Lecture 20: Topological bordism categories

We return to bordism and construct a more complicated "algebraic" invariant than the previous ones: a topological category. Of course, this is not purely algebraic, but rather a mix of algebra and topology. We begin with some preliminaries on the topology of function spaces. The Whitney theorem then gives a model of the classifying space of the diffeomorphism group of a compact manifold in terms of a space of embeddings. Then, following Galatius-Madsen-Tillmann-Weiss (GMTW), we construct the topological category of bordisms. We did not find a symmetric monoidal structure, though morally it should be there. It turns out that in any case the classifying space is the 0-space of a spectrum. For the bordism category with morphisms oriented 2-manifolds, this was first shown in [Ti]; the identity of that spectrum was conjectured in [MT] and first proved in [MW]. The GMTW Theorem is a generalization to all dimensions. As we shall see, it is a generalization of the classical Pontrjagin-Thom Theorem 10.33.

In this lecture we get as far as stating the GMTW Theorem [GMTW]. We discuss the proof in subsequent lectures.

Topology on function spaces

A reference for this subsection is [Hi, Chapter 2]. In particular, Hirsch uses jet spaces to describe the spaces of maps below as subspaces of function spaces with the standard compact-open topology. We pass immediately to C^∞ functions; it is somewhat easier to consider C^r functions for r finite and then take $r \to \infty$.

(20.1) *The Whitney topology.* Let Z, M be smooth manifolds with Z closed. We define a topology on the set $C^\infty(Z, M)$ of smooth maps $Z \to M$. The topology is generated[1] by sets $N\bigl(f,(U,z),(V,m),K,\epsilon\bigr)$ where $f\colon Z \to M$ is a smooth function; (U,z) is a chart on Z with $z\colon U \to \mathbb{A}^n$; (V,m) is a chart on M; $K \subset U$ is a compact set such that $f(K) \subset V$; and $\epsilon > 0$. To describe the sets we use multi-index notation $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \mathbb{Z}^{\ge 0}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, and

$$D^\alpha = \frac{\partial^{|\alpha|}}{(\partial z^1)^{\alpha_1} \cdots (\partial z^n)^{\alpha_n}}. \tag{20.2}$$

Then $N\bigl(f,(U,z),(V,m),K,\epsilon\bigr)$ consists of all smooth functions $f'\colon Z \to M$ such that $f'(K) \subset Z$ and for all multi-indices α and all $j = 1, \dots, \dim M$, we have

$$\bigl\| D^\alpha(m^j \circ f' \circ z^{-1}) - D^\alpha(m^j \circ f \circ z^{-1}) \bigr\|_{C^0(K)} < \epsilon. \tag{20.3}$$

The $C^0(K)$ norm is the sup norm, which is the maximum of the norm of a continuous function on the compact set K and $m^j\colon V \to \mathbb{R}$ is the coordinate function in the chart.

Remark 20.4. If Z is noncompact this is called the *weak* Whitney topology; there is also a strong Whitney topology.

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[1] In other words, the topology is the smallest topology which contains the sets N.

(20.5) *Embeddings and diffeomorphisms.* Topologize embeddings $\mathrm{Emb}(Z,M)\subset C^\infty(Z,M)$ using the subspace topology. Similarly, topologize the group of diffeomorphisms $\mathrm{Diff}(Z)\subset C^\infty(Z,Z)$ as a subspace. Composition and inversion are continuous, so $Diff(Z)$ is a topological group. For embeddings into affine space define

$$\mathrm{Emb}(Z,\mathbb{A}^\infty)=\operatorname*{colim}_{m\to\infty}\mathrm{Emb}(Z,\mathbb{A}^m). \tag{20.6}$$

An element of $\mathrm{Emb}(Z,\mathbb{A}^\infty)$ is an embedding $f\colon Z\to\mathbb{A}^m$ for some m, composed with the inclusion $\mathbb{A}^m\to\mathbb{A}^\infty$.

