Lecture 17: Invertible Topological Quantum Field Theories

In this lecture we introduce the notion of an *invertible* TQFT. These arise in both topological and non-topological quantum field theory as *anomaly theories*, a topic we might discuss at the end of the course. They are also interesting in homotopy theory, though not terribly much explored to date in that context. As usual, we need some preliminary discussion of algebra.

Group completion and universal properties

(17.1) *The group completion of a monoid.* Recall that a monoid M is a set with an associative composition law $M \times M \to M$ and a unit $1 \in M$.

Definition 17.2. Let M be a monoid. A *group completion* $(|M|, i)$ of M is a group $|M|$ and a homomorphism $i\colon M \to |M|$ of monoids which satisfies the following universal property: If G is a group and $f\colon M \to G$ a homomorphism of monoids, then there exists a unique map $\tilde{f}\colon |M| \to G$ which makes the diagram

(17.3)
$$\begin{array}{ccc} M & \xrightarrow{\;i\;} & |M| \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \tilde{f}} & \\ & G & \end{array}$$

commute.

The definition does not prove the existence of the group completion—we must provide a proof—but the universal property does imply a strong uniqueness property. Namely, if (H, i) and (H', i') are group completions of M, then there is a unique isomorphism $\phi\colon H \to H'$ of groups which makes the diagram

(17.4)
$$\begin{array}{ccc} M & \xrightarrow{\;i\;} & H \\ & {\scriptstyle i'}\searrow \quad \swarrow{\scriptstyle \phi} & \\ & H' & \end{array}$$

commute. The proof, the details of which I leave to the reader, involves four applications of the universal property (to $f = i$ and $f = i'$ to construct the isomorphism and its inverse, and then two more to prove the compositions are identity maps).

Example 17.5. If $M = \mathbb{Z}^{>0}$ under multiplication, then the group completion is $\mathbb{Q}^{>0}$ under multiplication.

Example 17.6. If $M = \mathbb{Z}^{\geq 0}$ under multiplication, then the group completion $(|M|, i)$ is the trivial group. For there exists $x \in |M|$ such that $x \cdot i(0) = 1$, and so for any $n \in M$ we have

$$i(n) = \bigl(x \cdot i(0)\bigr) \cdot i(n) = x \cdot \bigl(i(0) \cdot i(n)\bigr) = x \cdot i(0 \cdot n) = x \cdot i(0) = 1. \tag{17.7}$$

Now apply *uniqueness* of the factorization.

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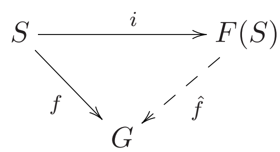
(17.8) *Free groups.* We recall that given a set S, there is a free group $F(S)$ generated by S, and it too is characterized by a universal property. In this setup a free group $\big(F(S), i\big)$ is a pair consisting of a group $F(S)$ and a map $i\colon S \to F(S)$ of sets such that for any group G and any map $f\colon S \to G$ of sets, there exists a unique homomorphism of groups $\tilde{f}\colon F(S) \to G$ which makes the diagram

(17.9)



commute. Again uniqueness follows immediately. Existence is something you must have seen when discussing van Kampen's theorem, for example. See [H1].

(17.10) *Construction of the group completion.* Now we prove that a group completion of a monoid M exists. Let $\big(F(M), i\big)$ be a free group on the set underlying M. Define N as the normal subgroup of $F(M)$ generated by elements

$$i(x_1x_2)i(x_2)^{-1}i(x_1)^{-1}, \qquad x_1, x_2 \in M. \tag{17.11}$$

Given a homomorphism $f\colon M \to G$ of monoids, for all $x_1, x_2 \in M$ we have $f(x_1x_2) = f(x_1)f(x_2)$. But $f = \hat{f}i$ from (17.9), and so it follows that

$$\hat{f}\big(i(x_1x_2)i(x_2)^{-1}i(x_1)^{-1}\big) = 1, \tag{17.12}$$

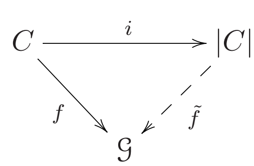
whence $\hat{f}$ factors down to a unique homomorphism $\tilde{f}\colon F(M)/N \to G$.

