimit of

$$T_q \xrightarrow{t_q} \Omega T_{q+1} \xrightarrow{\Omega t_{q+1}} \Omega^2 T_{q+2} \longrightarrow \cdots \tag{10.7}$$

which is computed as an union; see **(4.32)**. For the suspension spectrum of a pointed space X the 0-space is

$$(LT)_0 = \operatorname*{colim}_{\ell\to\infty} \Omega^\ell \Sigma^\ell X, \tag{10.8}$$

which is usually denoted QX; see (4.39) for QS^0.

Exercise 10.9. Prove that the homotopy groups of QX are the *stable* homotopy groups of X. (Recall Proposition 4.40.)

(10.10) *Homotopy and homology of prespectra.* Let $T_\bullet$ be a prespectrum. Define its homotopy groups by

$$\pi_n(T) = \operatorname*{colim}_{\ell\to\infty} \pi_{n+\ell} T_\ell, \tag{10.11}$$

where the colimit is over the sequence of maps

$$\pi_{n+\ell}T_\ell \xrightarrow{\pi_{n+\ell}t_\ell} \pi_{n+\ell}\Omega T_{\ell+1} \xrightarrow{\text{adjunction}} \pi_{n+\ell+1}T_{\ell+1} \tag{10.12}$$

Similarly, define the homology groups as the colimit

$$H_n(T) = \operatorname*{colim}_{\ell\to\infty} \widetilde{H}_{n+\ell} T_\ell, \tag{10.13}$$

where $\widetilde{H}$ denotes the reduced homology of a pointed space. We might be tempted to define the cohomology similarly, but that does not work.[2]

Exercise 10.14. Compute the homology groups of the sphere spectrum. More generally, compute the homology groups of the suspension spectrum of a pointed space X in terms of the reduced homology groups of X.

Exercise 10.15. Define maps of prespectra. Construct (in case the adjoint structure maps are inclusions) a map $T \to LT$ of prespectra and prove that it induces an isomorphism on homotopy and homology groups.

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[1] The notation 'L' indicates 'left adjoint'.

[2] Homotopy and homology commute with colimits, but cohomology does not: there is a derived functor $\lim^1$ which measures the deviation.

Thom spectra

(10.16) *Pullback of the universal bundle.* There is an inclusion

$$i\colon BO(q) \longrightarrow BO(q+1) \tag{10.17}$$

defined as the colimit of the inclusions of Grassmannians which are the vertical arrows

$$W \longmapsto \mathbb{R} \oplus W \tag{10.18}$$

in the diagram

$$\begin{array}{ccccccc} Gr_q(\mathbb{R}^m) & \longrightarrow & Gr_q(\mathbb{R}^{m+1}) & \longrightarrow & Gr_q(\mathbb{R}^{m+2}) & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ Gr_{q+1}(\mathbb{R}^{m+1}) & \longrightarrow & Gr_{q+1}(\mathbb{R}^{m+2}) & \longrightarrow & Gr_{q+1}(\mathbb{R}^{m+3}) & \longrightarrow & \cdots \end{array} \tag{10.19}$$

Recalling the definition of the tautological vector bundle $S(q) \to BO(q)$, as in (6.26), we see that there is a natural isomorphism

$$i^*S(q+1) \xrightarrow{\cong} \underline{\mathbb{R}} \oplus S(q) \tag{10.20}$$

over $BO(q)$.

Let $\mathcal{Y}$ be a stable tangential structure (Definition 9.45). Then we also have maps $i\colon \mathcal{Y}(q) \to \mathcal{Y}(q+1)$ and isomorphisms (10.20) of the pullbacks over $\mathcal{Y}(q)$.

(10.21) *Thom complexes and suspension.* Let $V \to Y$ be a real vector bundle, and fix a metric. Recall the Thom complex is the quotient $V/C_r(V)$, where $C_r(V)$ is the complement of the open disk bundle of radius $r > 0$. (The choice of radius is immaterial.)

