Lecture 11: Hirzebruch's signature theorem

In this lecture we define the signature of a closed oriented n-manifold for n divisible by four. It is a bordism invariant $\mathrm{Sign}\colon \Omega_n^{SO} \to \mathbb{Z}$. (Recall that we defined a $\mathbb{Z}/2\mathbb{Z}$-valued bordism invariant of non-oriented manifolds in Lecture 2.) The signature is a complete bordism invariant of closed oriented 4-manifolds (see **(2.28)**), as we prove here. It can be determined by tensoring with $\mathbb{Q}$, or even tensoring with $\mathbb{R}$. We use the general Pontrjagin-Thom Theorem 10.33 to convert the computation of this invariant to a homotopy theory problem. We state the theorem that all such bordism invariants can be determined on products of complex projective spaces. In this lecture we illustrate the techniques necessary to compute that $\Omega_4^{SO} \otimes \mathbb{Q}$ is a one-dimensional rational vector space. The general proof will be sketched in the next lecture. Here we also prove Hirzebruch's formula assuming the general result.

We sometimes tensor with $\mathbb{R}$ instead of tensoring with $\mathbb{Q}$. Tensoring with $\mathbb{R}$ has the advantage that real cohomology is represented by differential forms. Also, the computation of the real cohomology of BSO can be related to invariant polynomials on the orthogonal Lie algebra $\mathfrak{so}$.

Definition of signature

(11.1) *The fundamental class of an oriented manifold.* Let M be a closed oriented n-manifold for some $n \in \mathbb{Z}^{\geq 0}$. The orientation[1] defines a *fundamental class*

$$[M] \in H_n(M). \tag{11.2}$$

Here coefficients in $\mathbb{Z}$ are understood. The fundamental class depends on the orientation: the fundamental class of the oppositely oriented manifold satisfies

$$[-M] = -[M]. \tag{11.3}$$

The fundamental class is part of a discussion of duality in homology and cohomology; see [H1, §3.3]. The fundamental class determines a homomorphism

$$\begin{aligned} H^n(M;A) &\longrightarrow A \\ c &\longmapsto \langle c, [M]\rangle \end{aligned} \tag{11.4}$$

for any coefficient group A. When $A = \mathbb{R}$ we use the de Rham theorem to represent an element $c \in H^n(M;\mathbb{R})$ by a closed differential n-form ω. Then

$$\langle c, [M]\rangle = \int_M \omega. \tag{11.5}$$

(Recall that integration of differential forms depends on an orientation, and is consistent with (11.3).) For that reason the map (11.4) can be thought of as an integration operation no matter the coefficients.

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[1] We remark that any closed manifold (without orientation, or possibly nonorientable) has a fundamental class in mod 2 homology.

(11.6) *The intersection pairing.* Let M be a closed oriented n-manifold and suppose $n = 4k$ for some $k \in \mathbb{Z}^{\ge 0}$. To define the intersection pairing we use the cup product on cohomology. Consider, then, the integer-valued bihomomorphism

$$\begin{aligned} I_M\colon H^{2k}(M;\mathbb{Z}) \times H^{2k}(M;\mathbb{Z}) &\longrightarrow \mathbb{Z} \\ c_1, c_2 &\longmapsto \langle c_1 \smile c_2, [M]\rangle \end{aligned} \tag{11.7}$$

This *intersection form* is symmetric, by basic properties of the cup product The abelian group $H^{2k}(M;\mathbb{Z})$ is finitely generated, so has a finite torsion subgroup and a finite rank free quotient; the rank of the free quotient is the second *Betti number* $b_2(M)$.

Exercise 11.8. Prove that the torsion subgroup is in the kernel of the intersection form (11.7). This means that if c_1 is torsion, then $I(c_1, c_2) = 0$ for all c_2.

It follows that the intersection form drops to a pairing

$$\begin{aligned} \overline{I}_M\colon \operatorname{Free} H^{2k}(M;\mathbb{Z}) \times \operatorname{Free} H^{2k}(M;\mathbb{Z}) &\longrightarrow \mathbb{Z} \\ \bar c_1, \bar c_2 &\longmapsto \langle \bar c_1 \smile \bar c_2, [M]\rangle \end{aligned} \tag{11.9}$$

on the free quotient. *Poincaré duality* is the assertion that $\overline{I_M}$ is *nondegenerate*: if $\overline{I}_M(\bar c_1, \bar c_2) = 0$ for all $\bar c_2$, then $\bar c_1 = 0$. See [H1, §3.3] for a discussion.