Exercise 17.13. This last step uses a universal property which characterizes a quotient group. What is that universal property?

The groupoid completion of a category

Definition 17.14. Let C be a category. A *groupoid completion* $(|C|, i)$ of C is a groupoid $|C|$ and a homomorphism $i\colon C \to |C|$ of monoids which satisfies the following universal property: If $\mathcal{G}$ is a groupoid and $f\colon C \to \mathcal{G}$ a functor, then there exists a unique map $\tilde{f}\colon |C| \to \mathcal{G}$ which makes the diagram

(17.15)



commute.

Intuitively, $|C|$ is obtained from C by "inverting all of the arrows", much in the same way that the group completion of a monoid is constructed. In fact, notice that if C has one object, then Definition 17.14 reduces to Definition 17.2.

We give some examples below; see Theorem 17.41.

(17.16) *Uniqueness of $\tilde{f}$.* There is a choice whether to require that $\tilde{f}$ in (17.15) be unique. If so, then you should show that $(|C|, i)$ is unique up to unique isomorphism. We do make that choice. It has a consequence that the map i is an isomorphism $i_0\colon C_0 \to |C|_0$ on objects. For let $\mathcal{G}$ be the groupoid with objects $\mathcal{G}_0 = C_0$ and with a unique morphism between any two objects, so the set of morphisms is $\mathcal{G}_1 = C_0 \times C_0$. There is a unique functor $f\colon C \to \mathcal{G}$ which is the identity on objects, and applying the universal property we deduce that $i_0\colon C_0 \to |C|_0$ is injective. If there exists $y \in |C|_0$ not in the image of i_0, then we argue as follows. Let $\mathcal{G}'$ be the groupoid with two objects a, b and a unique morphisms between any two objects. Let $f\colon C \to \mathcal{G}'$ be the functor which sends all objects to a and all morphisms to id_a. Then the factorization $\tilde{f}$ cannot be unique. For if $\tilde{f}(y) = a$, then define a new factorization with $y \mapsto b$ and adjust all morphisms starting or ending at y accordingly.

(17.17) *Sketch of a construction for $|C|$.* Here is a sketch of the existence proof for $|C|$, which follows the argument in **(17.8)**, **(17.10)**. In the next lecture we give a proof using topology. Briefly, given sets C_0, C_1 and maps $s, t\colon C_1, C_0$, there is a *free groupoid* $F(C_0, C_1)$ generated. It has the set C_0 of objects. Let C_1' be the set C_1 equipped with maps $s' = t\colon C_1' \to C_0$, $t' = s\colon C_1' \to C_0$. They are formal inverses of the arrows in C_1. Then a morphism in $F(C_0, C_1)$ is a formal string of composable elements in $C_1 \amalg C_1'$. The composition and inverse operations in $F(C_0, C_1)$ are by amalgamation and order-reversal. If now C is a category, then we take the quotient of $F(C_0, C_1)$ which keeps the same objects C_0 and for every pair of composable arrows g, f in C_1 identifies the amagamation gf with the composition $g \circ f$ in $F(C_0, C_1)$. I didn't try to work out the details of this quotient construction.

Invertibility in symmetric monoidal categories

The following should be compared with Definition 15.19(i).

Definition 17.18. Let C be a symmetric monoidal category and $y \in C$. Then *invertibility data* for y is a pair (y', θ) consisting of $y' \in C$ and an isomorphism $\theta\colon 1_C \to y \otimes y'$. If invertibility data exists, then we say that y is *invertible*.

There is a category of invertibility data, and it is a contractible groupoid (Definition 15.24). So an inverse to y, if it exists, is unique up to unique isomorphism. We denote any choice of inverse as y^{-1}. Note that the set of invertible objects is closed under the tensor product and it contains the unit object 1_C.

Remark 17.19. An object $y \in Y$ is invertible if and only if the functor $y \otimes -\colon C \to C$ is an equivalence.

