Lecture 16: 1-dimensional TQFTs

In this lecture we determine the groupoid of 1-dimensional TQFTs of oriented manifolds with values in any symmetric monoidal category. This is a truncated version of the cobordism hypothesis, but illustrates a few of the basic underlying ideas.

Categorical preliminaries

We need three notions from category theory: a *full subcategory* of an arbitrary category, the *groupoid of units* of an arbitrary category, and the *dimension* of an object in a symmetric monoidal category.

Definition 16.1. Let C be a category and $C_0' \subset C_0$ a subset of objects. Then the *full subcategory* C' with set of objects C_0' has as hom-sets

$$C_1'(y_0, y_1) = C_1(y_0, y_1), \qquad y_0, y_1 \in C_0'. \tag{16.2}$$

There is a natural inclusion $C_0' \to C_0$ which is an isomorphism on hom-sets. We can describe the entire set of morphisms C_1' as a pullback:

$$\begin{array}{ccc} C_1' & \dashrightarrow & C_1 \\ \downarrow & & \downarrow{\scriptstyle s\times t} \\ C_0' \times C_0' & \xrightarrow{j\times j} & C_0 \times C_0 \end{array} \tag{16.3}$$

where s, t are the source and target maps (13.8) and $j\colon C_0' \hookrightarrow C_0$ is the inclusion.

We need a particular example of a full subcategory.

Definition 16.4. Let C be a symmetric monoidal category. Define $C^{\mathrm{fd}} \subset C$ as the full subcategory whose objects are the dualizable objects of C.

The notation 'fd' puts in mind 'finite dimensional', which is correct for the category Vect: the dualizable vector spaces are those which are finite dimensional. It also stands for 'fully dualizable'. The 'fully' is not (yet) relevant.

Recall that if M is a monoid, then the *group of units* $M^{\sim} \subset M$ is the subset of invertible elements. For example, if M is the monoid of $n \times n$ matrices under multiplication, then $M^{\sim}$ is the subset of invertible matrices, which form a group.

Definition 16.5. Let C be a category. Its *groupoid of units*[1] is the groupoid $C^{\sim}$ with same objects $C_0^{\sim} = C_0$ as in the category C and with morphisms $C_1^{\sim} \subset C_1$ the subset of invertible morphisms in C.

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[1]usually called the *maximal groupoid*

Notice that identity arrows are invertible and compositions of invertible morphisms are invertible, so $C^{\sim}$ is a category. Obviously, it is a groupoid.

The last definition applies only to symmetric monoidal categories.

Definition 16.6. Let C be a symmetric monoidal category and $y \in C$ a dualizable object. Then the *dimension* of y, denoted $\dim y \in C(1_C, 1_C)$, is the composition

$$\dim y\colon 1_C \xrightarrow{\;c\;} y \otimes y^{\vee} \xrightarrow{\;\sigma\;} y^{\vee} \otimes y \xrightarrow{\;e\;} 1_C, \tag{16.7}$$

where $(y^{\vee}, c, e)$ is duality data for y.

The reader can easily check that $\dim y$ is independent of the choice of duality data (Definition 15.19).

Classification of 1-dimensional oriented TQFTs

Recall from **(2.28)** that the oriented bordism group in dimension zero is the free abelian group on one generator: $\Omega_0^{SO} \cong \mathbb{Z}$. We can restate this in terms of bordism invariants. Let M be any commutative monoid. Then 0-dimensional bordism invariants with values in M is the commutative monoid $\operatorname{Hom}(\Omega_0^{SO}, M)$, where the sum $F+G$ of two bordism invariants is computed elementwise: $(F+G)(Y) = F(Y) + G(Y)$ for all compact 0-manifolds Y. Then $F(Y)$ is automatically invertible, since Ω_0^{SO} is a group.

Theorem 16.8 (cobordism hypothesis—set version)**.** *The map*

$$\begin{aligned} \Phi\colon \operatorname{Hom}(\Omega_0^{SO}, M) &\longrightarrow M^{\sim} \\ F &\longmapsto F(\mathrm{pt}_+) \end{aligned} \tag{16.9}$$

is an isomorphism *of abelian groups.*

This is the restatement.