(11.10) *Homology interpretation.* Another consequence of Poincaré duality is that there is a dual pairing on $\operatorname{Free} H_{2k}(M)$, and it is more geometric. In fact, the name 'intersection pairing' derives from the homology version. To compute it we represent two homology classes in the middle dimension by closed oriented submanifolds $C_1, C_2 \subset M$, wiggle them to be transverse, and define the intersection pairing as the oriented intersection number $I_M(C_1, C_2) \in \mathbb{Z}$.

(11.11) *de Rham interpretation.* Let A be a finitely generated abelian group of rank r. Then $A \to A \otimes \mathbb{R}$ has kernel the torsion subgroup of A. The codomain is a real vector space of dimension r, and the image is a full sublattice isomorphic to the free quotient $\operatorname{Free} A$. We apply this to the middle cohomology group. A part of the de Rham theorem asserts that wedge product of closed forms goes over to cup product of real cohomology classes, and so we can represent the intersection pairing $I_M \otimes \mathbb{R}$ in de Rham theory by the pairing

$$\begin{aligned} \widehat{I}_M\colon \Omega^{2k}(M) \times \Omega^{2k}(M) &\longrightarrow \mathbb{R} \\ \omega_1, \omega_2 &\longmapsto \int_M \omega_1 \wedge \omega_2 \end{aligned} \tag{11.12}$$

The pairing is symmetric and makes sense for all differential forms.

Exercise 11.13. Use Stokes' theorem to prove that (11.12) vanishes if one of the forms is closed and the other exact. Conclude that it induces a pairing on de Rham cohomology, hence by the de Rham theorem on real cohomology.

The induced pairing on real cohomology is $I_M \otimes \mathbb{R}$.

Definition 11.14. The *signature* $\mathrm{Sign}(M)$ is the signature of the symmetric bilinear form $I_M \otimes \mathbb{R}$.

Recall that a symmetric bilinear form B on a real vector space V has three numerical invariants which add up to the dimension of V: the *nullity* and two numbers b_+, b_-. There is a basis $e_1, \dots, e_n$ of V so that

$$\begin{aligned} B(e_i, e_j) &= 0, && i \neq j; \\ B(e_i, e_i) &= 1, && i = 1, \dots, b_+; \\ B(e_i, e_i) &= -1, && i = b_+ + 1, \dots, b_+ + b_-; \\ B(e_i, e_i) &= 0, && i = b_+ + b_- + 1, \dots, n. \end{aligned} \tag{11.15}$$

There is a subspace $\ker B \subset V$, the null space of B, whose dimension is the nullity. b_+ is the dimension of the maximal subspace on which B is positive definite; b_- is the dimension of the maximal subspace on which B is negative definite. See [HK], for example. The signature is defined to be the difference $\mathrm{Sign}(B) = b_+ - b_-$. Note B is nondegenerate iff $\ker B = 0$ iff the nullity vanishes.

Examples

The following depends on a knowledge of the cohomology ring in several cases, but you can also use the oriented intersection pairing. We begin with several 4-manifolds.

Example 11.16 (S^4)**.** Since $H^2(S^4; \mathbb{Z}) = 0$, we have $\mathrm{Sign}(S^4) = 0$.

Example 11.17 ($S^2 \times S^2$)**.** The second cohomology $H^2(S^2 \times S^2; \mathbb{Z})$ has rank two. In the standard basis the intersection form is represented by the matrix

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{11.18}$$

The 'H' stands for 'hyperbolic'. One way to see this is to compute in homology. The submanifolds $S^2 \times \mathrm{pt}$ and $\mathrm{pt} \times S^2$ represent generators of $H_2(S^2 \times S^2)$, each has self-intersection number zero, and the intersection number of one with the other is one. Diagonalize H to check that its signature is zero.