Proposition 10.22. *The Thom complex of $\underline{\mathbb{R}} \oplus V \to Y$ is homeomorphic to the suspension of the Thom complex of $V \to Y$.*

Note that the Thom complex of the 0-vector bundle—the identity map $Y \to Y$—is the disjoint union of Y and a single point, which is then the basepoint of the disjoint union. That disjoint union is denoted Y_+. Then Proposition 10.22 implies that the Thom complex of $\underline{\mathbb{R}} \to Y$ is ΣY_+, the suspension of Y_+. Iterating, and using the notation Y^V for the Thom complex of $V \to Y$, we have $Y^{\underline{\mathbb{R}}^\ell} \simeq \Sigma^\ell Y_+$. So the Thom complex is a "twisted suspension" of the base space.

Proof. Up to homeomorphism we can replace the disk bundle of $\underline{\mathbb{R}} \oplus V \to Y$ by the Cartesian product of the unit disk in $\mathbb{R}$ and the disk bundle of $V \to Y$. Crushing the complement in $\mathbb{R} \times V$ to a point is the same crushing which one does to form the suspension of Y^V, as in Figure 20. $\square$

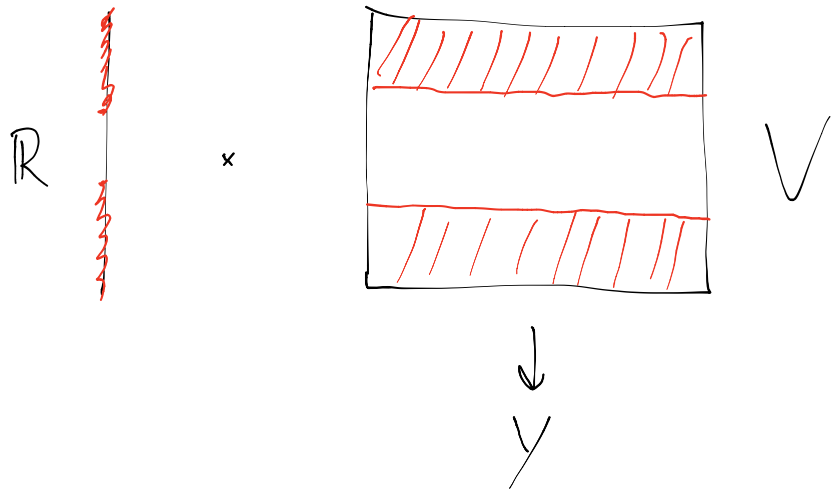

FIGURE 20. The Thom complex of $\underline{\mathbb{R}} \oplus V \to Y$

(10.23) *The Thom prespectrum.* Let $\mathcal{Y}$ be a stable tangential structure. Consider the diagram

$$\begin{array}{ccc} \underline{\mathbb{R}} \oplus S(q) & \longrightarrow & S(q+1) \\ \downarrow & & \downarrow \\ \mathcal{Y}(q) & \xrightarrow{\;i\;} & \mathcal{Y}(q+1) \end{array} \tag{10.24}$$

where we use (10.20). There is an induced map on Thom complexes, and by Proposition 10.22 this is a map

$$s_q\colon \Sigma\big(\mathcal{Y}(q)^{S(q)}\big) \longrightarrow \mathcal{Y}(q+1)^{S(q+1)}. \tag{10.25}$$

Definition 10.26.

(i) The *Thom prespectrum* $T\mathcal{Y}_\bullet$ of a stable tangential structure $\mathcal{Y}$ is defined by

$$T\mathcal{Y}_q = \mathcal{Y}(q)^{S(q)} \tag{10.27}$$

and the structure maps (10.25).

(ii) The *Thom spectrum* $M\mathcal{Y}_\bullet$ is $L(T\mathcal{Y}_\bullet)$.

Note that the maps (10.25) are inclusions, so $L(T\mathcal{Y}_\bullet)$ is defined in **(10.6)**.

