Lecture 13: Categories

We begin again. In Lecture 1 we used bordism to define an equivalence relation on closed manifolds of a fixed dimension n. The set of equivalence classes has an abelian group structure defined by disjoint union of manifolds. Now we extract a more intricate algebraic structure from bordisms. The equivalence relation only remembers the existence of a bordism; now we record the bordism itself. The bordism now has a direction: it is a map from one closed manifold to another. Gluing of bordisms, previously used to prove transitivity of the equivalence relation, is now recorded as a composition law on bordisms. To obtain an associative composition law we remember bordisms only up to diffeomorphism. (In subsequent lectures we will go further and remember the diffeomorphism.) The algebraic structure obtained is a *category* $\mathrm{Bord}_{\langle n-1,n\rangle}$, which here replaces the *set* of equivalence classes Ω_n. The notation for this category suggests more refinements to come later. Disjoint union provides an algebraic operation on $\mathrm{Bord}_{\langle n-1,n\rangle}$, which is then a *symmetric monoidal category*.

In this lecture we introduce categories, homomorphisms, natural transformations, and symmetric monoidal structures. Pay particular attention to the example of the fundamental groupoid (Example 13.14), which shares some features with the bordism category, though with one important difference: the bordism category is not a groupoid.

Categories

Definition 13.1. A *category* C consists of a collection of objects, for each pair of objects y_0, y_1 a set of morphisms $C(y_0, y_1)$, for each object y a distinguished morphism $\mathrm{id}_y \in C(y, y)$, and for each triple of objects y_0, y_1, y_2 a composition law

$$\circ\colon C(y_1, y_2) \times C(y_0, y_1) \longrightarrow C(y_0, y_2) \tag{13.2}$$

such that $\circ$ is associative and id_y is an identity for $\circ$.

The last phrase indicates two conditions: for all $f \in C(y_0, y_1)$ we have

$$\mathrm{id}_{y_1} \circ f = f \circ \mathrm{id}_{y_0} = f \tag{13.3}$$

and for all $f_1 \in C(y_0, y_1)$, $f_2 \in C(y_1, y_2)$, and $f_3 \in C(y_2, y_3)$ we have

$$(f_3 \circ f_2) \circ f_1 = f_3 \circ (f_2 \circ f_1). \tag{13.4}$$

We use the notation $y \in C$ for an object of C and $f\colon y_0 \to y_1$ for a morphism $f \in C(y_0, y_1)$.

Remark 13.5 (set theory). The words ‘collection’ and ‘set’ are used deliberately. The Russell paradox pointed out that the collection of all sets is not a set, yet we still want to consider a category whose objects are sets. For many categories the objects do form a set. In that case the moniker ‘small category’ is often used. In these lecture we will be sloppy about the underlying set theory and simply talk about a set of objects.

Bordism: Old and New (M392C, Fall ’12), Dan Freed, October 23, 2012

Definition 13.6. Let C be a category.

(i) A morphism $f \in C(y_0, y_1)$ is *invertible* (or an *isomorphism*) if there exists $g \in C(y_1, y_0)$ such that $g \circ f = \mathrm{id}_{y_0}$ and $f \circ g = \mathrm{id}_{y_1}$.
(ii) If every morphism in C is invertible, then we call C a *groupoid*.

(13.7) *Reformulation.* To emphasize that a category is an algebraic structure like any other, we indicate how to formulate the definition in terms of sets[1] and functions. Then a category C consists of a set C_0 of objects, a set C_1 of functions, and structure maps

$$\begin{aligned} i\colon C_0 &\longrightarrow C_1 \\ s,t\colon C_1 &\longrightarrow C_0 \\ c\colon C_1 \times_{C_0} C_1 &\longrightarrow C_1 \end{aligned} \tag{13.8}$$

which satisfy certain conditions. The map i attaches to each object y the identity morphism id_y, the maps s,t assign to a morphism $(f\colon y_0 \to y_1) \in C_1$ the source $s(f) = y_0$ and target $t(f) = y_1$, and c is the composition law. The fiber product $C_1 \times_{C_0} C_1$ is the set of pairs $(f_2, f_1) \in C_1 \times C_1$ such that $t(f_1) = s(f_2)$. The conditions (13.3) and (13.4) can be expressed as equations for these maps.

Examples of categories

Example 13.9 (monoid)**.** Let C be a category with a single object, i.e., $C_0 = \{*\}$. Then C_1 is a set with an identity element and an associative composition law. This is called a *monoid*. A groupoid with a single object is a *group*.[2]

Example 13.10 (set)**.** At the other extreme, suppose C is a category with only identity maps, i.e., $i\colon C_0 \to C_1$ is an isomorphism of sets (a 1:1 correspondence). Then C is given canonically by the set C_0 of objects, and we identify the category C as this set.