Theorem 20.7. $\mathrm{Emb}(Z,\mathbb{A}^\infty)$ *is contractible and* $\mathrm{Diff}(Z)$ *acts freely.*

Notice that a contractible space is nonempty; the nonemptiness is a nontrivial statement. The following argument may be found in [KM, Lemma 44.22].

Proof. $\mathrm{Emb}(Z,\mathbb{A}^\infty)$ is nonempty by Whitney's embedding theorem. The freeness of the diffeomorphism action is clear, since each embedding $f\colon Z\to\mathbb{A}^m$ is injective. For the contractibility consider the homotopy $H_t\colon\mathbb{A}^\infty\to\mathbb{A}^\infty$, $0\le t\le 1$, defined by

$$H_t(x^1,x^2,\dots)=\bigl(x^1,\dots,x^{n-1},x^n\cos\theta^n(t),x^n\sin\theta^n(t),x^{n+1}\cos\theta^n(t),x^{n+1}\sin\theta^n(t),\dots\bigr), \tag{20.8}$$

where n is determined by $\frac{1}{n+1}\le t\le\frac1n$ and

$$\theta^n(\frac{1}{n+1}+s)=\rho\bigl(n(n+1)s\bigr)\frac{\pi}{2}. \tag{20.9}$$

Here $\rho\colon[0,1]\to[0,1]$ is a smooth(ing) function with $\rho\bigl([0,\epsilon)\bigr)=0$, $\rho\bigl((1-\epsilon,\epsilon]\bigr)=1$ for some $\epsilon>0$. In fact, n is not uniquely determined if t is the reciprocal of an integer, but the formulas are consistent for the two choices. Since all but finitely many x^i vanish, the map $H\colon[0,1]\times\mathbb{A}^\infty\to\mathbb{A}^\infty$ is smooth. Also, $H_0=\mathrm{id}_{\mathbb{A}^\infty}$ and

$$\begin{aligned}H_{1/2}(x^1,x^2,\dots)&=(x^1,0,x^2,0,\dots)\\ H_1(x^1,x^2,\dots)&=(0,x^1,0,x^2,\dots).\end{aligned} \tag{20.10}$$

We use H to construct a contraction. Fix an embedding $i_0\colon Z\to\mathbb{A}^\infty$. Use $H_{0\to1/2}$ to homotop i_0 to an embedding $H_{1/2}\circ i_0$ which lands in $\mathbb{A}^\infty_{\mathrm{odd}}$. Now composition with H_1 is a map $\mathrm{Emb}(Z,\mathbb{A}^\infty)\to\mathrm{Emb}(Z,\mathbb{A}^\infty_{\mathrm{even}}\subset\mathbb{A}^\infty)$. Combining composition with $H_{0\to1}$ with the homotopy

$$K_u(z)=(1-u)(H_1\circ i)(z)+u\,(H_{1/2}\circ i_0)(z),\qquad i\in\mathrm{Emb}(Z,\mathbb{A}^\infty), \tag{20.11}$$

we obtain a contraction of $\mathrm{Emb}(Z,\mathbb{A}^\infty)$. ∎

Notice that averaging an embedding into $\mathbb{A}^\infty_{\mathrm{odd}}$ with an embedding into $\mathbb{A}^\infty_{\mathrm{even}}$ yields an embedding.

(20.12) *A classifying space for* Diff(Z). Let $B_\infty(Z)$ denote the quotient space of the free Diff(Z)-action on Emb$(Z,\mathbb{A}^\infty)$.

Proposition 20.13 ([BiFi]). *The map π in*

$$\begin{array}{ccc} \mathrm{Diff}(Z) & \longrightarrow & \mathrm{Emb}(Z,\mathbb{A}^\infty) \\ & & \big\downarrow \pi \\ & & B_\infty(Z) \end{array} \tag{20.14}$$

is a topological principal bundle with structure group Diff(Z)*.*

A point of $B_\infty(Z)$ is a submanifold $Y \subset \mathbb{A}^\infty$ which is diffeomorphic to Z. The topology on Emb$(Z,\mathbb{A}^\infty)$ induces a quotient topology on the set $B_\infty(Z)$ via the map π. There are smoothness statements one can make about the fiber bundle (20.14), but we are content with the topological assertion.[2] There is also a generalization for Z noncompact [KM, §44].