Example 11.19 ($K3$ surface)**.** The $K3$-surface was introduced in (**5.60**). You computed its total Chern class, so its Pontrjagin class, in Exercise 7.67. One can compute (I'm not giving techniques here for doing so) that the intersection form is

$$-E_8 \oplus -E_8 \oplus H \oplus H \oplus H, \tag{11.20}$$

where E_8 is an 8×8 symmetric positive definite matrix of integers derived from the Lie group E_8. Its signature is -16.

The $K3$ surface is spin(able), which follows from the fact that its first Chern class vanishes. (A related statement appears as Proposition 9.28.) The following important theorem of Rohlin applies.

Theorem 11.21 (Rohlin)**.** *Let M^n be a closed oriented manifold with $n \equiv 4 \pmod 8$. Then $\operatorname{Sign} M$ is divisible by 16.*

Example 11.22 ($\mathbb{CP}^2$)**.** The group $H^2(\mathbb{CP}^2;\mathbb{Z})$ is infinite cyclic and a positive generator is Poincareé dual to a projective line $\mathbb{CP}^1 \subset \mathbb{CP}^2$. The self-intersection number of that line is one, whence $\operatorname{Sign}\mathbb{CP}^2 = 1$.

Example 11.23 ($\overline{\mathbb{CP}^2}$)**.** This is the usual notation for the orientation-reversed manifold $-\mathbb{CP}^2$. By (11.3) we find $\operatorname{Sign}\overline{\mathbb{CP}^2} = -1$.

Obviously, neither $\mathbb{CP}^2$ nor $\overline{\mathbb{CP}^2}$ is spinable, as proved in Corollary 9.29 and now also follows from Theorem 11.21.

I leave several important facts to you.

Exercise 11.24. Prove that $\operatorname{Sign}\mathbb{CP}^{2\ell} = 1$ for all $\ell \in \mathbb{Z}^{>0}$.

Exercise 11.25. Show that the signature is additive under disjoint union and also connected sum. Prove that if M_1, M_2 have dimensions divisible by 4, then $\operatorname{Sign}(M_1 \times M_2) = \operatorname{Sign}(M_1)\operatorname{Sign}(M_2)$. In fact, the statement is true without restriction on dimension as long as we define $\operatorname{Sign} M = 0$ if $\dim M$ is not divisible by four.

Signature and bordism

We prove that the signature is a bordism invariant: if $M^{4k} = \partial N^{4k+1}$ and N is compact and oriented, then $\operatorname{Sign} M = 0$. We first prove two lemmas. The first should remind you of Stokes' theorem.

Lemma 11.26. *Let N^{4k+1} be a compact oriented manifold with boundary $i\colon M^{4k} \hookrightarrow N$. Suppose $c \in H^{4k}(N;A)$ for some abelian group A. Then*

$$\langle i^*(c), [M]\rangle = 0. \tag{11.27}$$

Proof. We have

$$\langle i^*(c), [M]\rangle = \langle c, i_*[M]\rangle = 0 \tag{11.28}$$

since $i_*[M] = 0$. (This is a property of duality; intuitively, the manifold M *is* a boundary, so too is its fundamental class.) □

This can also be proved using differential forms, via the de Rham theorem, if $A \subset \mathbb{R}$. Namely, if ω is a closed $4k$-form on N which represents the real image of c in $H^{4k}(N;\mathbb{R})$, then the pairing $\langle i^*(c), [M]\rangle$ can be computed as

$$\int_M i^*(\omega) = \int_N d\omega = 0 \tag{11.29}$$

by Stokes' theorem.

Lemma 11.30. *Let $B\colon V\times V\to\mathbb{R}$ be a nondegenerate symmetric bilinear form on a real vector space V. Suppose $W\subset V$ is isotropic—$B(w_1,w_2)=0$ for all $w_1,w_2\in W$—and $2\dim W=\dim V$. Then* $\operatorname{Sign}B=0$.