Example 17.20. Let $C = \mathrm{Vect}_k$ be the category of vector spaces over a field k with symmetric monoidal structure the tensor product. Then $V \in \mathrm{Vect}_k$ is invertible if and only if $\dim V = 1$. In this case we say V is a *line*.

Lemma 17.21. *Let C be a symmetric monoidal category.*

(i) *If $y \in C$ is invertible, then y is dualizable and y^{-1} is a dual object.*

(ii) *If C is a symmetric monoidal* groupoid *and $y \in C$ is dualizable, then y is invertible with inverse $y^\vee$.*

Proof. Part (ii) is trivial as the coevaluation $c\colon 1_C \to y \otimes y^\vee$ is invertible. For (i) we let $\theta\colon 1_C \to y \otimes y^{-1}$ be coevaluation and evaluation is, up to multiplication by an element of $C(1_C, 1_C)$, the composition

(17.22) $$y^{-1} \otimes y \xrightarrow{\ \sigma\ } y \otimes y^{-1} \xrightarrow{\ \theta^{-1}\ } 1_C,$$

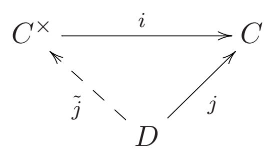
where σ is the symmetry of the symmetric monoidal structure. We leave the details to a homework problem. □

Definition 17.23. A *Picard groupoid* is a symmetric monoidal category in which all objects and morphisms are invertible.

Example 17.24. Given a field k, there is a Picard groupoid Line_k whose objects are k-lines and whose morphisms are isomorphisms of k-lines. Given a space X, there are Picard groupoids $\mathrm{Line}_{\mathbb{R}}(X)$ and $\mathrm{Line}_{\mathbb{C}}(X)$ of line bundles over X.

Definition 17.25. Let C be a symmetric monoidal category. An *underlying Picard groupoid* is a pair $(C^\times, i)$ consisting of a Picard groupoid $C^\times$ and a functor $i\colon C^\times \to C$ which satisfies the universal property: If D is any Picard groupoid and $j\colon D \to C$ a symmetric monoidal functor, then there exists a unique $\tilde{j}\colon D \to C^\times$ such that the diagram

(17.26)



commutes.

We obtain $C^\times$ from C by discarding all non-invertible objects and non-invertible morphisms. Recall (Definition 16.5) that C contains a subgroupoid $C^\sim \subset C$ of units, obtained by discarding all non-invertible morphisms. So we have $C^\times \subset C^\sim \subset C$.

(17.27) *Invariants of a Picard groupoid.* Associated to a Picard groupoid D are abelian groups $\pi_0 D$, $\pi_1 D$ and a *k-invariant*

(17.28) $$\pi_0 D \otimes \mathbb{Z}/2\mathbb{Z} \to \pi_1 D.$$

Define objects $y_0, y_1 \in D$ to be equivalent if there exists a morphism $y_0 \to y_1$. Then $\pi_0 D$ is the set of equivalence classes. The group law is given by the monoidal structure $\otimes$, and we obtain an abelian group since $\otimes$ is symmetric. Define $\pi_1 D = D(1_D, 1_D)$ as the automorphism group of the tensor unit. If $y \in D$ then there is an isomorphism

(17.29) $$- \otimes \mathrm{id}_y\colon\ \mathrm{Aut}(1_D) \longrightarrow \mathrm{Aut}(y)$$

where we write $\mathrm{Aut}(y) = D(y, y)$. The k-invariant on y is the symmetry $\sigma\colon y \otimes y \to y \otimes y$, which is an element of $\mathrm{Aut}(y \otimes y) \cong \mathrm{Aut}(1_D) = \pi_1 D$. We leave the reader to verify that this determines a homomorphism $\pi_0 D \otimes \mathbb{Z}/2\mathbb{Z} \to \pi_1 D$.

Invertible TQFTs

We distinguish the special subset of *invertible* topological quantum field theories.

