Lecture 15: Duality

We ended the last lecture by introducing one of the main characters in the remainder of the course, a topological quantum field theory (TQFT). At this point we should, of course, elaborate on the definition and give examples, background, motivation, etc. I will not do so in these notes. Instead I refer you to the expository paper [F1] as well as to the beginning sections of [L1]. There are many other references with great expository material.

In this lecture we explore the finiteness property satisfied by a TQFT, which is encoded via duality in symmetric monoidal categories.

Some categorical preliminaries

We begin with a standard notion which you'll find in any book which contains a chapter on categories, including books on category theory.

Definition 15.1. Let C, D be categories. A functor $F\colon C \to D$ is an *equivalence* if there exist a functor $G\colon D \to C$, and natural isomorphisms $G \circ F \to \mathrm{id}_C$ and $F \circ G \to \mathrm{id}_D$.

Proposition 15.2. *A functor $F\colon C \to D$ is an equivalence if and only if it satisfies:*

(i) *For each $d \in D$ there exist $c \in C$ and an isomorphism $\bigl(f(c) \to d\bigr) \in D$; and*

(ii) *For each $c_1, c_2 \in C$ the map of hom-sets $F\colon C(c_1, c_2) \to D\bigl(F(c_1), F(c_2)\bigr)$ is a bijection.*

If F satisfies (i) it is said to be *essentially surjective* and if it satisfies (ii) it is *fully faithful*.

Exercise 15.3. Prove Proposition 15.2.

Next we spell out the answer to Exercise 13.45. It is part of the definition of a TQFT.

Definition 15.4. Let C, D be symmetric monoidal categories and $F, G\colon C \to D$ symmetric monoidal functors. Then a *symmetric monoidal natural transformation* $\eta\colon F \to G$ is a natural transformation such that the diagrams

(15.5)
$$\begin{array}{ccc} & & F(1_C) \\ & \nearrow & \big\downarrow{\scriptstyle \eta(1_C)} \\ 1_D & & \\ & \searrow & G(1_C) \end{array}$$

and

(15.6)
$$\begin{array}{ccc} F(y_1) \otimes F(y_2) & \xrightarrow{\ \psi\ } & F(y_1 \otimes y_2) \\ {\scriptstyle \eta\otimes\eta}\big\downarrow & & \big\downarrow{\scriptstyle \eta} \\ G(y_1) \otimes G(y_2) & \xrightarrow{\ \psi\ } & G(y_1 \otimes y_2) \end{array}$$

commute for all $y_1, y_2 \in C$.

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TQFT's as a symmetric monoidal category

Fix a bordism category $B = \mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}$ and a symmetric monoidal category C. We now explain that topological quantum field theories $F\colon B \to C$ are objects in a symmetric monoidal category. A morphism $F \to G$ is as defined in Definition 15.4. The monoidal product of theories F_1, F_2 is defined by

$$\begin{aligned}(F_1 \otimes F_2)(Y) &= F_1(Y) \otimes F_2(Y)\\ (F_1 \otimes F_2)(X) &= F_1(X) \otimes F_2(X)\end{aligned} \tag{15.7}$$

for all objects $Y \in B$ and morphisms $(X\colon Y_0 \to Y_1) \in B$. The tensor unit $\mathbf{1}$ is the trivial theory

$$\begin{aligned}\mathbf{1}(Y) &= 1_C\\ \mathbf{1}(X) &= \mathrm{id}_{1_C}\end{aligned} \tag{15.8}$$

for all $Y \in B$ and $(X\colon Y_0 \to Y_1) \in B$.

We denote the symmetric monoidal category of TQFT's as

$$\mathrm{TQFT}_n = \mathrm{TQFT}^{\mathcal{X}(n)}_n[C] = \mathrm{Hom}^{\otimes}(\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}, C). \tag{15.9}$$

The short form of the notation is used if the tangential structure $\mathcal{X}(n)$ and codomain category C are clear.

(15.10) *Endomorphisms of the trivial theory.* Suppose $\eta\colon \mathbf{1} \to \mathbf{1}$ in TQFT_n. Then for all $Y \in \mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}$ we have $\eta(Y) \in C(1_C, 1_C) = \mathrm{End}(1_C)$. Note that if $C = \mathrm{Ab}$ is the category of abelian groups, then $\mathrm{End}(1_C) = \mathbb{Z}$. So η is a numerical invariant of closed $(n-1)$-manifolds. Furthermore, if $X\colon Y_0 \to Y_1$ then by the naturality condition (13.18) we find that $\eta(Y_0) = \eta(Y_1)$. This shows that η factors down to a homomorphism of monoids

$$\eta\colon \Omega^{\mathcal{X}(n)}_{n-1} \longrightarrow \mathrm{End}(1_C). \tag{15.11}$$

Now by Lemma 1.30, and its generalization to manifolds with tangential structure, we know that every element of $\Omega^{\mathcal{X}(n)}_{n-1}$ is invertible. It follows that the image of η consists of invertible elements. We say, simply, that η is invertible.