Proof. Let $e_1\in W$ be nonzero. Since B is nondegenerate there exists $f_1\in V$ such that $B(e_1,f_1)=1$. Shifting f_1 by a multiple of e_1 we can arrange that $B(f_1,f_1)=0$. In other words, the form B on the subspace $\mathbb{R}\{e_1,f_1\}\subset V$ is hyperbolic, so has signature zero. Let V_1 be the orthogonal complement to $\mathbb{R}\{e_1,f_1\}\subset V$ relative to the form B. Since B is nondegenerate we have $V=\mathbb{R}\{e_1,f_1\}\oplus V_1$. Also, $W_1:=W\cap V_1\subset V_1$ is isotropic and $2\dim W_1=\dim V_1$. Set $B_1=B\big|_{V_1}$. Then the data (V_1,B_1,W_1) satisfies the same hypotheses as (V,B,W) and has smaller dimension. So we can repeat and in a finite number of steps write B as a sum of hyperbolic forms. □

Theorem 11.31. *Let N^{4k+1} be a compact oriented manifold with boundary $i\colon M^{4k}\hookrightarrow N$. Then* $\operatorname{Sign}M=0$.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} H^{2k}(N;\mathbb{R}) & \xrightarrow{i^*} & H^{2k}(M;\mathbb{R}) & \longrightarrow & H^{2k+1}(N,M;\mathbb{R}) \\ \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\ H_{2k+1}(N,M;\mathbb{R}) & \longrightarrow & H_{2k}(M;\mathbb{R}) & \xrightarrow{i_*} & H_{2k}(N;\mathbb{R}) \end{array} \tag{11.32}$$

The rows are a stretch of the long exact sequences of the pair (N,M) in real cohomology and real homology. The vertical arrows are Poincaré duality isomorphisms. We claim that $\operatorname{image}(i^*)$ is isotropic for the real intersection pairing

$$I_M\otimes\mathbb{R}\colon H^{2k}(M;\mathbb{R})\times H^{2k}(M;\mathbb{R})\longrightarrow\mathbb{R} \tag{11.33}$$

and has dimension $\frac12\dim H^{2k}(M;\mathbb{R})$. The isotropy follows immediately from Lemma 11.26. This and the commutativity of (11.32) imply that (i) $\operatorname{image}(i^*)$ maps isomorphically to $\ker(i_*)$ under Poincaré duality, and (ii) $\operatorname{image}(i^*)$ annihilates $\ker(i_*)$ under the pairing of cohomology and homology. It is an easy exercise that these combine to prove $2\dim\operatorname{image}(i^*)=\dim H^{2k}(M;\mathbb{R})$. Now the theorem follows immediately from Lemma 11.30. □

Corollary 11.34. *For each $k\in\mathbb{Z}^{\ge0}$ the signature defines a homomorphism*

$$\operatorname{Sign}\colon\Omega^{SO}_{4k}\longrightarrow\mathbb{Z}. \tag{11.35}$$

That (11.35) is well-defined follows from Theorem 11.31; that it is a homomorphism follows from Exercise 11.25. In fact, defining the signature to vanish in dimensions not divisible by four, we see from Exercise 11.25 that

$$\operatorname{Sign}\colon\Omega^{SO}\longrightarrow\mathbb{Z} \tag{11.36}$$

is a ring homomorphism.

Any manifold with nonzero signature is not null bordant. In particular,

Proposition 11.37. *$\mathbb{CP}^{2\ell}$ is not null bordant, $l\in\mathbb{Z}^{>0}$.*

Exercise 11.38. Demonstrate explicitly that $\mathbb{CP}^{2\ell+1}$ is null bordant by exhibiting a null bordism.

Hirzebruch's signature theorem

(11.39) *Pontrjagin numbers.* Recall the Pontrjagin classes, defined in (7.68). For a smooth manifold M we have $p_i(M) \in H^{4i}(M;\mathbb{Z})$. Suppose M is closed and oriented. Then for any sequence $(i_1,\dots,i_r)$ of positive integers we define the Pontrjagin number

$$p_{i_1,\dots,i_r}(M) = \langle p_{i_1}(M) \smile \cdots \smile p_{i_r}(M), [M]\rangle. \tag{11.40}$$

By degree count, this vanishes unless $4(i_1+\dots+i_r) = \dim M$. In any case Lemma 11.26 immediately implies the following

Proposition 11.41. *The Pontrjagin numbers are bordism invariants*

$$p_{i_1,\dots,i_r}\colon \Omega_n^{SO} \longrightarrow \mathbb{Z}. \tag{11.42}$$

