Lecture 14: Bordism categories

The definition

Fix a nonnegative[1] integer n. Recall the basic Definition 1.19 of a bordism $X\colon Y_0 \to Y_1$ whose domain and codomain are closed $(n-1)$-manifolds. A bordism is a quartet $(X, p, \theta_0, \theta_1)$ in which X is a compact manifold with boundary, $p\colon \partial X \to \{0,1\}$ is a partition of the boundary,[2] and θ_0, θ_1 are boundary diffeomorphism. As usual we overload the notation and use 'X' to denote the full quartet of data.

Definition 14.1. Suppose $X, X'\colon Y_0 \to Y_1$ are bordisms between closed $(n-1)$-manifolds Y_0, Y_1. A *diffeomorphism* $F\colon X \to X'$ is a diffeomorphism of manifolds with boundary which commutes with p, θ_0, θ_1.

So, for example, we have a commutative diagram

(14.2)
$$\begin{array}{ccc} X & \xrightarrow{\;p\;} & \\ {\scriptstyle F}\downarrow & & \{0,1\} \\ X' & \xrightarrow{\;p'\;} & \end{array}$$

and similar commutative diagrams involving the θ's.

Definition 14.3. Fix $n \in \mathbb{Z}^{\geq 0}$. The *bordism category* $\mathrm{Bord}_{\langle n-1,n\rangle}$ is the symmetric monoidal category defined as follows.

(i) The objects are closed $(n-1)$-manifolds.
(ii) The hom-set $\mathrm{Bord}_{\langle n-1,n\rangle}(Y_0, Y_1)$ is the set of diffeomorphism classes of bordisms $X\colon Y_0 \to Y_1$.
(iii) Composition of morphisms is by gluing (Figure 2).
(iv) For each Y the bordism $[0,1] \times Y$ is $\mathrm{id}_Y\colon Y \to Y$.
(v) The monoidal product is disjoint union.
(vi) The empty manifold $\emptyset^{n-1}$ is the tensor unit (13.29).

The additional data α, σ, ι expresses the associativity and commutativity of disjoint union, which we suppress; but see **(1.16)**. In (iv) the partition of the boundary is projection $p\colon \{0,1\} \times Y \to \{0,1\}$ onto the first factor and the boundary diffeomorphisms are the identity on Y.

(14.4) *Isotopy.* Let $\mathrm{Diff}\, Y$ denote the group of smooth diffeomorphisms of a closed manifold Y. It is a topological group[3] if we use the compact-open topology.

Definition 14.5.

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[1] We allow $n = 0$. Recall that the empty manifold can have any dimension, and we allow $\emptyset^{-1}$ of dimension -1.

[2] This formulation is better than that written currently in Definition 1.19.

[3] A *topological group* G is simultaneously and compatibly a topological space and a group: composition and inversion are continuous maps $G \times G \to G$ and $G \to G$.

(i) An *isotopy* is a smooth map $F\colon [0,1]\times Y\to Y$ such that $F(t,-)\colon Y\to Y$ is a diffeomorphism for all $t\in[0,1]$.
(ii) A *pseudoisotopy* is a diffeomorphism $\widetilde{F}\colon [0,1]\times Y\to[0,1]\times Y$ which preserves the submanifolds $\{0\}\times Y$ and $\{1\}\times Y$.

Equivalently,[4] an isotopy is a path in $\operatorname{Diff} Y$. Diffeomorphisms f_0, f_1 are said to be *isotopic* if there exists an isotopy $F\colon f_0\to f_1$. Isotopy is an equivalence relation. The set of isotopy classes is $\pi_0 \operatorname{Diff} Y$, which is often called the *mapping class group* of Y. An isotopy induces a pseudoisotopy

$$\begin{aligned}\widetilde{F}\colon [0,1]\times Y &\longrightarrow [0,1]\times Y \\ (t,y) &\longmapsto \bigl(t,F(t,y)\bigr)\end{aligned} \tag{14.6}$$

We say $\widetilde{F}\colon f_0\to f_1$ if the induced diffeomorphisms of Y on the boundary of $[0,1]\times Y$ are f_0 and f_1.

Exercise 14.7. Prove that pseudoisotopy is an equivalence relation.

Remark 14.8. Pseudoisotopy is potentially a courser equivalence relation than isotopy: isotopic diffeomorphisms are pseudoisotopic. The converse is true for simply connected manifolds of dimension ≥ 5 by a theorem of Cerf.