Example 13.11 (action groupoid)**.** Let S be a set and G a group which acts on S. There is an associated groupoid $C = S/\!/G$ with objects $C_0 = S$ and morphisms $C_1 = G \times S$. The source map is projection to the first factor and the target map is the action $G \times S \to S$. We leave the reader to work out the composition and show that the axioms for a category are a direct consequence of those for a group action. See Figure 22.

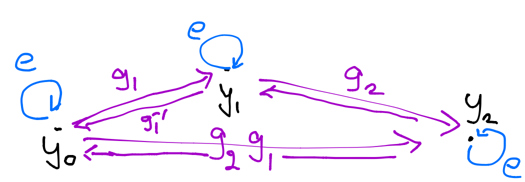


FIGURE 22. The action groupoid $S/\!/G$

[1] ignoring set-theoretic complications, as in Remark 13.5

[2] So, by analogy, you'd think instead of 'category' we'd use 'monoidoid'!

Example 13.12 (category of sets)**.** Assuming that the set theoretic difficulties alluded to in Remark 13.5 are overcome, there is a category Set whose objects are sets and whose morphisms are functions.

Example 13.13 (subcategories of Set)**.** There is a category Ab of abelian groups. An object $A \in \mathrm{Ab}$ is an abelian group and a morphism $f\colon A_0 \to A_1$ is a homomorphism of abelian groups. Similarly, there is a category Vect_k of vector spaces over a field k. There is also a category of rings and a category of R-modules for a fixed ring R. (Note Ab is the special case $R = \mathbb{Z}$.) Each of these categories is special in that the hom-sets are abelian groups. There is also a category Top whose objects are topological spaces Y and in which a morphism $f\colon Y_0 \to Y_1$ is a continuous map.

Example 13.14 (fundamental groupoid)**.** Let Y be a topological space. The simplest invariant is the *set* $\pi_0 Y$. It is defined by imposing an equivalence relation on the set Y underlying the topological space: points y_0 and y_1 in Y are equivalent if there exists a continuous path which connects them, i.e., a continuous map $\gamma\colon [0,1] \to Y$ which satisfy $\gamma(0) = y_0$, $\gamma(1) = y_1$.

The *fundamental groupoid* $C = \pi_{\le 1} Y$ is defined as follows. The objects $C_0 = Y$ are the points of Y. The hom-set $C(y_0, y_1)$ is the set of homotopy classes of maps $\gamma\colon [0,1] \to Y$ which satisfy $\gamma(0) = y_0$, $\gamma(1) = y_1$. The homotopies are taken "rel boundary", which means that the endpoints are fixed in a homotopy. Explicitly, a homotopy is a map

$$\Gamma\colon [0,1] \times [0,1] \longrightarrow Y \tag{13.15}$$

such that $\Gamma(s, 0) = y_0$ and $\Gamma(s, 1) = y_1$ for all $s \in [0,1]$. The composition of homotopy classes of paths is associative, and every morphism is invertible. Note that the *automorphism group* $C(y, y)$ is the fundamental group $\pi_1(Y, y)$. So $\pi_{\le 1} Y$ encodes both $\pi_0 Y$ and all of the fundamental groups.

Exercise 13.16. Given a groupoid C use the morphisms to define an equivalence relation on the objects and so a set $\pi_0 C$ of equivalence classes. Can you do the same for a category which is not a groupoid?

Functors and natural transformations

Definition 13.17. Let C, D be categories.

(i) A *functor* or *homomorphism* $F\colon C \to D$ is a pair of maps $F_0\colon C_0 \to D_0$, $F_1\colon C_1 \to D_1$ which commute with the structure maps (13.8).

(ii) Suppose $F, G\colon C \to D$ are functors. A *natural transformation* η from F to G is a map of sets $\eta\colon C_0 \to D_1$ such that for all morphisms $(f : y_0 \to y_1) \in C_1$ the diagram

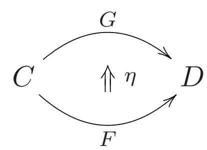
$$\begin{CD} Fy_0 @>{Ff}>> Fy_1 \\ @V{\eta(y_0)}VV @VV{\eta(y_1)}V \\ Gy_0 @>{Gf}>> Gy_1 \end{CD} \tag{13.18}$$

commutes. We write $\eta\colon F \to G$.

(iii) A natural transformation $\eta\colon F \to G$ is an *isomorphism* if $\eta(y)\colon Fy \to Gy$ is an isomorphism for all $y \in C$.