Definition 17.30. Fix a nonnegative integer n, a tangential structure $\mathcal{X}(n)$, and a symmetric monoidal category C. Then a topological quantum field theory $\alpha\colon \mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle} \to C$ is *invertible* if it factors through the underlying Picard groupoid of C:

(17.31)
$$\begin{array}{ccc} \mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle} & \xrightarrow{\ \alpha\ } & C \\ & \searrow & \nearrow \\ & C^{\times} & \end{array}$$

If α is invertible, it follows from the universal property of the groupoid completion (Definition 17.14) that there is a factorization

(17.32)
$$\begin{array}{ccc} \mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle} & \xrightarrow{\ \alpha\ } & C \\ \downarrow & & \uparrow \\ |\,\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}\,| & \xrightarrow{\ \tilde{\alpha}\ } & C^{\times} \end{array}$$

We will identify the invertible theory with the map $\tilde{\alpha}$ (and probably omit the tilde.) In the next lecture we will see that $\tilde{\alpha}$ can be identified with a map of spectra.

Lemma 17.33. *The groupoid completion* $|\,\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}\,|$ *of a bordism category is a Picard groupoid.*

Proof. By Theorem 15.29 an object of $\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}$ is dualizable, so by Lemma 17.21(ii) it is also invertible. □

Remark 17.34. A TQFT α is invertible if and only if it is an invertible object in the symmetric monoidal category (15.9) of TQFTs.

(17.35) *Super vector spaces.* We introduce the symmetric monoidal category of *super vector spaces*. For more detail on superalgebra I recommend [DeM]. The word 'super' is a synonym for '$\mathbb{Z}/2\mathbb{Z}$-graded'. A *super vector space* is a pair (V, ϵ) consisting of a vector space (over a field k of characteristic not equal[1] to 2) and an endomorphism $\epsilon\colon V \to V$ such that $\epsilon^2 = \mathrm{id}_V$. The $\pm$-eigenspaces of ϵ provide a decomposition $V = V^0 \oplus V^1$; elements of the subspace V^0 are called *even* and elements of the subspace V^1 are called *odd*. A morphism $(V, \epsilon) \to (V', \epsilon')$ is a linear map $T\colon V \to V'$ such that $T \circ \epsilon = \epsilon' \circ T$. It follows that T maps even elements to even elements and odd elements to odd elements. The monoidal structure is defined as

(17.36)
$$(V_1, \epsilon_1) \otimes (V_2, \epsilon_2) = (V_1 \otimes V_2, \epsilon_1 \otimes \epsilon_2)$$

[1]We can give a different description in that case.

What is novel is the symmetry σ. If $v\in V$ is a homogeneous element, define its *parity* $|v|\in\{0,1\}$ so that $v\in V^{|v|}$. Then for homogeneous elements $v_i\in V_i$ the symmetry is

$$\begin{aligned}\sigma\colon (V_1,\epsilon_1)\otimes(V_2,\epsilon_2) &\longrightarrow (V_2,\epsilon_2)\otimes(V_1,\epsilon_1)\\ v_1\otimes v_2 &\longmapsto (-1)^{|v_1|\,|v_2|}\,v_2\otimes v_1\end{aligned} \tag{17.37}$$

This is called the *Koszul sign rule*. Let $s\mathrm{Vect}_k$ denote the symmetric monoidal category of super vector spaces. The obvious forgetful functor $s\mathrm{Vect}_k\to\mathrm{Vect}_k$ is *not* a symmetric monoidal functor, though it is a monoidal functor.

(17.38) *Example of an invertible field theory.* According to Theorem 16.10 to define an oriented one-dimensional TQFT

$$\alpha\colon\ \mathrm{Bord}^{SO}_{\langle 0,1\rangle}\to s\mathrm{Vect}_k \tag{17.39}$$

we need only specify $\alpha(\mathrm{pt}_+)$. We let it be the *odd line* $(k,-1)$ whose underlying vector space is the trivial line k (the field as a one-dimensional vector space) viewed as odd: the endomorphism ϵ is multiplication by -1. We leave as a homework problem to prove that α is invertible and that $\alpha(S^1)=-1$.