Now we consider 1-dimensional oriented TQFTs.

Theorem 16.10 (cobordism hypothesis—1-categorical version)**.** *Let C be a symmetric monoidal category. Then the map*

$$\begin{aligned} \Phi\colon \mathrm{TQFT}^{SO}_{\langle 0,1\rangle}(C) &\longrightarrow (C^{\mathrm{fd}})^{\sim} \\ F &\longmapsto F(\mathrm{pt}_+) \end{aligned} \tag{16.11}$$

is an equivalence of groupoids.

The map Φ is well-defined by Theorem 15.36, which asserts in particular that $F(\mathrm{pt}_+)$ is dualizable. Recall (you shouldn't have forgotten in one page!) Definition 16.4 and Definition 16.5, which give meaning to the subgroupoid $(C^{\mathrm{fd}})^{\sim}$ of C.

The proof relies on the classification of closed 0-manifolds and compact 1-manifolds with boundary [M3]. Note that if Y_0, Y_1 are closed 0-manifolds which are diffeomorphic, then the set of

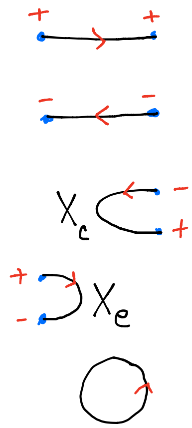


FIGURE 28. The five connected oriented bordisms in $\mathrm{Bord}^{SO}_{\langle 0,1\rangle}$

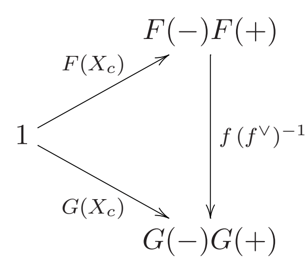
diffeomorphisms $Y_0 \to Y_1$ is a torsor for the group of permutations (of, say, Y_0). A connected compact 1-manifold with boundary is diffeomorphic to a circle or a closed interval, which immediately leads to the classification of connected morphisms in $\mathrm{Bord}^{SO}_{\langle 0,1\rangle}$, as illustrated in Figure 28: every connected oriented bordism is diffeomorphic to one of the five possibilities illustrated there.

Proof. We must show that Φ is fully faithful and essentially surjective. Recall that

First, if F, G are field theories and $\eta_1, \eta_2 \colon F \to G$ isomorphisms, and suppose that $\eta_1(\mathrm{pt}_+) = \eta_2(\mathrm{pt}_+)$. Since $\mathrm{pt}_- = \mathrm{pt}_+^\vee$, according to the formula proved in Proposition 15.34 we have $\eta(\mathrm{pt}_-) = \bigl(\eta(\mathrm{pt}_+)^\vee\bigr)^{-1}$ for any natural isomorphism η. It follows that $\eta_1(\mathrm{pt}_-) = \eta_2(\mathrm{pt}_-)$. Since any compact oriented 0-manifold Y is a finite disjoint union of copies of pt_+ and pt_-, it follows that $\eta_1(Y) = \eta_2(Y)$ for all Y, whence $\eta_1 = \eta_2$. This shows that Φ is faithful.

To show Φ is full, given F, G and an isomorphism $f \colon F(\mathrm{pt}_+) \to G(\mathrm{pt}_+)$ we must construct $\eta \colon F \to G$ such that $\eta(\mathrm{pt}_+) = f$. So define $\eta(\mathrm{pt}_+) = f$ and $\eta(\mathrm{pt}_-) = (f^\vee)^{-1}$. Extend using the monoidal structure in C to define $\eta(Y)$ for all compact oriented 0-manifolds Y. This uses the statement given before the proof that any such Y is diffeomorphic to $(\mathrm{pt}_+)^{\amalg n_+} \amalg (\mathrm{pt}_-)^{\amalg n_-}$ for unique $n_+, n_- \in \mathbb{Z}^{\ge 0}$. Also, the diffeomorphism is determined up to permutation, but because of coherence the resulting map $\eta(Y)$ is independent of the chosen diffeomorphism. It remains to show that η is a natural isomorphism, so to verify (13.18) for each morphism in $\mathrm{Bord}^{SO}_{\langle 0,1\rangle}$. It suffices to consider connected bordisms, so each of the morphisms in Figure 28. The first two are identity maps, for which (13.18) is trivial. The commutativity of the diagram

