Lecture 19: Γ-spaces and deloopings

To a topological category C we associate a topological space BC. We saw in (17.32) that an invertible field theory, defined on a discrete bordism category B, factors through the groupoid completion $|B|$ of B. Furthermore, by Corollary 18.33, the groupoid completion is the fundamental groupoid $|B|$ of the classifying space of B. In the next lecture we introduce *topological* bordism categories and a corresponding richer notion of a topological quantum field theory, with values in a symmetric monoidal topological category. In that case we will see that an *invertible* field theory factor through the classifying space of the topological bordism category. Now a topological bordism category has a symmetric monoidal structure, so we can ask what extra structure is reflected on the classifying space. In this lecture we will see that this extra structure is an infinite loop space structure. In other words, *the classifying space BC of a topological symmetric monoidal category is the 0-space of a prespectrum.* (Review Definition 10.2.)

There are many "delooping machines" which build the infinite loop space structure. Here we give an exposition of Segal's Γ-spaces [S2], though we use the observation of Anderson [A] that the opposite category Γ^{op} to Segal's category Γ is the category of finite pointed sets. Further accounts may be found in [BF] and [Sc]. So whereas in Lecture 18 we have the progression

(19.1) $$\text{Topological categories} \longrightarrow \text{Simplicial spaces} \longrightarrow \text{Spaces}$$

in this lecture we make a progression

(19.2) $$\text{Symmetric monoidal topological categories} \longrightarrow \Gamma\text{-spaces} \longrightarrow \text{Prespectra.}$$

In fact, we will only discuss a special type of symmetric monoidal structure, called a *permutative* structure, which is rigid in the sense that the associativity and identity maps (13.31) and (13.33) are equalities. Our treatment follows [Ma2]; see also [EM, §4].

Motivating example: commutative monoids

(19.3) *Segal's category.* Segal [S2] defined a category Γ whose opposite (Definition 18.5) is easier to work with.

Definition 19.4. Γ^{op} is the category whose objects are finite pointed sets and whose morphisms are maps of finite sets which preserve the basepoint.

Any finite pointed set is isomorphic to

(19.5) $$n^+ = \{*, 1, 2, \dots, n\}$$

for some $n \in \mathbb{Z}^{\geq 0}$. We also use the notation

(19.6) $$S^0 = 1^+ = \{*, 1\}.$$

There are also categories Set_*, Top_* of pointed sets and pointed topological spaces, and $\Gamma^{\text{op}} \subset \text{Set}_*$ is a subcategory.

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(19.7) *The* Γ*-set associated to a commutative monoid.* Let M be a commutative monoid, which we write additively. Forgetting the addition we are left with a pointed set $(M, 0)$. Define the functor

$$\begin{aligned} A_M\colon \Gamma^{\mathrm{op}} &\longrightarrow \mathrm{Set}_* \\ S &\longmapsto \mathrm{Set}_*(S, M) \end{aligned} \tag{19.8}$$

This defines A_M on objects: there is a canonical isomorphism $A_M(n^+) = M^{\times n}$. Note in particular that we recover the commutative monoid as

$$A_M(S^0) = M. \tag{19.9}$$

Given a map $(S_0 \xrightarrow{\theta} S_1) \in \Gamma^{\mathrm{op}}$, we must produce $\bigl(\mathrm{Set}_*(S_0, M) \xrightarrow{\theta_* = A_M(\theta)} \mathrm{Set}_*(S_1, M)\bigr)$. This is not composition, but rather is a "wrong-way map", or integration. It is defined as

$$\theta_*(\mu)(s_1) = \begin{cases} 0, & s_1 = *; \\ \sum\limits_{s_0 \in \theta^{-1}(s_1)} \mu(s_0), & s_1 \neq *, \end{cases} \tag{19.10}$$

where $\mu\colon S_0 \to M$ is a pointed map ($\mu(*) = 0$) and $s_1 \in S_1$. This pushforward map is illustrated in Figure 36. Note that the map $\alpha\colon 2^+ \to 1^+$ with $\alpha(1) = \alpha(2) = 1$ maps to addition $M^{\times 2} \to M$, and said addition is necessarily commutative and associative, which one proves by applying A_M to the commutative diagrams

$$\begin{array}{ccc} 2^+ & \xrightarrow{\ \tau\ } & 2^+ \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\alpha} & \\ & 1^+ & \end{array} \qquad\qquad \begin{array}{ccc} 3^+ & \longrightarrow & 2^+ \\ \downarrow & & \downarrow \\ 2^+ & \longrightarrow & 1^+ \end{array} \tag{19.11}$$