In (i) the commutation with the structure maps means that F is a homomorphism in the usual sense of algebra: it preserves compositions and takes identities to identities. A natural transformation is often depicted in a diagram

(13.19)



with a double arrow.

Example 13.20 (functor categories)**.** Show that for fixed categories C, D there is a category $\mathrm{Hom}(C, D)$ whose objects are functors and whose morphisms are natural transformations.

Remark 13.21. Categories have one more layer of structure than sets. Intuitively, elements of a set have no "internal" structure, whereas objects in a category do, as reflected by their self-maps. Numbers have no internal structure, whereas sets do. Try that intuition out on each of the examples above. Anything to do with categories has an extra layer of structure. This is true for homomorphisms of categories: they form a category (Example 13.20) rather than a set. Below we see that when we define a monoidal structure there is an extra layer of data before conditions enter.

Example 13.22. There is a functor $**\colon \mathrm{Vect} \to \mathrm{Vect}$ which maps a vector space V to its double dual V^{**}. But this is not enough to define it—we must also specify the map on morphisms, which in this case are linear maps. Thus if $f\colon V_0 \to V_1$ is a linear map, there is an induced linear map $f^{**}\colon V_0^{**} \to V_1^{**}$. (Recall that $f^*\colon V_1^* \to V_0^*$ is defined by $\langle f^*(v_1^*), v_0\rangle = \langle v_1^*, f(v_0)\rangle$ for all $v_0 \in V_0$, $V_1^* \in V_1^*$. Then define $f^{**} = (f^*)^*$.) Now there is a natural transformation $\eta\colon \mathrm{id}_{\mathrm{Vect}} \to **$ defined on a vector space V as

$$\begin{aligned} \eta(V)\colon V &\longrightarrow V^{**} \\ v &\longmapsto \bigl(v^* \mapsto \langle v^*, v\rangle\bigr) \end{aligned} \tag{13.23}$$

for all $v^* \in V^*$. I encourage you to check (13.18) carefully.

Example 13.24 (fiber functor)**.** Let Y be a topological space and $\pi\colon Z \to Y$ a covering space. Then there is a functor

$$\begin{aligned} F_\pi\colon \pi_{\le 1}Y &\longrightarrow \mathrm{Set} \\ y &\longrightarrow \pi^{-1}(y) \end{aligned} \tag{13.25}$$

which maps each point of y to the fiber over y. Again, this is not a functor until we tell how morphisms map. For that we need to use the theory of covering spaces. Any path $\gamma\colon [0,1] \to$

Y "lifts" to an isomorphism $\tilde{\gamma}\colon \pi^{-1}(y_0) \to \pi^{-1}(y_1)$, and the isomorphism is unchanged under homotopy. A map

$$\begin{array}{ccc} Z_0 & \xrightarrow{\ \varphi\ } & Z_1 \\ & {\scriptstyle \pi_0}\searrow \quad \swarrow{\scriptstyle \pi_1} & \\ & Y & \end{array} \tag{13.26}$$

of covering spaces induces a natural transformation $\eta_\varphi\colon F_{\pi_0} \to F_{\pi_1}$.

Symmetric monoidal categories

A category is an enhanced version of a set; a *symmetric monoidal category* is an enhanced version of a commutative monoid. Just as a commutative monoid has data (composition law, identity element) and conditions (associativity, commutativity, identity property), so too does a symmetric monoidal category have data and conditions. Only now the conditions of a commutative monoid become data for a symmetric monoidal category. The conditions are new and numerous. We do not spell them all out, but defer to the references.

(13.27) *Product categories.* If C', C'' are categories, then there is a *Cartesian product* category $C = C' \times C''$. The set of objects is the Cartesian product $C_0 = C_0' \times C_0''$ and the set of objects is likewise the Cartesian product $C_1 = C_1' \times C_1''$. We leave the reader to work out the structure maps (13.8).

Definition 13.28. Let C be a category. A *symmetric monoidal structure* on C consists of an object

$$1_C \in C, \tag{13.29}$$

a functor

$$\otimes\colon C \otimes C \longrightarrow C \tag{13.30}$$

and natural isomorphisms

$$C \times C \times C \ \underset{(-\otimes-)\otimes-}{\overset{-\otimes(-\otimes-)}{\rightrightarrows}}\ C \quad (\Uparrow \alpha), \tag{13.31}$$

$$C \times C \ \underset{-\otimes-}{\overset{(-\otimes-)\circ\tau}{\rightrightarrows}}\ C \quad (\Uparrow \sigma), \tag{13.32}$$

and

(13.33)
$$C \underset{1_C\otimes -}{\overset{\mathrm{id}_C}{\Uparrow \iota}} C\,.$$

The quintuple $(1_C, \otimes, \alpha, \sigma, \iota)$ is required to satisfy the axioms indicated below.

