Lecture 18: Groupoids and spaces

The simplest algebraic invariant of a topological space T is the set $\pi_0 T$ of path components. The next simplest invariant, which encodes more of the topology, is the fundamental groupoid $\pi_{\le 1} T$. In this lecture we see how to go in the other direction. There is nothing to say for a set T: it is already a discrete topological space. If $\mathcal{G}$ is a groupoid, then we can ask to construct a space $B\mathcal{G}$ whose fundamental groupoid $\pi_{\le 1} B\mathcal{G}$ is equivalent to $\mathcal{G}$. We give such a construction in this section. More generally, for a category C we construct a space BC whose fundamental groupoid $\pi_{\le 1} BC$ is equivalent to the groupoid completion (Definition 17.14) of C. The space BC is called the *classifying space* of the category C. As we will see in the next lecture, if $T_\bullet$ is a spectrum, then its fundamental groupoid $\pi_{\le 1} T_\bullet$ is a Picard groupoid, and conversely the classifying space of a Picard groupoid is a spectrum.

As an intermediate between categories and spaces we introduce *simplicial sets*. These are combinatorial models for spaces, and are familiar in some guise from the first course in topology. We only give a brief introduction and refer to the literature—e.g. [S2, Fr] for details. One important generalization is that we allow *spaces* of simplices rather than simply discrete sets of simplices. In other words, we also consider *simplicial spaces*. This leads naturally to *topological categories*, which we also introduce in this lecture.

In subsequent lectures we will apply these ideas to bordism categories. Lemma 17.33 asserts that the groupoid completion of a bordism category is a Picard groupoid, and we can ask to identify its classifying spectrum. To make the problem more interesting we will yet again extract from smooth manifolds and bordism a more intricate algebraic invariant: a topological category.

Simplices

Let S be a nonempty finite ordered set. For example, we have the set

$$[n] = \{0, 1, 2, \dots, n\} \tag{18.1}$$

with the given total order. Any S is uniquely isomorphic to $[n]$, where the cardinality of S is $n+1$. Let $A(S)$ be the affine space generated by S and $\Sigma(S) \subset A(S)$ the simplex with vertex set S. So if $S = \{s_1, s_1, \dots, s_n\}$, then $A(S)$ consists of formal sums

$$p = t^0 s_0 + t^1 s_1 + \cdots + t^n s_n, \qquad t^i \in \mathbb{R}, \quad t^0 + t^1 + \cdots + t^n = 1, \tag{18.2}$$

and $\Sigma(S)$ consists of those sums with $t^i \ge 0$. We write $\mathbb{A}^n = A([n])$ and $\Delta^n = \Sigma([n])$.

Let Δ be the category whose objects are nonempty finite ordered sets and whose morphisms are order-preserving maps (which may be neither injective nor surjective). The category Δ is generated by the morphisms

$$[0] \rightleftarrows [1] \rightleftarrows [2] \ \cdots \tag{18.3}$$

Bordism: Old and New (M392C, Fall '12), Dan Freed, November 9, 2012 where the right-pointing maps are injective and the left-pointing maps are surjective. For example, the map $d_i\colon [1] \to [2]$, $i = 0, 1, 2$ is the unique injective order-preserving map which does not contain $i \in [2]$ in its image. The map $s_i\colon [2] \to [1]$, $i = 0, 1$, is the unique surjective order-preserving map for which $s_i^{-1}(i)$ has two elements. Any morphism in Δ is a composition of the maps d_i, s_i and identity maps.

Each object $S \in \Delta$ determines a simplex $\Sigma(S)$, as defined above. This assignment extends to a functor

$$\Sigma\colon S \longrightarrow \mathrm{Top} \tag{18.4}$$

to the category of topological spaces and continuous maps. A morphism $\theta\colon S_0 \to S_1$ maps to the affine extension $\theta_*\colon \Sigma(S_0) \to \Sigma(S_1)$ of the map θ on vertices.

Simplicial sets and their geometric realizations

Recall the definition (**13.7**) of a category.

Definition 18.5. Let C be a category. The *opposite category* C^{op} is defined by

$$C_0^{\mathrm{op}} = C_0, \quad C_1^{\mathrm{op}} = C_1, \quad s^{\mathrm{op}} = t, \quad t^{\mathrm{op}} = s, \quad i^{\mathrm{op}} = i, \tag{18.6}$$

and the composition law is reversed: $g^{\mathrm{op}} \circ f^{\mathrm{op}} = (f \circ g)^{\mathrm{op}}$.