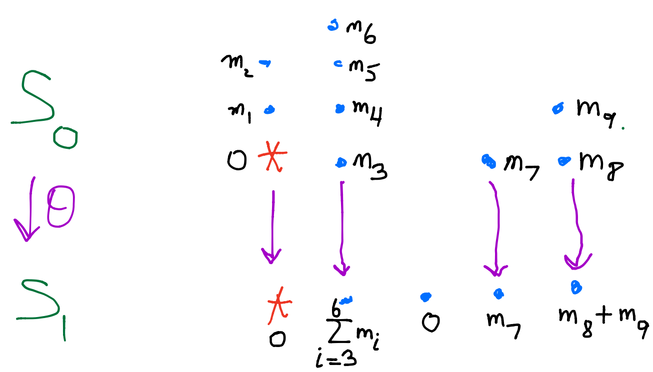

FIGURE 36. The pushforward θ_* associated to $\theta\colon S_0 \to S_1$

The functor A_M is a *special* Γ*-set.*

Definition 19.12.

(i) A Γ-*set* is a functor $A\colon \Gamma^{\mathrm{op}} \to \mathrm{Set}_*$ such that $A(\{*\}) = \{*\}$.

(ii) A is *special* if the natural map

$$A(S_1 \vee S_2) \longrightarrow A(S_1) \times A(S_2) \tag{19.13}$$

is an isomorphism of pointed sets.

In (i) the pointed set $\{*\} \in \Gamma^{\mathrm{op}} \subset \mathrm{Set}_*$ is the special object with a single point. A specification of this object makes Γ^{op} and Set_* into *pointed categories*, that is, categories with a distinguished object.[1] So the requirement in (i) is that A be a pointed map of pointed categories. The map (19.13) is induced from the collapse maps

$$S_1 \vee S_2 \longrightarrow S_1 \quad \text{and} \quad S_1 \vee S_2 \longrightarrow S_2. \tag{19.14}$$

Remark 19.15. For any category C a functor $C^{\mathrm{op}} \to \mathrm{Set}$ is called a *presheaf* on C. So a special Γ-set is a pointed presheaf on Γ.

Remark 19.16. We view a (special) Γ-set A as a set $A(S^0)$ with extra structure. So for $A = A_M$ we have the set M in (19.9) with the extra structure of a basepoint $A(\{*\})$ and a commutative associative composition law $A(2^+ \xrightarrow{\alpha} 1^+)$. A similar picture holds for Γ-spaces below.

Example 19.17. A *representable* Γ-set is defined by $A(S) = \Gamma^{\mathrm{op}}(T, S)$ for some fixed $T \in \Gamma^{\mathrm{op}}$. Taking $T = S^0$ we have the special Γ-set

$$\mathbb{S}(S) = \Gamma^{\mathrm{op}}(S^0, S). \tag{19.18}$$

Notice that $\mathbb{S}(S^0) = S^0$, so that $\mathbb{S}$ is the set S^0 with extra structure. Spoiler alert![2]

At the end of the lecture we give a similar construction (a bit heuristic) in which we replace the commutative monoid M with a symmetric monoidal category C. In that case $\mu\colon S \to C$ assigns an object of C to each element of S and the addition in (19.10) is replaced by the tensor product in C.

Γ-spaces

It is a small leap to generalize Definition 19.12 to spaces. We just need to be careful to replace isomorphisms with weak homotopy equivalences.

Definition 19.19.

(i) A Γ-*space* is a functor $A\colon \Gamma^{\mathrm{op}} \to \mathrm{Top}_*$ such that $A(\{*\})$ is contractible.

[1] A standard definition of 'pointed category' also requires that for every object y there be a unique map $* \to y$ and a unique map $y \to *$. We do not make that requirement, though it is true here.