(10.28) *Stable tangential structures from reduction of structure group.* Let $\{G(n)\}_{n\in\mathbb{Z}^{>0}}$ be a sequence of Lie groups and $G(n) \longrightarrow G(n+1)$, $\rho(n)\colon G(n) \to O(n)$ sequences of homomorphisms such that the diagram

$$\begin{array}{ccccccc} \cdots \longrightarrow & G(n) & \longrightarrow & G(n+1) & \longrightarrow \cdots \\ & \downarrow{\scriptstyle \rho(n)} & & \downarrow{\scriptstyle \rho(n+1)} & \\ \cdots \longrightarrow & O(n) & \longrightarrow & O(n+1) & \longrightarrow \cdots \end{array} \tag{10.29}$$

commutes. There is an stable tangential structure $BG \to BO$ which is the colimit of the induced sequence of maps of classifying spaces

(10.30)
$$\begin{array}{ccccccc} \cdots & \longrightarrow & BG(n) & \longrightarrow & BG(n+1) & \longrightarrow & \cdots \\ & & \Big\downarrow{\scriptstyle B\rho(n)} & & \Big\downarrow{\scriptstyle B\rho(n+1)} & & \\ \cdots & \longrightarrow & BO(n) & \longrightarrow & BO(n+1) & \longrightarrow & \cdots \end{array}$$

The corresponding bordism groups are denoted $\Omega^G_\bullet$, consistent with the notation in (**2.23**) for $G(n) = SO(n)$ and the obvious inclusion maps.

Exercise 10.31. Show that the tangential structures in Example 9.49, Example 9.50, Example 9.52, and Example 9.53 are all of the form BG for a suitable $G = \operatorname{colim}_{n\to\infty} G(n)$.

Exercise 10.32. Show that the Thom spectrum of the stable framing tangential structure (Example 9.50) is the sphere spectrum.

The general Pontrjagin-Thom theorem

This general form of the Pontrjagin-Thom theorem was introduced by Lashof [La]; see [St, §2] for an exposition.

Theorem 10.33. *Let $\mathcal{X}$ be a stable tangential structure. Then for each $n \in \mathbb{Z}^{\ge 0}$ there is an isomorphism*

(10.34)
$$\phi\colon \pi_n(M\mathcal{X}^\perp) \longrightarrow \Omega^{\mathcal{X}}_n.$$

The perp stable tangential structure $\mathcal{X}^\perp$ is defined in (**9.62**) and its Thom spectrum in Definition 10.26. Our notation for the bordism group indicates the stable *tangential* structure, which is not standard in the literature.

Remark 10.35. I do not know an example in which $\mathcal{X}^\perp \neq \mathcal{X}$. I would like to know one.

Lemma 10.36. *Let $\mathcal{X} = BSO$ be the stable tangential structure of orientations. Then $\mathcal{X}^\perp = \mathcal{X}$.*

Proof. BSO is a colimit of Grassmannians $Gr^{SO}_n(\mathbb{R}^m)$ of oriented subspaces of $\mathbb{R}^m$. Let the vector space $\mathbb{R}^m$ have its standard orientation. Then the orthogonal complement of an oriented subspace inherits a natural orientation,[3] and this gives a lift

(10.37)
$$\begin{array}{ccc} Gr^{SO}_n(\mathbb{R}^m) & \longrightarrow & Gr^{SO}_{m-n}(\mathbb{R}^m) \\ \Big\downarrow & & \Big\downarrow \\ Gr_n(\mathbb{R}^m) & \longrightarrow & Gr_{m-n}(\mathbb{R}^m) \end{array}$$

of (9.63) in which the vertical maps are double covers which forget the orientation. The double colimit of (10.37) gives an equivalence $\mathcal{X}^\perp \approx \mathcal{X}$. □

[3] Check the signs carefully to construct an involution in the following.

Corollary 10.38. *There is an isomorphism*

$$\phi\colon \pi_n(MSO) \longrightarrow \Omega_n^{SO}. \tag{10.39}$$

In the next lecture we compute the rational vector space obtained by tensoring the left hand side of (10.39) with $\mathbb{Q}$; then $\phi \otimes \mathbb{Q}$ gives an isomorphism to $\Omega_n^{SO} \otimes \mathbb{Q}$.

Exercise 10.40. Generalize Lemma 10.36 to the tangential structures described in (**10.28**).

Exercise 10.41. Check that Theorem 10.33 reduces to Corollary 5.22.