Here recall that C_0 is the set of objects, C_1 the set of morphisms, and $s, t\colon C_1 \to C_0$ the source and target maps.

The following definition is slick, and at first encounter needs unpacking (see [Fr], for example).

Definition 18.7. A *simplicial set* is a functor

$$X\colon \Delta^{\mathrm{op}} \longrightarrow \mathrm{Set} \tag{18.8}$$

It suffices to specify the sets $X_n = X([n])$ and the basic maps (18.3) between them. Thus we obtain a diagram

$$X_0 \;\substack{\longleftarrow \\ \dashrightarrow \\ \longleftarrow}\; X_1 \;\substack{\longleftarrow \\ \dashrightarrow \\ \longleftarrow \\ \dashrightarrow \\ \longleftarrow}\; X_2 \;\cdots \tag{18.9}$$

We label the maps d_i and s_i as before. The d_i are called *face maps* and the s_i *degeneracy maps*. The set X_n is a set of abstract simplices. An element of X_n is degenerate if it lies in the image of some s_i.

The morphisms in an abstract simplicial set are gluing instructions for concrete simplices.

Definition 18.10. Let $X\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ be a simplicial set. The *geometric realization* is the topological space $|X|$ obtained as the quotient of the disjoint union

$$\coprod_S X(S) \times \Sigma(S) \tag{18.11}$$

by the equivalence relation

$$(\sigma_1, \theta_* p_0) \sim (\theta^* \sigma_1, p_0), \qquad \theta\colon S_0 \to S_1, \quad \sigma_1 \in X(S_1), \quad p_0 \in \Sigma(S_0). \tag{18.12}$$

The map $\theta_* = \Sigma(\theta)$ is defined after (18.4) and $\theta^* = X(\theta)$ is part of the data of the simplicial set X. Alternatively, the geometric realization map be computed from (18.9) as

$$\coprod_n X_n \times \Delta^n \;/\!\sim, \tag{18.13}$$

where the equivalence relation is generated by the face and degeneracy maps.

Remark 18.14. The geometric realization is a CW complex.

Examples

Example 18.15. Let X be a simplicial set whose nondegenerate simplices are

$$X_0 = \{A, B, C, D\}, \qquad X_1 = \{a, b, c, d\}. \tag{18.16}$$

The face maps are as indicated in Figure 33. For example $d_0(a) = B$, $d_1(a) = A$, etc. (This requires a choice not depicted in Figure 33.) The level 0 and 1 subset of the disjoint union (18.13) is pictured in Figure 34. The 1-simplices a, b, c, d glue to the 0-simplices A, B, C, D to give the space depicted in Figure 33. The red 1-simplices labeled A, B, C, D are degenerate, and they collapse under the equivalence relation (18.12) applied to the degeneracy map s_0.

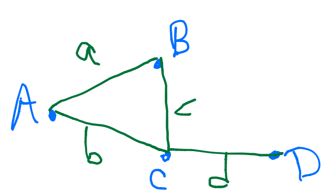

Figure 33. The geometric realization of a simplicial set

Example 18.17. Let T be a topological space. Then there is a simplicial set $\mathrm{Sing}\, T$ of singular simplices, defined by

$$(\mathrm{Sing}\, T)(S) = \mathrm{Top}\big(\Sigma(S), T\big), \tag{18.18}$$

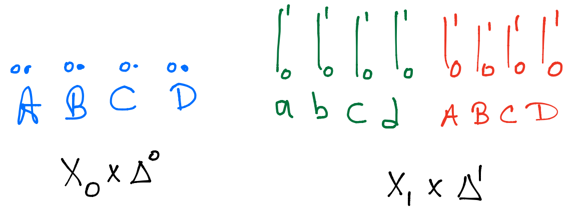

FIGURE 34. Gluing the simplicial set

where $\mathrm{Top}\bigl(\Sigma(S),T\bigr)$ is the set of continuous maps $\Sigma(S)\to T$, i.e., the set of singular simplices with vertex set S. The boundary maps are the usual ones. The evaluation map

$$(\mathrm{Sing}\,T)(S)\times\Sigma(S)=\mathrm{Top}\bigl(\Sigma(S),T\bigr)\times\Sigma(S)\longrightarrow T, \tag{18.19}$$

passes through the equivalence relation to induce a continuous map

$$|\,\mathrm{Sing}\,T|\longrightarrow T. \tag{18.20}$$

A basic theorem in the subject asserts that this map is a weak homotopy equivalence.