(11.43) *Tensoring with* $\mathbb{Q}$. The following simple observation is crucial: the map $\mathbb{Z} \longrightarrow \mathbb{Z}\otimes\mathbb{Q}$ is injective. For this is merely the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$. This means that (11.35) and (11.42) are determined by the linear functionals

$$\mathrm{Sign}\colon \Omega_{4k}^{SO}\otimes\mathbb{Q} \longrightarrow \mathbb{Q} \tag{11.44}$$

and

$$p_{i_1,\dots,i_r}\colon \Omega_n^{SO}\otimes\mathbb{Q} \longrightarrow \mathbb{Q} \tag{11.45}$$

obtained by tensoring with $\mathbb{Q}$. This has the advantage that the vector space $\Omega_n^{SO}\otimes\mathbb{Q}$ is easier to compute than the abelian group Ω_n^{SO}. In fact, we already summarized the main results about Ω^{SO} in Theorem 2.24. These follow by applying the Pontrjagin-Thom theorem of Lecture 10, specifically Corollary 10.38. We recall just the statement we need here and present the proof in the next lecture.

Theorem 11.46. *There is an isomorphism*

$$\mathbb{Q}[y_4, y_8, y_{12}, \dots] \xrightarrow{\;\cong\;} \Omega^{SO}\otimes\mathbb{Q} \tag{11.47}$$

under which y_{4k} maps to the oriented bordism class of the complex projective space $\mathbb{CP}^{2k}$.

Assuming Theorem 11.46 for now, we can prove the main theorem of this lecture.

Theorem 11.48 (Hirzebruch)**.** *Let M^{4k} be a closed oriented manifold. Then*

$$\mathrm{Sign}\, M = \langle L(M), [M]\rangle, \tag{11.49}$$

where $L(M) \in H^\bullet(M;\mathbb{Q})$ is the L-class (7.61).

Proof. It suffices to check the equation (11.49) on a basis of the rational vector space $\Omega_{4k}^{SO}\otimes\mathbb{Q}$. By Theorem 11.46 this is given by a product of projective spaces $M_{k_1,\dots k_r} := \mathbb{CP}^{2k_1}\times\cdots\times\mathbb{CP}^{2k_r}$ for $k_1+\cdots+k_r=k$. By Exercise 11.24 and Exercise 11.25 we see that

$$\operatorname{Sign} M_{k_1,\dots,k_r} = 1. \tag{11.50}$$

On the other hand, by Proposition 8.8 we have

$$\langle L(\mathbb{CP}^{2k_i}), [\mathbb{CP}^{k_i}]\rangle = 1 \tag{11.51}$$

for all i. Since

$$L(\mathbb{CP}^{2k}) = L(\mathbb{CP}^{2k_1})\cdots L(\mathbb{CP}^{2k_r}) \tag{11.52}$$

and

$$[\mathbb{CP}^{2k_1}\times\cdots\times\mathbb{CP}^{2k_r}] = [\mathbb{CP}^{2k_1}]\times\cdots\times[\mathbb{CP}^{2k_r}], \tag{11.53}$$

it follows that

$$\langle L(M_{k_1,\dots,k_r}), [M_{k_1,\dots,k_r}]\rangle = 1. \tag{11.54}$$

(The product on the right hand side of (11.53) is the tensor product in the Kunneth theorem for the rational homology vector space $H_{2k}(\mathbb{CP}^{2k_1}\times\cdots\times\mathbb{CP}^{2k_r};\mathbb{Q})$.) The theorem now follows from (11.50) and (11.54). □

Integrality

For a 4-manifold M^4 the signature formula (11.49) asserts

$$\operatorname{Sign} M = \langle p_1(M)/3, [M]\rangle. \tag{11.55}$$

In particular, since the left hand side is an integer, so is the right hand side. *A priori* this is far from clear: whereas $p_1(M)$ is an integral cohomology class, $\frac{1}{3}p_1(M)$ is definitely not—it is only a rational class. Also, there exist real vector bundles $V\to M$ over 4-manifolds so that $\langle p_1(V)/3, [M]\rangle$ is not an integer.

Exercise 11.56. Find an example. Even better, find an example in which M is a spin manifold.

So the integrality is special to the tangent bundle.

This is the tip of an iceberg of integrality theorems.