Sketch proof. We must prove (20.14) is locally trivial, so produce local sections of π. Fix an embedding $i\colon Z \to \mathbb{A}^\infty$ and let $U \subset \mathbb{A}^\infty$ be a tubular neighborhood around the image $i(Z)$. It is equipped with a submersion $p\colon U \to i(Z) \cong Z$. Then we claim

$$\{i' \in \mathrm{Emb}(Z,\mathbb{A}^\infty) : i'(Z) \subset U,\ p\circ i' = i\} \tag{20.15}$$

is an open subset of Emb$(Z,\mathbb{A}^\infty)$ on which π is injective and whose image under π is an open neighborhood of $\pi(i)$. We defer to the references for the proofs of these claims. □

(20.16) *The associated bundle.* The topological group Diff(Z) has a left action on Z by evaluation: $f \in \mathrm{Diff}(Z)$ acts on $y \in Z$ to give $f(y) \in Z$. We can "mix" this left action with the right Diff(Z)-action in (20.14) to produce the associated fiber bundle

$$\begin{array}{ccc} Z & \longrightarrow & E_\infty(Z) \\ & & \big\downarrow \\ & & B_\infty(Z) \end{array} \tag{20.17}$$

in which

$$E_\infty(Z) = \mathrm{Emb}(Z,\mathbb{A}^\infty) \times_{\mathrm{Diff}(Z)} Z. \tag{20.18}$$

Note there is a natural map $E_\infty(Z) \to \mathbb{A}^\infty$ which is an embedding on each fiber. The fiber bundle (20.17) is universal for fiber bundles with fiber (diffeomorphic to) Z embedded in $\mathbb{A}^\infty$. Because of the embedding, the classifying map of such a fiber bundle is unique.

[2]We should not be complacent, however. In the next lecture we will need to speak of smooth maps into $B_\infty(Z)$, as for example discussed in [KM].

The topological bordism category

The *discrete* category $\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}$ of Definition 14.3 uses abstract manifolds and bordisms. To define a *topological* category we use manifolds and bordisms which are embedded in affine space. Also, we do not identify diffeomorphic bordisms.

Definition 20.19. Fix $n \in \mathbb{Z}^{\ge 0}$. The *topological bordism category* ${}^t\mathrm{Bord}_{\langle n-1,n\rangle}$ is defined as follows.

(i) An object is a pair (a, Y) consisting of a real number $a \in \mathbb{R}$ and a closed $(n-1)$-submanifold $Y \subset \mathbb{A}^\infty$;

(ii) A morphism $X\colon (a_0, Y_0) \to (a_1, Y_1)$ is either the identity, if $a_0 = a_1$ and $Y_0 = Y_1$, or if $a_0 < a_1$ a compact n-dimensional neat submanifold $X \subset [a_0, a_1] \times \mathbb{A}^\infty$ such that for some $\delta > 0$ we have

$$\begin{aligned} X \cap \bigl([a_0, a_0 + \delta) \times \mathbb{A}^\infty\bigr) &= [a_0, a_0 + \delta) \times Y_0, \\ X \cap \bigl((a_1 - \delta, a_1] \times \mathbb{A}^\infty\bigr) &= (a_1 - \delta, a_1] \times Y_1. \end{aligned} \tag{20.20}$$

(iii) Composition of non-identity morphisms is the union, as illustrated in Figure 38.

(iv) The set $\coprod_Z \bigl(\mathbb{R} \times B_\infty(Z)\bigr)$ of objects is topologized using the quotient topology on $B_\infty(Z)$, as in **(20.12)**. The disjoint union runs over diffeomorphism types of closed $(n-1)$-manifolds.

(v) There is a similar topology on the set of morphisms, as discussed in [GMTW, §2].