Categories and simplicial sets

(18.21) *The nerve.* Let C be a category, which in part is encoded in the diagram

$$C_0 \;\substack{\longleftarrow \\ \dashrightarrow \\ \longleftarrow}\; C_1 \tag{18.22}$$

The solid left-pointing arrows are the source s and target t of a morphism; the dashed right-pointing arrow i assigns the identity map to each object. This looks like the start of a simplicial set, and indeed there is a simplicial set NC, the *nerve* of the category C, which begins precisely this way: $NC_0=C_0$, $NC_1=C_1$, $d_0=t$, $d_1=s$, and $s_0=i$. A slick definition runs like this: a finite nonempty ordered set S determines a category with objects S and a unique arrow $s\to s'$ if $s\le s'$ in the order. Then

$$NC(S)=\mathrm{Fun}(S,C) \tag{18.23}$$

where $\mathrm{Fun}(-,-)$ denotes the set of functors. As is clear from Figure 35, $NC([n])$ consists of sets of n composable arrows in C. The degeneracy maps in NC insert an identity morphism. The face map d_i omits the i^{th} vertex and composes the morphisms at that spot; if i is an endpoint $i=0$ or $i=n$, then d_i omits one of the morphisms.

Example 18.24. Let M be a monoid, regarded as a category with a single object. Then

$$NM_n=M^{\times n}. \tag{18.25}$$

It is a good exercise to write out the face maps.

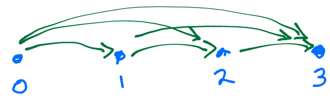

Figure 35. A totally ordered set as a category

Definition 18.26. Let C be a category. The *classifying space* BC of C is the geometric realization $|NC|$ of the nerve of C.

Remark 18.27. A homework exercise will explain the nomenclature 'classifying space'.

Example 18.28. Suppose $G = \mathbb{Z}/2\mathbb{Z}$ is the cyclic group of order two, viewed as a category with one object. Then NG_n has a single nondegenerate simplex $(g, \dots, g)$ for each n, where $g \in \mathbb{Z}/2\mathbb{Z}$ is the non-identity element. So BG is glued together with a single simplex in each dimension. We leave the reader to verify that in fact $BG \simeq \mathbb{RP}^\infty$.

Theorem 18.29. *Let M be a monoid. Then $\pi_1 BM$ is the group completion of M.*

The nerve NM has a single 0-simplex, which is the basepoint of BM.

Proof. The fundamental group of a CW complex B is computed from its 2-skeleton B^2. Assuming there is a single 0-cell, the 1-skeleton is a wedge of circles, so its fundamental group is a free group F. The homotopy class of the attaching map $S^1 \to B^1$ of a 2-cell is a word in F, and the fundamental group of B is the quotient F/N, where N is the normal subgroup generated by the words of the attaching maps of 2-cells. For $B = BM$ the set of 1-cells is M, so $\pi_1 BM^1 \cong F(M)$ is the free group generated by the set M. The homotopy class of the 2-cell (m_1, m_2) is the word $(m_1m_2)m_2^{-1}m_1^{-1}$. By (17.10) the quotient $F(M)/N$ is the group completion of M. □

We next prove an important proposition [S2].

Proposition 18.30. *Let $F, G\colon C \to D$ be functors and $\eta\colon F \to G$ a natural transformation. Then the induced maps $|F|, |G|\colon |C| \to |D|$ on the geometric realizations are homotopic.*

Proof. Consider the ordered set [1] as a category, as in Figure 35. Its classifying space is homeomorphic to the closed interval $[0, 1]$. Define a functor $H\colon [1] \times C \to D$ which on objects of the form $(0, -)$ is equal to F, on objects of the form $(1, -)$ is equal to G, and which maps the unique morphism $(0 \to 1)$ to the natural transformation η. Then $|H|\colon [0, 1] \times |C| \to |D|$ is the desired homotopy. □

Remark 18.31. The proof implicitly uses that the classifying space of a Cartesian product of categories is the Cartesian product of the classifying spaces. That is not strictly true in general; see [S2] for discussion.

