Lecture 12: More on the signature theorem

Here we sketch the proof of Theorem 11.46. In the last lecture we indicated most of the techniques involved by proving the theorem for 4-manifolds. There are two additional inputs necessary for the general case. First, we need to know that the rational cohomology of BSO is the polynomial ring on the Pontrjagin classes. We simply quote that result here, but remark that it follows from Theorem 7.72. In fact, all we really end up using is the graded dimension of the rational cohomology—its dimension in each degree. The second input is purely algebraic, to do with symmetric functions. We indicate what the issue is and refer the reader to the literature.

As we are about to leave classical bordism, we begin with a comment—thanks to a student question and off-topic with respect to the signature theorem—which could have been made right at the beginning of the course.

Bordism as a generalized homology theory

The basic building blocks of singular homology theory are continuous maps

$$f\colon \Delta^q \longrightarrow X \tag{12.1}$$

from the standard q-simplex Δ^q to a topological space X. Chains are formal sums of such maps, and there is a boundary operator, so a notion of closed chains, or cycles. From this one builds a chain complex and homology. A crucial case is $X = \mathrm{pt}$. Then the homology question comes down to whether a closed simplicial complex is a boundary. It is: one can simply cone off the simplicial complex σ to construct a new simplicial complex $C\sigma$ whose boundary is σ.

In bordism theory—as a generalized homology theory—one replaces (12.1) by continuous maps

$$f\colon M^q \longrightarrow X \tag{12.2}$$

out of a closed q-dimensional manifold M. Now rather than defining a formal abelian group of "chains", we define the equivalence relation of bordism: $f_i\colon M_i \to X$, $i = 0, 1$, are equivalent if there exists a compact $(q+1)$-manifold N with $\partial N = M_0 \amalg M_1$ and a continuous map $f\colon N \to X$ whose restriction to the boundary is $f_0 \amalg f_1$. (Of course, we should make a more elaborate definition modeled on Definition 1.19.) The equivalence classes turn out to be an abelian group, which we denote $\Omega_q(X)$. Then the graded abelian group $\Omega_\bullet(X)$ satisfies all of the axioms of homology theory except for the specification of $\Omega_\bullet(\mathrm{pt})$. What we have been studying *is* $\Omega_\bullet(\mathrm{pt})$. But I want you to know that there is an entire homology theory there. See [DK] for one account.

I remark that there is a variation $\Omega^{\mathcal{X}}_\bullet(X)$ for every stable tangential structure $\mathcal{X}$.

Mising steps

We begin with an important result in its own right.

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(12.3) *The cohomology of BSO.*

Theorem 12.4. *The rational cohomology ring of the classifying space of the special orthogonal group is the polynomial ring generated by the Pontrjagin classes:*

$$H^\bullet(BSO;\mathbb{Q}) \cong \mathbb{Q}[p_1, p_2, \dots]. \tag{12.5}$$

One proof follows from Theorem 7.72, which identifies real[1] cohomology classes on $BSO(q)$ with invariant polynomials on the orthogonal algebra $\mathfrak{o}(q)$. The latter is the Lie algebra of real skew-symmetric matrices. 'Invariant' means invariant under conjugation by an orthogonal matrix. So for a skew-symmetric matrix A we must produce a polynomial $P(A) \in \mathbb{R}$ so that $P(OAO^{-1}) = P(A)$ for every orthogonal matrix O. This is easy to do. Define

$$Q_t(A) = \det(I - tA) = 1 + P_1(A)t^2 + P_2(A)t^4 + \cdots , \tag{12.6}$$

where I is the identity matrix. Then $Q_t(A)$ is a polynomial in t with real coefficients, and by the skew-symmetry of A we can show $Q_{-t}(A) = Q_t(A)$, so only even powers of t occur. (Prove it!) The coefficients P_i are invariant polynomials in A, and up to a factor they correspond to the universal Pontrjagin classes.

Exercise 12.7 ([Kn])**.** Here are some hints—using some theory of compact Lie groups—towards a proof of Theorem 7.72.[2] Let $T \subset G$ be a maximal torus, $N \subset G$ its normalizer, and $W = N/T$ the Weyl group. Identify G-invariant polynomials on $\mathfrak{g}$ with W-invariant polynomials on the Lie algebra $\mathfrak{t}$ of T. Consider the iterated fibration $EG/T \to EG/N \to EG/G$, which is $BT \to BN \to BG$. The first map is a finite cover, and induces an isomorphism in rational cohomology. The fiber of $BN \to BG$ is G/N, which has the rational cohomology of a sphere.