In other words, an endomorphism of $\mathbf{1}$ is a bordism invariant of the type studied in the first half of the course. A topological quantum field theory, then, is a "categorified" bordism invariant.

Exercise 15.12. What is a topological quantum field theory whose codomain category has as objects the set of integers and only identity arrows?

The invertibility observed in **(15.10)** is quite general.

Theorem 15.13. *A morphism* $(\eta\colon F \to G) \in \mathrm{TQFT}_n$ *is invertible.* TQFT_n *is a groupoid.*

The two statements are equivalent. We prove Theorem 15.13 at the end of this lecture.

(15.14) *Central problem.* Given a dimension n, a tangential structure $\mathcal{X}(n)$, and a codomain category C we can ask to "compute" the groupoid $\mathrm{TQFT}_n^{\mathcal{X}(n)}[C]$. This is a vague problem whose solution is an equivalent groupoid which is "simpler" than the groupoid of topological quantum field theories. It has a nice answer when $n = 1$. In the oriented case it is a generalization of the theorem that Ω_0^{SO} is the free abelian group with a single generator pt_+. There is also a nice answer in the oriented case for $n = 2$.

Finiteness in TQFT

To motivate the abstract formulation of finiteness in symmetric monoidal categories, we prove the following simple proposition. For simplicity we omit any tangential structure.

Proposition 15.15. *Let* $F\colon \mathrm{Bord}_{\langle n-1,n\rangle} \to \mathrm{Vect}_{\mathbb{C}}$ *be a TQFT. Then for all* $Y \in \mathrm{Bord}_{\langle n-1,n\rangle}$ *the vector space* $F(Y)$ *is finite dimensional.*

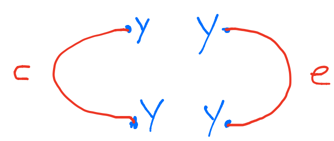

FIGURE 26. Some elementary bordisms

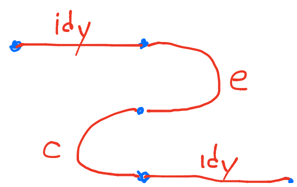

FIGURE 27. The S-diagram

Proof. Fix $Y \in \mathrm{Bord}_{\langle n-1,n\rangle}$ and let $V = F(Y)$. Let $c\colon \emptyset^{n-1} \to Y \amalg Y$ and $e\colon Y \amalg Y \to \emptyset^{n-1}$ be the bordisms pictured in Figure 26. The manifold Y is depicted as a point, and each bordism has underlying manifold with boundary $[0,1] \times Y$. The composition depicted in Figure 27 is diffeomorphic to the identity bordism $\mathrm{id}_Y : Y \to Y$. Under F it maps to $\mathrm{id}_V : V \to V$ (see (13.40)). On the other hand, the composition maps to

$$V \xrightarrow{\mathrm{id}_V \otimes F(c)} V \otimes V \otimes V \xrightarrow{F(e)\otimes \mathrm{id}_V} V \tag{15.16}$$

Let the value of $F(c)\colon \mathbb{C} \to V \otimes V$ on $1 \in \mathbb{C}$ be $\sum_i v_i' \otimes v_i''$ for some finite set of vectors $v_i', v_i'' \in V$. Then equating (15.16) with the identity map we find that for all $\xi \in V$ we have

$$\xi = \sum_i e(\xi, v_i')v_i'', \tag{15.17}$$

and so the finite set of vectors $\{v_i''\}$ spans V. This proves that V is finite dimensional. □

Exercise 15.18. Prove that $F(c)$ and $F(e)$ are inverse bilinear forms.

Duality data and dual morphisms

We abstract the previous argument by singling out those objects in a symmetric monoidal category which obey a finiteness condition analogous to that of a finite dimensional vector space.

Definition 15.19. Let C be a symmetric monoidal category and $y \in C$.

(i) *Duality data* for y is a triple of data $(y^\vee, c, e)$ in which $y^\vee$ is an object of C and c, e are morphisms $c\colon 1_C \to y \otimes y^\vee$, $e\colon y^\vee \otimes y \to 1_C$. We require that the compositions

$$y \xrightarrow{c\otimes \mathrm{id}_y} y \otimes y^\vee \otimes y \xrightarrow{\mathrm{id}_y \otimes e} y \tag{15.20}$$

and

$$y^\vee \xrightarrow{\mathrm{id}_{y^\vee} \otimes c} y^\vee \otimes y \otimes y^\vee \xrightarrow{e\otimes \mathrm{id}_{y^\vee}} y^\vee \tag{15.21}$$

be identity maps. If duality data exists for y, we say that y is *dualizable*.