Proposition 18.32. *Let $\mathcal{G}$ be a groupoid. Then the natural functor $i_{\mathcal{G}}\colon \mathcal{G} \to \pi_{\le 1} B\mathcal{G}$ is an equivalence of groupoids.*

The objects of $\mathcal{G}$ are the 0-skeleton of $B\mathcal{G}$, and $i_{\mathcal{G}}$ is the inclusion of the 0-skeleton on objects. The 1-cells of $B\mathcal{G}$ are indexed by the morphisms of $\mathcal{G}$, and imposing a standard parametrization we obtain the desired map $i_{\mathcal{G}}$.

Proof. Any groupoid $\mathcal{G}$ is equivalent to a disjoint union of groups. To construct an equivalence choose a section of the quotient map $\mathcal{G}_0 \to \pi_0\mathcal{G}$ and take the disjoint union of the automorphism groups of the objects in the image of that section.

$\square$

Corollary 18.33. *Let C be a category. The fundamental groupoid $\pi_{\le 1}BC$ is equivalent to the groupoid completion of C.*

Proof. As explained after the statement of Proposition 18.32 there is a natural map $C \xrightarrow{i_C} \pi_{\le 1}BC$. We check the universal property (17.15). Suppose $f\colon C \to \mathcal{G}$ is a functor from C to a groupoid. There is an induced continuous maps $Bf\colon BC \to B\mathcal{G}$ and then an induced functor $\pi_{\le 1}Bf\colon \pi_{\le 1}BC \to \pi_{\le 1}B\mathcal{G}$ such that the diagram

$$\begin{array}{ccc} C & \xrightarrow{\;i_C\;} & \pi_{\le 1}BC \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\le 1}Bf} \\ \mathcal{G} & \xrightarrow{\;i_{\mathcal{G}}\;} & \pi_{\le 1}B\mathcal{G} \end{array} \tag{18.34}$$

By Proposition 18.32 the map $i_{\mathcal{G}}$ is an equivalence of groupoids, and composition with an inverse equivalence gives the required factorization $\tilde{f}$. $\square$

Remark 18.35. A *skeleton* of $\pi_{\le 1}BC$ is a groupoid completion as in Definition 17.14; its set of objects is isomorphic to C_0. There is a canonical skeleton: the full subcategory whose set of objects is $i_C(C_0)$.

Simplicial spaces and topological categories

A simplicial set describes a space—its geometric realization—as the gluing of a discrete set of simplices. However, we may also want to glue together a space from continuous families of simplices.

Definition 18.36. A *simplicial space* is a functor

$$X\colon \Delta^{\mathrm{op}} \longrightarrow \mathrm{Top} \tag{18.37}$$

More concretely, a simplicial space is a sequence $\{X_n\}$ of topological spaces with continuous face and degeneracy maps as in (18.9). The construction of the geometric realization (Definition 18.10) goes through verbatim.

We can also promote the sets and morphisms of a (discrete) category from sets to spaces.

Definition 18.38. A *topological category* consists of topological spaces C_0, C_1 and continuous maps

$$\begin{aligned} i\colon C_0 &\longrightarrow C_1 \\ s,t\colon C_1 &\longrightarrow C_0 \\ c\colon C_1 \times_{C_0} C_1 &\longrightarrow C_1 \end{aligned} \tag{18.39}$$

which satisfy the algebraic relations of a discrete category.

These are described following (13.8). Thus the partially defined composition law c is associative and $i(y)$ is an identity morphism with respect to the composition.

Example 18.40. Let M be a topological monoid. So M is both a monoid and a topological space, and the composition law $M \times M \to M$ is continuous. Then M may be regarded as a topological category with a single object.

Example 18.41. At the other extreme, a topological space T may be regarded as a topological category with only identity morphisms.

Example 18.42. There is a topological category whose objects are finite dimensional vector spaces and whose spaces of morphisms are spaces of linear maps (with the usual topology).

Example 18.43. Let M be a smooth manifold and G a Lie group. Then there is a topological category whose objects are principal G-bundles with connection and whose morphisms are flat bundle isomorphisms.

Definition 18.44. Let C be a topological category. Its nerve NC and classifying space BC are defined as in (**18.21**) and Definition 18.26, *verbatim*.

Notice that the nerve is a simplicial *space*.

References

[Fr] Greg Friedman, *Survey article: an elementary illustrated introduction to simplicial sets*, Rocky Mountain J. Math. **42** (2012), no. 2, 353–423, `arXiv:0809.4221`. 1, 2

[S2] Graeme Segal, *Classifying spaces and spectral sequences*, Inst. Hautes Études Sci. Publ. Math. (1968), no. 34, 105–112. 1, 5