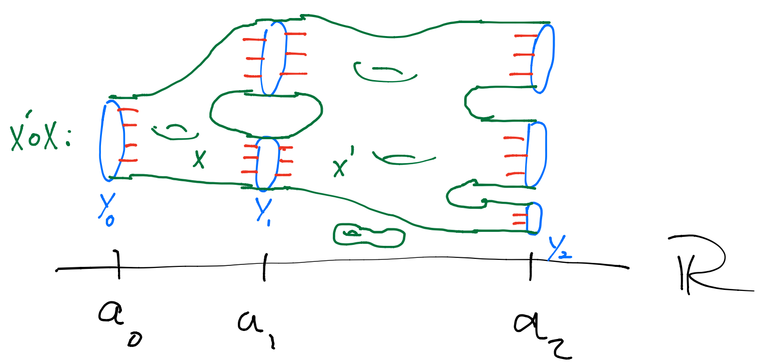

FIGURE 38. Composition of morphisms

(20.21) *Symmetric monoidal structure: discussion.* We would like to introduce a symmetric monoidal structure on ${}^t\mathrm{Bord}_{\langle n-1,n\rangle}$ using disjoint union as usual. Then the discussion in Lecture 19 on Γ-spaces would imply that the classifying space $B\bigl({}^t\mathrm{Bord}_{\langle n-1,n\rangle}\bigr)$ is the 0-space of a spectrum. Unfortunately, we don't see how to introduce that structure, though it *is* true that $B\bigl({}^t\mathrm{Bord}_{\langle n-1,n\rangle}\bigr)$ is an infinite loop space.

Since the manifolds are embedded, we must make the disjoint union concrete. One technique is to introduce the map

$$\begin{aligned} m\colon \mathbb{A}^\infty \times \mathbb{A}^\infty &\longrightarrow \mathbb{A}^\infty \\ (x^1, x^2, \dots)\,,\ (y^1, y^2, \dots) &\longmapsto (x^1, y^1, x^2, y^2, \dots) \end{aligned} \tag{20.22}$$

and then define the monoidal product as $(a_1, Y_1) \otimes (a_2, Y_2) = \big(a_1 + a_2, m(Y_1, Y_2)\big)$. The tensor unit is $(0, \emptyset^{n-1})$. Unfortunately, this is not strictly associative nor is the unit strict—in other words, this is not a permutative category—and there are not enough morphisms in ${}^t\mathrm{Bord}_{\langle n-1,n\rangle}$ to define an associator and a map (13.33), much less a symmetry. Naive modifications do not seem to work either. Fortunately, we do not need to use the symmetric monoidal structure to define and identify the classifying space.

We remark that the bordism *multi*-category we will discuss in the last few lectures does have a symmetric monoidal structure[3]

Finally, we would like to define a continuous TQFT as a symmetric monoidal functor from ${}^t\mathrm{Bord}_{\langle n-1,n\rangle}$ into a symmetric monoidal topological category, but absent the symmetric monoidal structure on ${}^t\mathrm{Bord}_{\langle n-1,n\rangle}$ we cannot do so. Nonetheless, we can still motivate interest in the classifying space $B\big({}^t\mathrm{Bord}_{\langle n-1,n\rangle}\big)$ by asserting that an *invertible* continuous TQFT is a map of topological Picard groupoids $B\big({}^t\mathrm{Bord}_{\langle n-1,n\rangle}\big) \to C$ for a topological Picard groupoid C.

(20.23) $\mathcal{X}(n)$*-structures.* We use $BO(n) = Gr_n(\mathbb{R}^\infty)$ as a model for the classifying space of the orthogonal group **(6.22)**. This is convenient since if $i\colon Y \hookrightarrow \mathbb{A}^\infty$ is an $(n-1)$-dimensional submanifold, then the "Gauss map"