Remarks about the proof of Theorem 10.33. The tools from differential topology which go into the proof were all employed in the first lectures for the special case of stably framed manifolds; see especially the proof of Theorem 3.9. So we content ourselves of reminding the reader of the map ϕ and its inverse map ψ.

The map ϕ: A class in $\pi_n(M\mathcal{X}^\perp)$ is represented by

$$f\colon S^{n+q} \longrightarrow T\mathcal{X}_q^\perp = \mathcal{X}^\perp(q)^{S(q)} \tag{10.42}$$

for some $q \in \mathbb{Z}^{>0}$. We choose f so that it is smooth and transverse to the zero section $Z(q) \subset \mathcal{X}^\perp(q)^{S(q)}$. Define $M := f^{-1}\bigl(Z(q)\bigr) \subset S^{n+q}$. The normal bundle $\nu \to M$ to $M \subset S^{n+q}$ is a rank q bundle isomorphic to the pullback of the normal bundle to $Z(q) \subset \mathcal{X}^\perp(q)^{S(q)}$, which is $S(q) \to Z(q)$, so inherits the $\mathcal{X}^\perp$-structure

$$M \xrightarrow{\;f\;} Z(q) \cong \mathcal{X}^\perp(q) \longrightarrow \mathcal{X}^\perp \tag{10.43}$$

on its normal bundle, so on its stable normal bundle. By (**9.66**) this is equivalent to an $\mathcal{X}$-structure on the stable tangent bundle to M.

The inverse map ψ: We refer to Figure 21. Suppose M is a closed n-manifold with a stable tangential $\mathcal{X}$-structure, or equivalently a stable normal $\mathcal{X}^\perp$-structure. Choose an embedding $M \hookrightarrow S^{n+q}$ for some $q \in \mathbb{Z}^{>0}$ and a tubular neighborhood $U \subset S^{n+q}$. The normal structure induces—possibly after suspending to increase q—a classifying map

$$\begin{array}{ccc} \nu \approx U & \longrightarrow & S(q) \\ \downarrow & & \downarrow \\ M & \longrightarrow & \mathcal{X}^\perp(q) \end{array} \tag{10.44}$$

The Pontrjagin-Thom collapse, which maps the complement of U to the basepoint, induces a map

$$S^{n+q} \to \mathcal{X}^\perp(q)^{S(q)} \tag{10.45}$$

to the Thom complex, and this represents a class in $\pi_n(\mathcal{X}^\perp)$. □

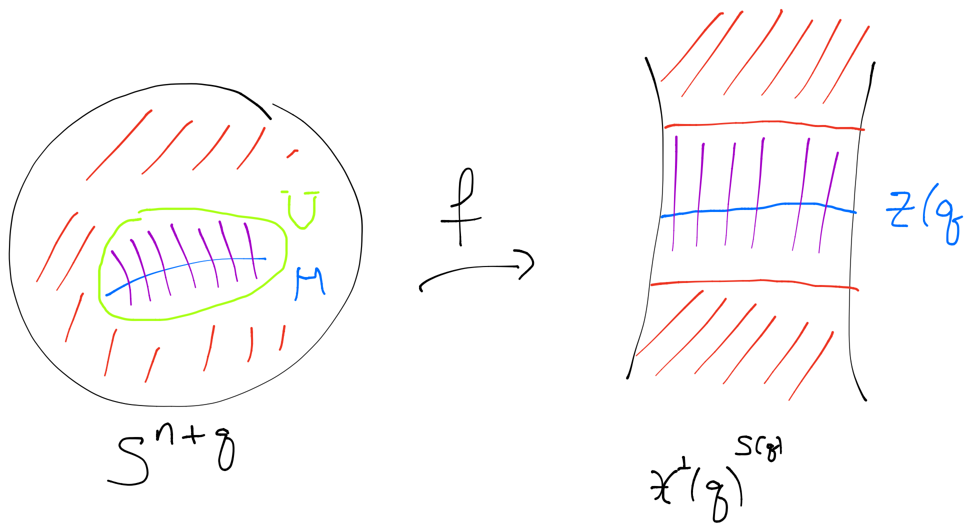

FIGURE 21. The Pontrjagin-Thom collapse

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