(16.12)



for coevaluation X_c follows from the commutativity of

$$\begin{array}{ccccc} 1 & \xrightarrow{F(X_c)} & F(-)F(+) & \xrightarrow{1f} & F(-)G(+) \\ \downarrow{\scriptstyle G(X_c)} & & \downarrow{\scriptstyle 11G(X_c)} & & \uparrow{\scriptstyle 1G(X_e)1} \\ G(-)G(+) & \xrightarrow{F(X_c)11} & F(-)F(+)G(-)G(+) & \xrightarrow{1f11} & F(-)G(+)G(-)G(+) \end{array} \tag{16.13}$$

In these diagrams we use '+' and '−' for 'pt_+' and 'pt_-', and also denote identity maps as '1'. The argument for evaluation X_e is similar, and that for the circle follows since the circle is $X_e \circ \sigma \circ X_c$. Notice that the commutative diagram for the circle S^1 asserts $F(S^1) = G(S^1)$.

Finally, we must show that Φ is essentially surjective. Given $y \in C$ dualizable, we must[2] construct a field theory F with $F(\mathrm{pt}_+) = y$. Let $(y^\vee, c, e)$ be duality data for y. Define $F(\mathrm{pt}_+) = y$, $F(\mathrm{pt}_-) = y^\vee$, and

$$F\bigl((\mathrm{pt}_+)^{\amalg n_+} \amalg (\mathrm{pt}_-)^{\amalg n_-}\bigr) = y^{\otimes n_+} \otimes (y^\vee)^{\otimes n_-}. \tag{16.14}$$

Any compact oriented 0-manifold Y is diffeomorphic to some $(\mathrm{pt}_+)^{\amalg n_+} \amalg (\mathrm{pt}_-)^{\amalg n_-}$, and again by coherence the choice of diffeomorphism does not matter. Now any oriented bordism $X\colon Y_0 \to Y_1$ is diffeomorphic to a disjoint union of the bordisms in Figure 28, and for these standard bordisms we define $F(X_c) = c$, $F(X_e) = e$, and $F(S^1) = e \circ \sigma \circ c$; the first two bordisms in the figure are identity maps, which necessarily map to identity maps. We map X to a tensor product of these basic bordisms. It remains to check that F is a functor, i.e., that compositions map to compositions. When composing in $\mathrm{Bord}^{SO}_{\langle 0,1\rangle}$ the only nontrivial compositions are those indicated in Figure 29. The first composition is what we use to define $F(S^1)$. The S-diagram relations (15.20) and (15.21) show that the last compositions are consistent under F. ☐

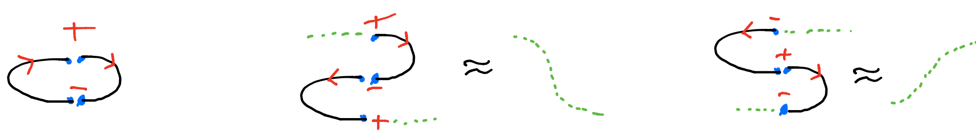


FIGURE 29. Nontrivial compositions in $\mathrm{Bord}^{SO}_{\langle 0,1\rangle}$

References

[M3] John W. Milnor, *Topology from the differentiable viewpoint*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1997. Based on notes by David W. Weaver, Revised reprint of the 1965 original. 2

[2] In fact, we only need construct F with $F(\mathrm{pt}_+) \cong y$, but we will construct one where $F(\mathrm{pt}_+)$ *equals* y.