The groupoid completion of one-dimensional bordism categories

Of course, by Lemma 17.33 the groupoid completion $|B|$ of a bordism category B is a Picard groupoid, so has invariants described in **(17.27)**. We compute them for the bordism categories

$$\begin{aligned}B &= \mathrm{Bord}_{\langle 0,1\rangle}\\ B^{SO} &= \mathrm{Bord}^{SO}_{\langle 0,1\rangle}\end{aligned} \tag{17.40}$$

Theorem 17.41. *For the group completion of the unoriented bordism category*

$$\pi_0|B|\cong\mathbb{Z}/2\mathbb{Z},\qquad \pi_1|B|=0 \tag{17.42}$$

and for the group completion of the oriented bordism category

$$\pi_0|B^{SO}|\cong\mathbb{Z},\qquad \pi_1|B^{SO}|\cong\mathbb{Z}/2\mathbb{Z}, \tag{17.43}$$

with nontrivial k-invariant.

Proof. The arguments for π_0 are straightforward and amount to Proposition 1.31 and the assertion $\Omega_0^{SO}\cong\mathbb{Z}$.

To compute $\pi_1 B$ we argue as follows. First, $1_B=\emptyset^0$ is the empty 0-manifold, so $\mathrm{End}(1_B)=B(1_B,1_B)$ consists of diffeomorphism classes of closed 1-manifolds. Therefore, there is an isomorphism of commutative monoids $\mathrm{End}(1_B)\cong\mathbb{Z}^{\geq 0}$ which counts the number of components of a

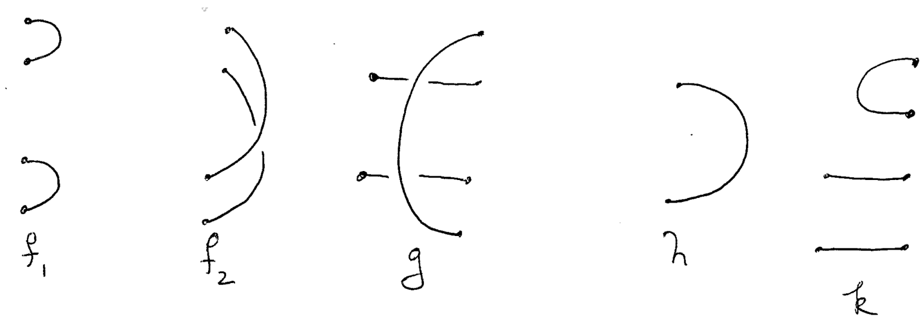


FIGURE 30. Some unoriented 1-dimensional bordisms

bordism X. Let X_n denote the disjoint union of n circles. Then using the bordisms defined in Figure 30 we have

(17.44) $$f_1 \circ g = f_2 \circ g = h$$

as morphisms in B. In the groupoid completion $|B|$ we can compose on the right with the inverse to g to conclude that $i(f_1) = i(f_2)$, where $i\colon B \to |B|$. That implies that in $|B|$ we have

(17.45) $$i(X_1) \circ i(h) = i(f_1) \circ i(k) = i(f_2) \circ i(k) = i(h),$$

whence $i(X_1) = i(\emptyset^0) = 1_{|B|}$. It remains to show that every morphism $\emptyset^0 \to \emptyset^0$ in $|B|$ is equivalent to a union of circles and their formal inverses. Observe first that the inverse of the "right elbow" h is the "left elbow", since their composition in one order is the circle, which is equivalent to the identity map. Next, any morphism $\emptyset^0 \to \emptyset^0$ in $|B|$ is the composition of a finite number of morphisms $Y_{2k} \to Y_{2\ell}$ and inverses of such morphisms, where Y_n is the 0-manifold consisting of n points. Furthermore, each such morphism is the disjoint union of circles, identities, right elbows, left elbows, and their inverses. Identities are self-inverse and the elbows are each other's inverse, hence carrying out the compositions of elbows and identities we obtain a union of circles and their inverses, as desired. This proves $\pi_1|B|$ is the abelian group with a single element.