[2] The associated prespectrum is the sphere spectrum, after completing to a spectrum as in (**10.6**).

(ii) A is *special* if the natural map

$$A(S_1 \vee S_2) \longrightarrow A(S_1) \times A(S_2) \tag{19.20}$$

is a weak homotopy equivalence of pointed spaces.

Some authors require the stronger condition that $A(\{*\}) = \{*\}$.

Γ and Δ

Recall that Δ is the category of nonempty finite ordered sets and nondecreasing maps. Any object is isomorphic to

$$[n] = \{0 < 1 < 2 < \cdots < n\} \tag{19.21}$$

for some $n \in \mathbb{Z}^{\geq 0}$. We now define a functor

$$\kappa\colon \Delta^{\mathrm{op}} \longrightarrow \Gamma^{\mathrm{op}} \tag{19.22}$$

Composing with κ we obtain a functor from Γ-spaces to simplicial spaces (recall Definition 18.36).

(19.23) *Definition of* κ. The functor κ on objects is straightforward. If $S \in \Delta$ is a nonempty finite ordered set, let $* \in S$ be the minimum, and consider the pair $\kappa(S) = (S, *)$ as a finite pointed set, forgetting the ordering.

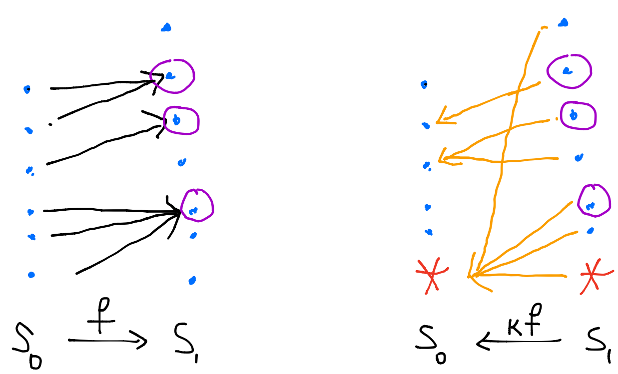

FIGURE 37. The functor $\Delta^{\mathrm{op}} \to \Gamma^{\mathrm{op}}$ on morphisms

What is trickier is the action of κ on morphisms. We illustrate the general definition in Figure 37. On the left is shown a non-decreasing map $f\colon S_0 \to S_1$ of finite ordered sets. The induced map $\kappa(f)$ of pointed sets maps in the opposite direction. We define it by moving in S_1 from the smallest to the largest element. The smallest element $* \in S_1$ necessarily maps to $* \in S_0$. For each successive element $s_1 \in S_1$ we find the minimal $s_1' \in f(S_0) \subset S_1$ such that $s_1' \geq s_1$; then define $\kappa f(s_))$ as the minimal element of $f^{-1}(s_1')$. Finally, if no element $s_1' \geq s_1$ is in the image of f, then set $\kappa f(s_1) = *$.

(19.24) *Motivation.* The category Δ is generated by injective/surjective= face/degeneracy maps, as depicted in (18.9). So let's see what κ does on face and degeneracy maps, and we go a step further and apply to the Γ-set A_M defined in **(19.7)**. We leave the reader to check that if $d\colon [n] \to [n+1]$ is the injective map which misses $i \in [n+1]$, then the induced face map $d^*\colon M^{\times(n+1)} \to M^{\times n}$ sends

$$(m_1, m_2, \dots m_{n+1}) \longmapsto (m_1, \dots, m_i + m_{i+1}, \dots, m_{n+1}), \tag{19.25}$$

where $m_j \in M$. Similarly, if $s\colon [n] \to [n-1]$ is the surjective map which sends both i and $i+1$ to the same element, then the induced degeneracy map $s^*\colon [n-1] \to [n]$ sends

$$(m_1, \dots, m_{n-1}) \longmapsto (m_1, \dots, 0, \dots, m_{n-1}), \tag{19.26}$$

where 0 is inserted in the i^{th} spot. These are the face and degeneracy maps of the nerve of the category with one object whose set of morphisms is M; see Example 18.24.