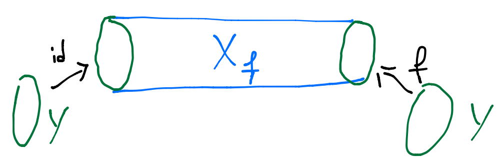

FIGURE 23. The bordism associated to a diffeomorphism

(14.9) *Embedding diffeomorphisms in the bordism category.* Let Y be a closed $(n-1)$-manifold and $f\colon Y\to Y$ a diffeomorphism. There is an associated bordism $(X_f,p,\theta_0,\theta_1)$ with (i) $X_f=[0,1]\times Y$, $p\colon\{0,1\}\times Y\to Y$ projection, (iii) $\theta_0=\mathrm{id}_Y$, and (iv) $\theta_1=f$, as depicted in Figure 23. If $F\colon f_0\to f_1$ is an isotopy, then we claim that the bordisms X_{f_0} and X_{f_1} are equal in the homset $\operatorname{Bord}_{\langle n-1,n\rangle}(Y,Y)$. For the isotopy F determines a diffeomorphism and the composition of bordisms in the top row of
Figure 24 is X_{f_1}. Of course, Figure 24 shows that *pseudo*isotopic diffeomorphisms determine equal bordisms in $\operatorname{Bord}_{\langle n-1,n\rangle}(Y,Y)$.

Exercise 14.10. Show that if X_{f_0} and X_{f_1} are equal in $\operatorname{Bord}_{\langle n-1,n\rangle}(Y,Y)$, then f_0 is pseudoisotopic to f_1.

Summarizing, there is a homomorphism

$$\pi_0(\operatorname{Diff} Y)\longrightarrow \operatorname{Bord}_{\langle n-1,n\rangle}(Y,Y) \tag{14.11}$$

which is not necessarily injective.

Exercise 14.12. Is (14.11) injective for $n=1$ and $Y=\mathrm{pt}\amalg\mathrm{pt}$?

[4]With what we have introduced we can talk about *continuous* paths in $\operatorname{Diff} Y$, which correspond to maps F which are only continuous in the first variable. But then we can approximate by a smooth map. In any case we can in a different framework discuss *smooth* maps of smooth manifolds into $\operatorname{Diff} Y$.

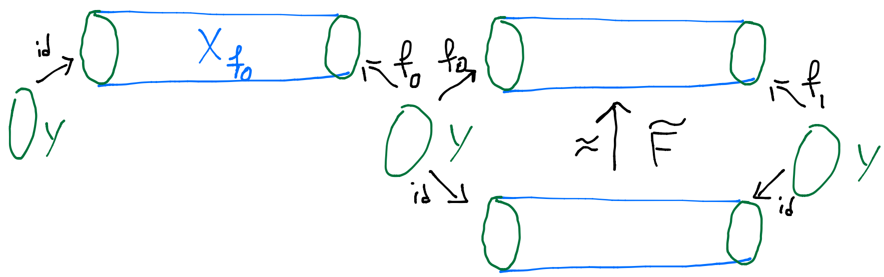

FIGURE 24. Isotopic diffeomorphisms give diffeomorphic bordisms

(14.13) *Bordism categories with tangential structures.* Recall Definition 9.45: an n-dimensional tangential structure is a fibration $\mathcal{X}(n) \to BO(n)$. There is a universal rank n bundle $S(n) \to \mathcal{X}(n)$ with $\mathcal{X}(n)$-structure, and an $\mathcal{X}(n)$-structure on a manifold M of dimension $k \le n$ is a commutative diagram

$$\begin{array}{ccc} \underline{\mathbb{R}^{n-k}} \oplus TM & \longrightarrow & S(n) \\ \downarrow & & \downarrow \\ M & \longrightarrow & \mathcal{X}(n) \end{array} \tag{14.14}$$

There is a bordism category $\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}$ analogous to $\mathrm{Bord}_{\langle n-1,n\rangle}$ as defined in Definition 14.3, but all manifolds Y, X are required to carry $\mathcal{X}(n)$-structures. Examples include *stable* tangential structures, such as orientation and spin, as well as *unstable* tangential structures, such as n-framings. We follow the notational convention of Exercise 9.71.

Examples of bordism categories

Example 14.15 ($\mathrm{Bord}_{\langle -1,0\rangle}$)**.** There is a unique (-1)-dimensional manifold—the empty manifold $\emptyset^{-1}$—so $\mathrm{Bord}_{\langle -1,0\rangle}$ is a category with a single object, hence a monoid (Example 13.9). The monoid is the set of morphisms $\mathrm{Bord}_{\langle -1,0\rangle}(\emptyset^{-1}, \emptyset^{-1})$ under composition. In fact, the symmetric monoidal structure gives a second composition law, but it is equal to the first which is necessarily commutative. This follows from general principles, but is easy to see in this case. Namely, the monoid consists of diffeomorphism classes of closed 0-manifolds, so finite unions of points. The set of diffeomorphism classes is $\mathbb{Z}^{\ge 0}$. Composition and the monoidal product are both disjoint union, which induces addition in $\mathbb{Z}^{\ge 0}$.

Example 14.16 ($\mathrm{Bord}^{SO}_{\langle -1,0\rangle}$)**.** Now all manifolds are oriented, so the morphisms are finite unions of pt_+ and pt_-, up to diffeomorphism. Let x_+, x_- denote the diffeomorphism class of $\mathrm{pt}_+, \mathrm{pt}_-$. Then the monoid $\mathrm{Bord}^{SO}_{\langle -1,0\rangle}$ is the free commutative monoid generated by x_+, x_-.