To compute $\pi_1|B^{SO}|$ we make a similar argument using the bordisms in Figure 31. Let $\widetilde{X}_n$ be the disjoint union of n *oriented* circles. Note that the circle has a unique orientation up to orientation-preserving diffeomorphism. In this case we conclude that $i(\widetilde{X}_2) = i(\emptyset^0) = 1_{|B^{SO}|}$. To rule out the possibility that $i(\widetilde{X}_1)$ is also the tensor unit, we use the TQFT in **(17.38)**. It maps the oriented circle $\widetilde{X}_1$ to a non-tensor unit (which necessarily has order two).

Figure 32 illustrates the computation of the k-invariant (17.28) of $|B^{SO}|$. The nontrivial element of $\pi_0|B^{SO}|$ is represented by pt_+, and the top part of the diagram is the symmetry $\sigma\colon \mathrm{pt}_+ \amalg \mathrm{pt}_+ \to \mathrm{pt}_+ \amalg \mathrm{pt}_+$ in B^{SO}. We then tensor with the identity on the inverse of pt_+, which is pt_-; that is represented by the disjoint union of the top and bottom four strands. The left and right ends implement the isomorphism $1_{|B^{SO}|} \cong \mathrm{pt}_+ \amalg \mathrm{pt}_+ \amalg \mathrm{pt}_- \amalg \mathrm{pt}_-$. The result is the oriented circle $\widetilde{X}_1$, which is the generator of $\pi_1|B^{SO}|$. ∎

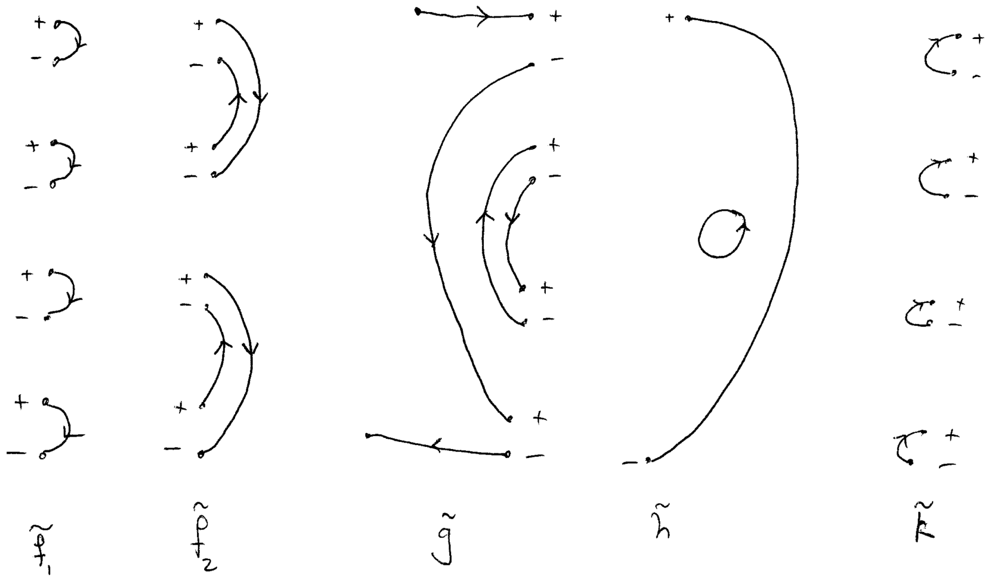


FIGURE 31. Some oriented 1-dimensional bordisms

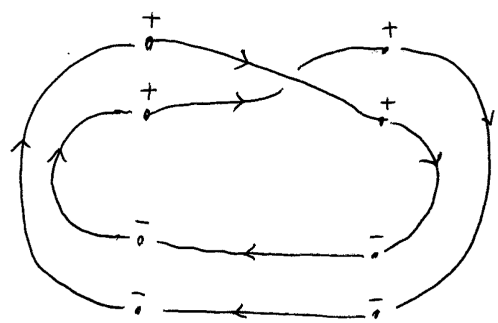


FIGURE 32. The k-invariant of $|\,\mathrm{Bord}^{SO}_{\langle 0,1\rangle}\,|$

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