The functor τ in (13.31) is transposition:

(13.34)
$$\begin{aligned} \tau\colon C\times C &\longrightarrow C\times C \\ y_1, y_2 &\longmapsto y_2, y_1 \end{aligned}$$

A crucial axiom is that

(13.35)
$$\sigma^2 = \mathrm{id}\,.$$

Thus for any $y_1, y_2 \in C$, the composition

(13.36)
$$y_1\otimes y_2 \xrightarrow{\ \sigma\ } y_2\otimes y_1 \xrightarrow{\ \sigma\ } y_1\otimes y_2$$

is $\mathrm{id}_{y_1\otimes y_2}$. The other axioms express compatibility conditions among the extra data (13.29)–(13.33). For example, we require that for all $y_1, y_2 \in C$ the diagram

(13.37)
$$\begin{array}{ccc} & (1_C\otimes y_1)\otimes y_2 & \\ {}^{\alpha}\swarrow & & \searrow^{\iota} \\ 1_C\otimes(y_1\otimes y_2) & \xrightarrow{\qquad\iota\qquad} & y_1\otimes y_2 \end{array}$$

commutes. We can state the axioms informally as asserting the equality of any two compositions of maps built by tensoring α, σ, ι with identity maps. These compositions have domain a tensor product of objects $y_1, \dots, y_n$ and any number of identity objects 1_C—ordered and parenthesized arbitrarily—to a tensor product of the same objects, again ordered and parenthesized arbitrarily. Coherence theorems show that there is a small set of conditions which needs to be verified; then arbitrary diagrams of the sort envisioned commute. You can find precise statements and proof in [Mac, JS]

(13.38) *Symmetric monoidal functor.* This is a homomorphism between symmetric monoidal categories, but as is typical for categories the fact that the identity maps to the identity and tensor products to tensor products is expressed via data, not as a condition. Then there are higher order conditions.

Definition 13.39. Let C, D be symmetric monoidal categories. A *symmetric monoidal functor* $F\colon C \to D$ is a functor with two additional pieces of data, namely an isomorphism

$$1_D \longrightarrow F(1_C) \tag{13.40}$$

and a natural isomorphism

$$C \times C \underset{F(-)\otimes F(-)}{\overset{F(-\otimes -)}{\rightrightarrows}} C \quad (\Uparrow \psi) . \tag{13.41}$$

There are many conditions on this data.

The first condition expresses compatibility with the associativity morphisms: for all $y_1, y_2, y_3 \in C$ the diagram

$$\begin{array}{ccc} \big(F(y_1)\otimes F(y_2)\big)\otimes F(y_3) & \xrightarrow{\ \psi\ } & F(y_1\otimes y_2)\otimes F(y_3) \\ \downarrow{\scriptstyle \alpha_D} & & \downarrow{\scriptstyle \psi} \\ F(y_1)\otimes\big(F(y_2)\otimes F(y_3)\big) & & F\big((y_1\otimes y_2)\otimes y_3\big) \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle F(\alpha_C)} \\ F(y_1)\otimes F(y_2\otimes y_3) & \xrightarrow{\ \psi\ } & F\big(y_1\otimes(y_2\otimes y_3)\big) \end{array} \tag{13.42}$$

is required to commute. Next, there is compatibility with the identity data ι: for all $y \in C$ we requre that

$$\begin{array}{ccc} F(1_C)\otimes F(y) & \xrightarrow{\ F(\psi)\ } & F(1_C\otimes y) \\ \uparrow{\scriptstyle (13.40)} & & \downarrow{\scriptstyle F(\iota)} \\ 1_D\otimes F(y) & \xrightarrow{\ \iota\ } & F(y) \end{array} \tag{13.43}$$

commute. The final condition expresses compatibility with the symmetry σ: for all $y_1, y_2 \in C$ the diagram

$$\begin{array}{ccc} F(y_1)\otimes F(y_2) & \xrightarrow{\ \sigma_D\ } & F(y_2)\otimes F(y_1) \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} \\ F(y_1\otimes y_2) & \xrightarrow{\ F(\sigma_C)\ } & F(y_2\otimes y_1) \end{array} \tag{13.44}$$

Exercise 13.45. Define a natural transformation of symmetric monoidal functors.

References

[JS] André Joyal and Ross Street, *Braided tensor categories*, Adv. Math. **102** (1993), no. 1, 20–78.

[Mac] Saunders Mac Lane, *Categories for the working mathematician*, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998.