(19.27) *The realization of a Γ-space.* To a Γ-space A is associated a simplicial space $A \circ \kappa$ and so its geometric realization $|A \circ \kappa|$, a topological space. We simply use the notation $|A|$ for this space. Observe that $|A|$ is a *pointed* space. For the set of n-simplices is the pointed space $A(n^+)$, and its basepoint is the degenerate simplex built by successively applying degeneracy maps to the basepoint of $A(0^+)$. The basepoint of $A(0^+)$ gives a distinguished 0-simplex in the geometric realization (18.13), which is then the basepoint of $|A|$. We will now define additional structure on the geometric realization in the form of a Γ-space BA such that $BA(S^0) = |A|$.

The classifying space of a Γ-space

Definition 19.28. Let A be a Γ-space. Its *classifying space* BA is the Γ-space

$$BA(S) = |T \longmapsto A(S \wedge T)|\,. \tag{19.29}$$

The vertical bars denote the geometric realization of the simplicial space underlying a Γ-space; we prove in the lemma below that the map inside the vertical bars *is* a Γ-space. Note $S, T \in \Gamma^{\text{op}}$. Also, there is a canonical isomorphism

$$BA(S^0) = |A|, \tag{19.30}$$

and $BA(\{*\})$ is the basepoint of $|A|$.

Remark 19.31. There are modified geometric realizations of a simplicial space which have better technical properties; see the appendix to [S2]. Also, see [D] for another version of geometric realization. Depending on the realization, it may be that $BA(\{*\})$ is a contractible space which contains the basepoint of $|A|$.

Lemma 19.32. *Let A be a Γ-space and $S \in \Gamma^{\text{op}}$. Then $T \mapsto A(S \wedge T)$ is a Γ-space, special if A is special.*

Proof. Observe that $T \mapsto S \wedge T$ is a functor $\Gamma^{\mathrm{op}} \to \Gamma^{\mathrm{op}}$, and that $0^+ \mapsto S \wedge 0^+ = 0^+$. For the special statement, if $T_1, T_2 \in \Gamma^{\mathrm{op}}$, then

$$T_1 \vee T_2 \longmapsto S \wedge (T_1 \vee T_2) = (S \wedge T_1) \vee (S \wedge T_2). \tag{19.33}$$

Now use the special property of A and the fact that the realization of a product is the product of the realizations. □

The prespectrum associated to a Γ-space

(19.34) *Iteration.* Let A be a Γ-space. We iterate the classifying space construction to obtain a sequence

$$A,\ BA,\ B^2A,\ B^3A,\ \dots \tag{19.35}$$

of Γ-spaces, and so too a sequence

$$A(S^0),\ BA(S^0),\ B^2A(S^0),\ B^3A(S^0),\ \dots \tag{19.36}$$

of pointed topological spaces.

(19.37) *Prespectrum structure.* We define for any Γ-space A a continuous map

$$s\colon \Sigma(A(S^0)) \longrightarrow BA(S^0) = |A|. \tag{19.38}$$

Applying this to each space in (19.36) we obtain a prespectrum. The simplicial space associated to A has $A(0^+) = A(\{*\})$ as its space of 0-simplices. Assume for simplicity that $A(\{*\}) = *$; in any case $* \in A(\{*\})$ and the same construction applies. Now the geometric realization of a simplicial space has a natural filtration by subspaces; the q^{th} stage of the filtration is obtained by taking the disjoint union over $n = 0, 1, \dots, q$ in (18.13). Under the hypothesis just made on A, the 0^{th} stage of the filtration is a single point $*$. The 1^{st} stage of the filtration is obtained by gluing on $A(1^+) = A(S^0)$ using the two face maps and single degeneracy map. We leave the reader to check that we exactly obtain the (reduced) suspension $\Sigma(A(S^0))$. Hence the map s is the inclusion of the 1^{st} stage of the filtration of $|A|$.

(19.39) *Monoid structure on $\pi_0 A(S^0)$.* If A is a special Γ-space, then the composition

$$\Gamma^{\mathrm{op}} \xrightarrow{\ A\ } \mathrm{Top}_* \xrightarrow{\ \pi_0\ } \mathrm{Set}_* \tag{19.40}$$

is a special Γ-set. You will prove in the homework that the Γ-set structure gives $\pi_0 A(S^0)$ the structure of a commutative monoid.