(12.8) *The proof.* Now we sketch a proof of most of Theorem 11.46, which we restate here. The statement about complex projective spaces is deferred to a later subsection.

Theorem 12.9. *There is an isomorphism*

$$\mathbb{Q}[x^1, x^2, x^3, \dots] \xrightarrow{\ \cong\ } \Omega^{SO} \otimes \mathbb{Q}. \tag{12.10}$$

All we really need from the statement is that the dimension of $\Omega^{SO}_{4k} \otimes \mathbb{Q}$ is $p(k)$, the number of partitions of k.

Proof. The rational homology of BSO is the dual vector space to the rational cohomology, so

$$H_\bullet(BSO;\mathbb{Q}) \cong \mathbb{Q}[p^1, p^2, \dots] \tag{12.11}$$

[1] The result over the rationals is stronger, but follows since the Pontrjagin classes are rational.

[2] The Lie group G in the theorem should be assumed connected.

for dual homology classes $p^1, p^2, \dots$. The Thom isomorphism theorem, as in the derivation of (11.72), and the definition (10.13) of the homology of a spectrum, imply

$$H_\bullet(MSO;\mathbb{Q}) \cong \mathbb{Q}[q^1, q^2, \dots] \tag{12.12}$$

for some classes $q^k \in H_{2k}(MSO;\mathbb{Q})$. Finally, $MSO(q)$ is $(q-1)$-connected, which by $\mathbb{Q}$-Hurewicz implies that the map

$$\eta_i \otimes \mathbb{Q}\colon \pi_i\bigl(MSO(q)\bigr) \otimes \mathbb{Q} \longrightarrow H_i\bigl(MSO(q);\mathbb{Q}\bigr) \tag{12.13}$$

is an isomorphism for $1 \le i \le 2q-2$. In the limit $q \to \infty$ we obtain an isomorphism for all i. □

(12.14) *A very nice exercise.* The following is a great test of your understanding of the Pontrjagin-Thom construction.

Exercise 12.15. Suppose that M is a closed oriented $4k$-manifold whose rational bordism class is the sum[3] $c_{i_1\cdots i_r} x^{i_1} \cdots x^{i_r}$ under the isomorphism (11.47). Recall the Pontrjagin number (11.42). Prove that $c_{i_1\cdots i_r}$ is the Pontrjagin number $p_{i_1\cdots i_r}$ of the stable normal bundle to M. You will need, of course, to use the generators x^i defined in the proof.

Complex projective spaces as generators

The content of Theorem 12.9 is that $\Omega_{4k}^{SO} \otimes \mathbb{Q}$ is a rational vector space of dimension $p(k)$, the number of *partitions* of k. Recall that a partition of a positive integer k is a finite unordered set $\{i_1, \dots, i_r\}$ of positive integers such that $i_1 + \cdots + i_r = k$. For example, $\Omega_8^{SO} \otimes \mathbb{Q}$ is 2-dimensional. The remaining statement we must prove is the following.

Proposition 12.16. *Let $k \in \mathbb{Z}^{\ge 1}$. The manifolds $M_{i_1\cdots i_r} := \mathbb{CP}^{2i_1} \times \cdots \mathbb{CP}^{2i_r}$ form a basis of $\Omega_{4k}^{SO} \otimes \mathbb{Q}$, where $\{i_1, \dots, i_r\}$ ranges over all partitions of k.*

The case $k = 1$ is easy, as we used in Lecture 11. For $k = 2$ we must show that the classes of $\mathbb{CP}^4$ and $\mathbb{CP}^2 \times \mathbb{CP}^2$ are linearly independent. We can use the Pontrjagin numbers p_1^2, p_2 to show that: the matrix

$$\begin{pmatrix} 25 & 10 \\ 18 & 9 \end{pmatrix} \tag{12.17}$$

is nondegenerate. The rows represent the manifolds $\mathbb{CP}^4$, $\mathbb{CP}^2 \times \mathbb{CP}^2$ and the colums the Pontrjagin numbers p_1^2, p_2. This sort of argument does not easily generalize. Rather than repeat the necessary algebra of symmetric functions here, we defer to [MS, §16].

[3] over a basis of polynomials of degree $4k$

References

[DK] James F. Davis and Paul Kirk, *Lecture notes in algebraic topology*, Graduate Studies in Mathematics, vol. 35, American Mathematical Society, Providence, RI, 2001.

[Kn] Alan Knutson, http://mathoverflow.net/questions/61784/cohomology-of-bg-g-compact-lie-group.

[MS] John W. Milnor and James D. Stasheff, *Characteristic classes*, Princeton University Press, Princeton, N. J., 1974. Annals of Mathematics Studies, No. 76.