(ii) A *morphism of duality data* $(y^\vee, c, e) \to (\tilde{y}^\vee, \tilde{c}, \tilde{e})$ is a morphism $y^\vee \xrightarrow{f} \widetilde{y^\vee}$ such that the diagrams

$$\begin{array}{ccc} & & y \otimes y^\vee \\ & \overset{c}{\nearrow} & \big\downarrow{\scriptstyle \mathrm{id}_y \otimes f} \\ 1_C & & \\ & \underset{\tilde{c}}{\searrow} & \\ & & y \otimes \widetilde{y^\vee} \end{array} \tag{15.22}$$

and

$$\begin{array}{ccc} y^\vee \otimes y & & \\ & \overset{e}{\searrow} & \\ {\scriptstyle f\otimes \mathrm{id}_y}\big\downarrow & & 1_C \\ & \underset{\tilde{e}}{\nearrow} & \\ \widetilde{y^\vee} \otimes y & & \end{array} \tag{15.23}$$

commute.

c is called *coevaluation* and e is called *evaluation*.

We now express the uniqueness of duality data. As duality data is an object in a category, as defined in Definition 15.19, we cannot say there is a unique object. Rather, here we have the strongest form of uniqueness possible in a category: duality data is unique up to unique isomorphism.

Definition 15.24. Let C be a category.

(i) If for each pair $y_0, y_1 \in C$ the hom-set $C(y_0, y_1)$ is either empty or contains a unique element, we say that C *is a discrete groupoid*.

(ii) If for each pair $y_0, y_1 \in C$ the hom-set $C(y_0, y_1)$ has a unique element, we say that C *is contractible*.

A discrete groupoid is equivalent to a set (Example 13.10). A contractible groupoid is equivalent to a category with one object and one morphism, the categorical analog of a point.

Proposition 15.25. *Let C be a symmetric monoidal category and $y \in C$. Then the category of duality data for y is either empty or is contractible.*

The proof is a homework problem (Problem Set #2).

A morphism between dualizable objects has a dual.

Definition 15.26. Let $y_0, y_1 \in C$ be dualizable objects in a symmetric monoidal category and $f\colon y_0 \to y_1$ a morphism. The *dual morphism* $f^\vee\colon y_1^\vee \to y_0^\vee$ is the composition

$$y_1^\vee \xrightarrow{\mathrm{id}_{y_1^\vee}\otimes c_0} y_1^\vee \otimes y_0 \otimes y_0^\vee \xrightarrow{\mathrm{id}_{y_1^\vee}\otimes f \otimes \mathrm{id}_{y_0^\vee}} y_1^\vee \otimes y_1 \otimes y_0^\vee \xrightarrow{e_1 \otimes \mathrm{id}_{y_0^\vee}} y_0^\vee \tag{15.27}$$

In the definition we choose duality data $(y_0^\vee, c_0, e_0)$, $(y_1^\vee, c_1, e_1)$ for y_0, y_1.

Exercise 15.28. Check that this definition agrees with that of a dual linear map for $C = \mathrm{Vect}$. Also, spell out the consequence of Proposition 15.25 for the dual morphism.

Duality in bordism categories

We already encountered dual manifolds and dual bordisms in Definition 1.22, Remark 1.24, and (**2.20**). In this subsection we prove the following.

Theorem 15.29. *Every object in a bordism category* $\mathrm{Bord}_{\langle n-1,n\rangle}^{\mathcal{X}(n)}$ *is dualizable.*

Proof. If $\mathcal{X}(n)$ is the trivial tangential structure $BO(n) \to BO(n)$, so $\mathrm{Bord}_{\langle n-1,n\rangle}^{\mathcal{X}(n)} = \mathrm{Bord}_{\langle n-1,n\rangle}$ is the bordism category of (unoriented) manifolds, then for any closed $(n-1)$-manifold Y we have $Y^\vee = Y$ with coevaluation and evaluation as in Figure 26. In the general case, an object $(Y, \theta) \in \mathrm{Bord}_{\langle n-1,n\rangle}^{\mathcal{X}(n)}$ is a closed $(n-1)$-manifold Y equipped with a classifying map

$$\begin{array}{ccc} \underline{\mathbb{R}} \oplus TY & \xrightarrow{\theta} & S(n) \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \mathcal{X}(n) \end{array} \tag{15.30}$$

to the universal bundle (9.59); cf. (9.60). Its dual $(Y,\theta)^\vee = (Y, \theta^\vee)$ has the same underlying manifold and classifying map $\theta^\vee$ the composition

$$\begin{array}{ccccc} \underline{\mathbb{R}} \oplus TY & \xrightarrow{-1 \oplus \mathrm{id}_{TY}} & \underline{\mathbb{R}} \oplus TY & \xrightarrow{\theta} & S(n) \\ \downarrow & & \downarrow & & \downarrow \\ Y & \xrightarrow{\mathrm{id}_Y} & Y & \longrightarrow & \mathcal{X}(n) \end{array} \tag{15.31}$$