Theorem 19.41 ([S2]). *If the commutative monoid $\pi_0 A(S^0)$ is an abelian group, then the adjoint*

$$t\colon A(S^0) \longrightarrow \Omega BA(S^0) \tag{19.42}$$

is a weak homotopy equivalence.

Corollary 19.43. *For $k > 0$ the space $B^k A(S^0)$ is weakly equivalent to $\Omega B^{k+1} A(S^0)$.*

Proof. For $B^k A(S^0)$ is the geometric realization of the Γ-space $B^{k-1}A$ which has a contractible space of 0-simplices, and therefore π_0 trivial. ☐

The necessity of the condition in Theorem 19.41 is clear. For if $A(S^0)$ is equivalent to a loop space, then the loop product (on π_0) has additive inverses: reverse the parametrization of the loop. A standard argument, which you encountered encountered in the second problem set, proves that the loop product is equal to the product given by the Γ-space structure. If $\pi_0 A(S^0)$ is an abelian group, then (19.36) is an Ω-prespectrum.

We do not provide a proof of Theorem 19.41 in this version of the notes.

Example 19.44. Let A be a discrete abelian group. The Ω-prespectrum (19.35) associated to the Γ-set (19.8) defined by A (viewed as a commutative monoid) is an *Eilenberg-MacLane spectrum*.

Example 19.45. The prespectrum associated to the Γ-set $\mathbb{S}$ is the *sphere spectrum*. (Better: the sphere spectrum is the completion of that Ω-prespectrum to a spectrum.)

Γ-categories

The next definition is analogous to Definition 19.19. Recall that a *pointed category* is a category with a distinguished object. The collection of (small) pointed categories forms a category Cat_*; morphisms are functors and we require associativity on the nose.[3]

Definition 19.46.

(i) A Γ-*category* is a functor $D\colon \Gamma^{\mathrm{op}} \to \mathrm{Cat}_*$ such that $D(\{*\})$ is equivalent to the trivial category with a single object and the identity morphism.

(ii) D is *special* if the natural map

$$D(S_1 \vee S_2) \longrightarrow D(S_1) \times D(S_2) \tag{19.47}$$

is an equivalence of pointed categories.

(19.48) *From Γ-categories to Γ-spaces and prespectra.* Let D be a Γ-category. Then composing with the classifying space construction $B\colon \mathrm{Cat}_* \to \mathrm{Top}_*$ we obtain a Γ-space BD and then a prespectrum whose 0-space is $B(D(S^0))$, the classifying space of the category $D(S^0)$.

[3] Categories are more naturally objects in a *2-category*. Namely, functors are like sets, and there is an extra layer of structure: natural transformations between functors. So it is rather rigid to demand that composition of functors be associative on the nose.

(19.49) *Permutative categories.* We would like to associate a Γ-category to a symmetric monoidal category, but we need to assume additional rigidity to do so. A theorem of Isbell [I] asserts that every symmetric monoidal category is equivalent to a permutative category, so this is not really a loss of generality. A permutative category is a symmetric monoidal category with a strict unit and strict associativity.

Definition 19.50. A *permutative category* is a quartet $(C, 1_C, \otimes, \sigma)$ consisting of a pointed category $(C, 1_C)$, a functor $\otimes\colon C \times C \to C$, and a natural transformation σ as in (13.32) such that for all $y, y_1, y_2, y_3 \in C$

(i) $1_C \otimes y = y \otimes 1_C = y$;
(ii) $(y_1 \otimes y_2) \otimes y_3 = y_1 \otimes (y_2 \otimes y_3)$;
(iii) the composition $y_1 \otimes y_2 \xrightarrow{\sigma} y_2 \otimes y_1 \xrightarrow{\sigma} y_1 \otimes y_2$ is the identity; and
(iv) the diagrams

(19.51)
$$\begin{array}{ccc} 1 \otimes y & \xrightarrow{\sigma} & y \otimes 1 \\ & {\scriptstyle =}\searrow \quad \swarrow{\scriptstyle =} & \\ & y & \end{array} \qquad\qquad \begin{array}{ccc} y_1 \otimes y_2 \otimes y_2 & \xrightarrow{\sigma} & y_3 \otimes y_1 \otimes y_2 \\ & {\scriptstyle 1\otimes\sigma}\searrow \quad \swarrow{\scriptstyle \sigma\otimes 1} & \\ & y_1 \otimes y_3 \otimes y_2 & \end{array}$$

commute.