Exercise 11.57. Work out the formula for the signature in 8 and 12 dimensions in terms of Pontrjagin numbers. Note that the denominators grow rapidly.

Hurewicz theorems

A basic tool for the computation is the Hurewicz theorem, which relates homotopy and homology groups.

(11.58) *The integral Hurewicz theorem.* Let (X, x) be a pointed topological space.[2] The *Hurewicz map*

$$\eta_n\colon \pi_n X \longrightarrow H_n X \tag{11.59}$$

sends a homotopy class represented by a pointed map $f\colon S^n \to X$ to the homology class $f_*[S^n]$. You probably proved in the prelim class that for $n = 1$ the Hurewicz map is surjective with kernel the commutator subgroup $[\pi_1 X, \pi_1 X] \subset \pi_1 X$, i.e., $H_1 X$ is the abelianization of $\pi_1 X$. For higher n we have the following. Recall that a pointed space is *k-connected*, $k \in \mathbb{Z}^{>0}$, if it is path connected and if $\pi_i X = 0$ for $i \le k$.

Theorem 11.60 (Hurewicz)**.** *Let X be a pointed space which is $(n-1)$-connected for $n \in \mathbb{Z}^{\ge 2}$. Then the Hurewicz homomorphism η_n is an isomorphism.*

We refer the reader to standard texts (e.g. [H1], [Ma]) for a proof of the Hurewicz theorem. The following is immediate by induction.

Corollary 11.61. *Let X be a 1-connected pointed space which satisfies $H_i X = 0$ for $i = 2, 3, \dots, n-1$. Then X is $(n-1)$-connected and (11.59) is an isomorphism.*

(11.62) *The rational Hurewicz theorem.* There is also a version of the Hurewicz theorem over $\mathbb{Q}$. We state it here and refer to [KK] for an "elementary" proof. (It truly is more elementary than other proofs!)

Theorem 11.63 ($\mathbb{Q}$-Hurewicz)**.** *Let X be a 1-connected pointed space, and assume that $\pi_i X \otimes \mathbb{Q} = 0$, $2 \le i \le n-1$, for some $n \in \mathbb{Z}^{\ge 2}$. Then the rational Hurewicz map*

$$\eta_i \otimes \mathbb{Q}\colon \pi_i X \otimes \mathbb{Q} \longrightarrow H_i(X; \mathbb{Q}) \tag{11.64}$$

is an isomorphism for $1 \le i \le 2n-2$.

It is also true that η_{2n-1} is surjective, but we do not need this.

Computation for 4-manifolds

By Corollary 10.38 there is an isomorphism

$$\phi\colon \pi_4(MSO) \longrightarrow \Omega_4^{SO}. \tag{11.65}$$

Recall that $\pi_4(MSO) \cong \pi_{4+q} MSO(q)$ for q sufficiently large. And **(11.43)** it suffices to compute $\pi_4(MSO) \otimes \mathbb{Q}$.

[2] I didn't mention earlier the technical issue that the basepoint should be nondegenerate in a certain sense: the inclusion $\{x\} \hookrightarrow X$ should be a *cofibration*. See [Ma] for details.

Theorem 11.66. *If $q \geq 6$, then* $\dim_{\mathbb{Q}} \pi_{4+q}\,(MSO(q) \otimes \mathbb{Q}) = 1$.

Proof. Recall that there is a diffeomorphism $SO(3) \simeq \mathbb{RP}^3$, so its rational homotopy groups are isomorphic to those of the double cover S^3, the first few of which are

$$\pi_i SO(3) \otimes \mathbb{Q} \cong \begin{cases} 0, & i = 1, 2; \\ \mathbb{Q}, & i = 3. \end{cases} \tag{11.67}$$