The coevaluation and evaluation $(X, p, \theta_0, \theta_1)$ are as depicted in Figure 26. In both cases $X = [0,1] \times Y$. For the coevaluation $p\colon \partial X \to \{0,1\}$ is the constant function 1, and for the evaluation it is the constant function 0. For the evaluation

$$\begin{aligned} \theta_0\colon [0,1) \times Y \ \amalg \ [0,1) \times Y &\longrightarrow [0,1] \times Y \\ \text{(t,y)} \qquad\qquad\qquad\quad &\longmapsto (t/4, y) \\ \text{(t,y)} \quad &\longmapsto (1 - t/4, y) \end{aligned} \tag{15.32}$$

with natural lifts to the $\mathfrak{X}(n)$-structures. The formula for θ_1 for the coevaluation is similar. $\square$

Exercise 15.33. Write the map on $\mathfrak{X}(n)$-structures explicitly. Note the $-$ sign in the differential of the last formula in (15.32) matches the -1 in the first map of (15.31).

Proof of Theorem 15.13

We first prove the following.

Proposition 15.34. *Let B, C be symmetric monoidal categories, $F, G\colon B \to C$ symmetric monoidal functors, and $y \in B$ dualizable. Then*

(i) *$F(y) \in C$ is dualizable.*
(ii) *If $\eta\colon F \to G$ is a symmetric monoidal natural transformation, then $\eta(y)\colon F(y) \to G(y)$ is invertible.*

Proof. If $(y^\vee, c, e)$ is duality data for y, then $\bigl(F(y^\vee), F(c), F(e)\bigr)$ is duality data for $F(y)$. This proves (i).

For (ii) we claim that $\eta(y^\vee)^\vee$ is inverse to $\eta(y)$. Note that by Definition 15.26, $\eta(y^\vee)^\vee$ is a map $G(y^\vee)^\vee \to F(y^\vee)^\vee$, and since $G(y^\vee) = G(y)^\vee$ it may be interpreted as a map $G(y) \to F(y)$. Let $c\colon 1_B \to y \otimes y^\vee$ and $e\colon y^\vee \otimes y \to 1_B$ be coevaluation and evaluation. Consider the diagram

(15.35)

$$\begin{array}{ccccccc} G(y) & \xrightarrow{\operatorname{id}\otimes F(c)} & G(y)\otimes F(y^\vee)\otimes F(y) & \xrightarrow{\operatorname{id}\otimes \eta(y^\vee)\otimes \operatorname{id}} & G(y)\otimes G(y^\vee)\otimes F(y) & \xrightarrow{G(e)\otimes \operatorname{id}} & F(y) \\ & \searrow^{\operatorname{id}\otimes G(c)} & & \searrow^{\operatorname{id}\otimes \eta(y^\vee)\otimes \eta(y)} & \downarrow{\scriptstyle \operatorname{id}\otimes\operatorname{id}\otimes\eta(y)} & & \downarrow{\scriptstyle \eta(y)} \\ & & & & G(y)\otimes G(y^\vee)\otimes G(y) & \xrightarrow{G(e)\otimes \operatorname{id}} & G(y) \end{array}$$

We claim it commutes. The left triangle commutes due to the naturality of η applied to the coevaluation $c\colon 1_B \to y \otimes y^\vee$. The next triangle and the right square commute trivially. Now starting on the left, the composition along the top and then down the right is the composition $\eta(y) \circ \eta(y^\vee)^\vee$. The composition diagonally down followed by the horizontal map is the identity, by G applied to the S-diagram relation (15.20) (and using (13.40)). A similar diagram proves that $\eta(y^\vee)^\vee \circ \eta(y) = \operatorname{id}$. $\square$

Theorem 15.13 is an immediate consequence of Theorem 15.29 and part (ii) of (11.71). Part (i) implies the following.

Theorem 15.36. *Let C be a symmetric monoidal category and $F\colon \mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle} \to C$ be a topological quantum field theory. Then for all $Y \in \mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}$, the object $F(Y) \in C$ is dualizable.*

References

[F1] D. S. Freed, *The cobordism hypothesis*. http://www.ma.utexas.edu/users/dafr/cobordism.pdf.

[L1] Jacob Lurie, *On the classification of topological field theories*, Current developments in mathematics, 2008, Int. Press, Somerville, MA, 2009, pp. 129–280. arXiv:0905.0465.