Example 19.52. The category Γ^{op} of finite pointed sets has a permutative structure if we take a model in which the set of objects is precisely $\{n^+ : n \in \mathbb{Z}^{\ge 0}\}$. Then define ${n_1}^+ \otimes {n_2}^+ = (n_1 + n_2)^+$. The tensor unit is 0^+ and we leave the reader to define the symmetry σ.

(19.53) *The* Γ*-category associated to a permutative category.* As we said earlier, this construction is analogous to **(19.7)**. We give the basic definitions and leave to the reader the detailed verifications. Let C be a permutative category. We define an associated Γ-category D as follows. For $S \in \Gamma^{\mathrm{op}}$ a finite pointed set let $D(S)$ be the category whose objects are pairs (c, ρ) in which (i) $c(T) \in C$ for each pointed subset $T \subset S$ and (ii) the map

(19.54)
$$\rho(T_1, T_2)\colon c(T_1) \otimes c(T_2) \longrightarrow c(T_1 \vee T_2)$$

is an isomorphism for each pair of pointed subsets with $T_1 \cap T_2 = \{*\}$. These data must satisfy several conditions:

(i) $c(\{*\}) = 1_C$;
(ii) $\rho(\{*\}, T) = \mathrm{id}_T$ for all T; and
(iii) for all T_1, T_2, T_3 with correct intersections the diagrams

(19.55)
$$\begin{array}{ccc} c(T_1) \otimes c(T_2) & \xrightarrow{\rho(T_1,T_2)} & c(T_1 \vee T_2) \\ {\scriptstyle \sigma}\downarrow & & \Vert \\ c(T_2) \otimes c(T_1) & \xrightarrow{\rho(T_2,T_1)} & c(T_2 \vee T_1) \end{array}$$

and

$$c(T_1)\otimes c(T_2)\otimes c(T_3) \xrightarrow{\rho(T_1,T_2)\otimes \mathrm{id}} c(T_1\vee T_2)\otimes c(T_3)$$
$$\downarrow \mathrm{id}\otimes\rho(T_2,T_3) \qquad\qquad \downarrow \rho(T_1\vee T_2,T_3)$$
$$c(T_1)\otimes c(T_2\vee T_3) \xrightarrow{\rho(T_1,T_2\vee T_3)} c(T_1\vee T_2\vee T_3) \tag{19.56}$$

commute.

Exercise 19.57. Define a morphism $\bigl((c,\rho)\to(c',\rho')\bigr)\in D(S)$. The data is, for each pointed $T\subset S$, a morphism $\bigl(c(T)\to c'(T)\bigr)\in C$. What is the condition that these morphisms must satisfy?

This completes the definition of the category $D(S)$ associated to $S\in\Gamma^{\mathrm{op}}$. Now we must define a functor $D(S_0)\xrightarrow{\theta_*}D(S_1)$ for each morphism $(S_1\xrightarrow{\theta}S_1)\in\Gamma^{\mathrm{op}}$. The definition follows Figure 36. To streamline the notation, for $T\subset S_1$ a pointed subset define the modified inverse image to be the pointed subset

$$\widetilde{\theta^{-1}(T)} := \{*\}\cup\theta^{-1}\bigl(T\setminus\{*\}\bigr). \tag{19.58}$$

Now given $(c,\rho)\in D(S_0)$ define $(c',\rho')=\theta_*(c,\rho)$ by

$$c'(T)=c\bigl(\widetilde{\theta^{-1}(T)}\bigr) \tag{19.59}$$

We leave the reader to supply the definition of ρ' and of θ_* on morphisms.

Observe that there is a natural isomorphism of categories

$$D(S^0)\xrightarrow{\cong}C. \tag{19.60}$$

In this sense the Γ-category D is the category C with extra structure, which encodes its permutative structure.

Exercise 19.61. Work out the Γ-space associated to the permutative category of Example 19.52. How does it compare to $\mathbb{S}$?

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