Now for any integer $q \geq 3$ the group $SO(q+1)$ acts transitively on S^q with stabilizer of a point in S^q the subgroup $SO(q)$. So there is a fiber bundle $SO(q) \to SO(q+1) \to S^q$, which is in fact a principal $SO(q)$-bundle.[3] The induced long exact sequence of homotopy groups[4] has a stretch

$$\pi_{i+1} SO(q+1) \longrightarrow \pi_{i+1} S^q \longrightarrow \pi_i SO(q) \longrightarrow \pi_i SO(q+1) \longrightarrow \pi_i S^q \longrightarrow \pi_{i-1} SO(q) \longrightarrow \cdots \tag{11.68}$$

and it remains exact after tensoring with $\mathbb{Q}$. First use it to show $\pi_2 SO(q) \otimes \mathbb{Q} = 0$ for all[5] $q \geq 3$. Set $q = 3$. Then, using the result that $\pi_4 S^3 \cong \mathbb{Z}/2\mathbb{Z}$, so that $\pi_4 S^3 \otimes \mathbb{Q} = 0$, we deduce that $\pi_3 SO(4) \otimes \mathbb{Q}$ has dimension 2. Now set $q = 4$ and deduce that $\pi_3 SO(5) \otimes \mathbb{Q}$ has dimension 1. You will need to also use the result that $\pi_5 S^3 \otimes \mathbb{Q} = 0$. By induction on $q \geq 5$ we then prove

$$\pi_i SO(q) \otimes \mathbb{Q} \cong \begin{cases} 0, & i = 1, 2; \\ \mathbb{Q}, & i = 3 \end{cases} \tag{11.69}$$

for all $q \geq 5$.

Next, use the universal fiber bundle $G \to EG \to BG$ for $G = SO(q)$, $q \geq 5$, which is a special case of (6.60), and the fact that EG is contractible, so has vanishing homotopy groups, to deduce

$$\pi_i BSO(q) \otimes \mathbb{Q} \cong \begin{cases} 0, & i = 1, 2, 3; \\ \mathbb{Q}, & i = 4 \end{cases} \tag{11.70}$$

from the long exact sequence of homotopy groups. Then the $\mathbb{Q}$-Hurewicz Theorem 11.63 implies

$$H_i\big(BSO(q); \mathbb{Q}\big) \cong \begin{cases} 0, & i = 1, 2, 3; \\ \mathbb{Q}, & i = 4; \\ 0, & i = 5, 6 \end{cases} \tag{11.71}$$

for $q \geq 5$.

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[3] We construct it here by fixing a point in S^q. Can you construct an isomorphic principal $SO(q)$-bundle without choosing a basepoint? what is the geometric meaning of the total space?

[4] We have used this before; see [H1, Theorem 4.41] or, for a quick review, [BT, §17].

[5] In fact, $\pi_2 G = 0$ for any finite dimensional Lie group G.

The proof of the Thom isomorphism theorem, Proposition 8.35, gives a cell structure for the Thom complex. The resulting Thom isomorphism on homology implies

$$H_i\bigl(MSO(q);\mathbb{Q}\bigr) \cong \begin{cases} 0, & i = 1,\dots,q-1; \\ \mathbb{Q}, & i = q; \\ 0, & i = 1+q, 2+q, 3+q \\ \mathbb{Q}, & i = 4+q \\ 0, & i = 5+q, 6+q. \end{cases} \tag{11.72}$$

The cell structure also implies that the Thom complex $MSO(q)$ of the universal bundle $S(q) \to BSO(q)$ is $(q-1)$-connected. The $\mathbb{Q}$-Hurewicz theorem then implies that the $\mathbb{Q}$-Hurewicz map $\pi_i MSO(q) \otimes \mathbb{Q} \to H_i\bigl(MSO(q);\mathbb{Q}\bigr)$ is an isomorphism for $1 \le i \le 2q-2$, whence if $q \ge 6$ we deduce in particular

$$\pi_{4+q}\bigl(MSO(q);\mathbb{Q}\bigr) \cong \mathbb{Q}. \tag{11.73}$$

$\square$

By Proposition 11.37 the class of $\mathbb{CP}^2$ in $\Omega_4^{SO} \otimes \mathbb{Q}$ is nonzero. (We need a bit more: $\mathbb{CP}^2$ has infinite order in Ω_4^{SO} because its signature is nonzero and the signature (11.35) is a homomorphism.) Since $\pi_4(MSO) \otimes \mathbb{Q}$ is one-dimensional, the class of $\mathbb{CP}^2$ is a basis. Finally, we prove (11.55) by checking both sides for $M = \mathbb{CP}^2$ using Example 11.22, Proposition 7.51, and the definition (7.68) of the Pontrjagin classes.

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