Exercise 14.17. Prove that $\mathrm{Diff}\, S^1$ has two components, each of which deformation retracts onto a circle.

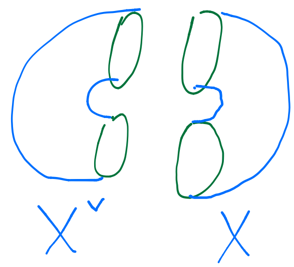

FIGURE 25. Some bordisms in $\mathrm{Bord}_{\langle 1,2\rangle}$

Example 14.18 ($\mathrm{Bord}_{\langle 1,2\rangle}$)**.** Objects are closed 1-manifolds, so finite unions of circles. As depicted in Figure 25 the cylinder can be interpreted as a bordism $X\colon (S^1)^{\amalg 2} \to \emptyset^1$; the dual bordism $X^\vee$ (Definition 1.22) is a map $X^\vee\colon \emptyset^1 \to (S^1)^{\amalg 2}$. Let $\rho\colon S^1 \to S^1$ be reflection, $f = 1 \amalg \rho$ the indicated diffeomorphism of $(S^1)^{\amalg 2}$, and X_f the associated bordism **(14.9)**. Then

$$\begin{aligned} X \circ X_{\mathrm{id}} \circ X^\vee &\simeq \text{torus} \\ X \circ X_f \circ X^\vee &\simeq \text{Klein bottle} \end{aligned} \tag{14.19}$$

These diffeomorphism become equations in the monoid $\mathrm{Bord}_{\langle 1,2\rangle}(\emptyset^1, \emptyset^1)$ of diffeomorphism classes of closed 2-manifolds.

Topological quantum field theories

Just as we study abstract groups via their representations, so too we study bordism categories via representations. There are linear actions of groups on vector spaces, and also nonlinear actions on more general spaces. Similarly, there are linear and nonlinear representations of bordism categories.

Definition 14.20. Fix $n \in \mathbb{Z}^{\ge 0}$ and $\mathcal{X}(n)$ an n-dimensional tangential structure. Let C be a symmetric monoidal category. An *n-dimensional topological quantum field theory of $\mathcal{X}(n)$-manifolds with values in C* is a symmetric monoidal functor

$$F\colon\ \mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle} \longrightarrow C \tag{14.21}$$

Symmetric monoidal functors are defined in **(13.38)**. We use the acronym 'TQFT' for 'topological quantum field theory'. We do not motivate the use of 'quantum field theory' for Definition 14.20 here; see instead the discussion in [F1]. I also strongly recommend the beginning sections of [L1]. The definition originates in the mathematics literature in [A1], which in turn was inspired by [S]. There is a nice thorough discussion in [**?**].

Remark 14.22. Let Top denote the symmetric monoidal category whose objects are topological spaces and whose morphisms are continuous maps. The monoidal structure is disjoint union. Let Ab denote the category whose objects are abelian groups and whose morphisms are group homomorphisms. Homology theory gives symmetric monoidal functors

$$H_q\colon (\mathrm{Top}, \amalg) \longrightarrow (\mathrm{Ab}, \oplus) \tag{14.23}$$

for all nonnegative integers q. Note that the symmetric monoidal structure on Ab is *direct sum*: the homology of a disjoint union is the direct sum of the homologies. One should think of the direct sum as *classical*; for *quantum* field theories we will use instead tensor product. In vague terms quantization, which is the passage from classical to quantum, is a sort of exponentiation which turns sums to products.

For this reason we keep the 'quantum' in 'TQFT'.

(14.24) *Codomain categories.* Typical "linear" choices for C are: (i) the symmetric monoidal category $(\mathrm{Vect}_k, \otimes)$ of vector spaces over a field k, (ii) the symmetric category $({}_R\mathrm{Mod}, \otimes)$ of left modules over a commutative ring R, and the special case (iii) the symmetric monoidal category $(\mathrm{Ab}, \otimes)$ of abelian groups under tensor product. On the other hand, we can take as codomain a bordism category, which is decidedly nonlinear. For example, if M is a closed k-manifold, then there is a symmetric monoidal functor

$$- \times M \colon \mathrm{Bord}_{\langle n-1,n\rangle} \longrightarrow \mathrm{Bord}_{\langle n+k-1,n+k\rangle} \tag{14.25}$$

which, I suppose, can be called a TQFT. If $F\colon \mathrm{Bord}_{\langle n+k-1,n+k\rangle} \to C$ is any $(n+k)$-dimensional TQFT, then composition with (14.25) gives an n-dimensional TQFT, the *dimensional reduction of F along M*.

References

[A1] Michael Atiyah, *Topological quantum field theories*, Inst. Hautes Études Sci. Publ. Math. (1988), no. 68, 175–186 (1989).
[F1] D. S. Freed, *The cobordism hypothesis*. http://www.ma.utexas.edu/users/dafr/cobordism.pdf.
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