(20.24)
$$\begin{array}{ccccc} TY & \xrightarrow{\;i_*\;} & \underline{\mathbb{R}} \oplus S(n-1) & \longrightarrow & S(n) \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & Gr_{n-1}(\mathbb{R}^\infty) & \longrightarrow & Gr_n(\mathbb{R}^\infty) \end{array}$$

is a classifying map for the once stabilized tangent bundle. Let $\mathcal{X}(n) \to Gr_n(\mathbb{R}^\infty)$ be an n-dimensional tangential structure. Then an $\mathcal{X}(n)$-structure on $Y \subset \mathbb{A}^\infty$ is a lift of (20.24) to a map

(20.25)
$$\begin{array}{ccc} TY & \longrightarrow & S(n) \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \mathcal{X}(n) \end{array}$$

The definition of an $\mathcal{X}(n)$-structure on a morphism is similar. There is a topological bordism category ${}^t\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}$ whose objects and morphisms are as in Definition 20.19, now with the addition of an $\mathcal{X}(n)$-structure. The equalities in (20.20) now include the $\mathcal{X}(n)$-structure as well. For a fixed $Y \subset \mathbb{A}^\infty$ there is a space of $\mathcal{X}(n)$-structures, and that space enters into the topologization of the sets of objects and morphisms. We refer to [GMTW, §5] for details.

(20.26) *The main question.* Identify the classifying space $B\big({}^t\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}\big)$.

[3]though the technical details on defining symmetric monoidal (∞, n)-categories may not be written down as of this writing.

Madsen-Tillmann spectra

(20.27) *Heuristic definition.* Fix a nonnegative integer n and an n-dimensional tangential structure $\mathcal{X}(n) \to BO(n)$. Recall the universal bundle $S(n) \to \mathcal{X}(n)$, which is pulled back from $BO(n)$.

Definition 20.28. The *Madsen-Tillmann spectrum* $MT\mathcal{X}(n)$ is the Thom spectrum $\mathcal{X}(n)^{-S(n)}$.

(20.29) *Precise definition.* As we have only defined the Thom spectrum associated to $S(n)$ (Definition 10.26), not the *virtual* Thom spectrum associated to $-S(n)$, we need something more concrete. Recall that $BO(n)$ is the colimit **(6.22)** of Grassmannians. Define $\mathcal{X}(n, n+q)$ as the pullback

(20.30)
$$\begin{array}{ccc} \mathcal{X}(n, n+q) & \dashrightarrow & \mathcal{X}(n) \\ \downarrow & & \downarrow \\ Gr_n(\mathbb{R}^{n+q}) & \longrightarrow & BO(n) \end{array}$$

Use the standard metric on $\mathbb{R}^{n+q}$ so that there is a direct sum $\underline{\mathbb{R}^{n+q}} = S(n) \oplus Q(q)$ of vector bundles over $Gr_n(\mathbb{R}^{n+q})$ and, by pullback, over $\mathcal{X}(n, n+q)$.

Definition 20.31. The *Madsen-Tillmann spectrum* $MT\mathcal{X}(n)$ is the spectrum completion of the prespectrum whose $(n+q)^{\text{th}}$ space is the Thom space of $Q(q) \to \mathcal{X}(n, n+q)$. The structure maps are obtained by applying the Thom space construction to the map

(20.32)
$$\begin{array}{ccc} \underline{\mathbb{R}} \oplus Q(q) & \longrightarrow & Q(q+1) \\ \downarrow & & \downarrow \\ \mathcal{X}(n, n+q) & \longrightarrow & \mathcal{X}(n, n+q+1) \end{array}$$

of vector bundles.

This prespectrum has spaces defined for integers $\geq n$, which is allowed; see the remarks following Definition 10.2. The intuition here is that, as formal bundles, $Q(q) = -S(n) + \underline{\mathbb{R}^{n+q}}$, so the Thom space of the vector bundle $Q(q) \to \mathcal{X}(n, n+q)$ represents the 0-space of the $(n+q)^{\text{th}}$ suspension of the spectrum defined in Definition 20.28. The latter is equally the $(n+q)^{\text{th}}$ space of the unsuspended MT spectrum.

(20.33) *Notation.* The 'MT' notation is due to Mike Hopkins. It not only stands for 'Madsen-Tillmann', but also for a Tangential variant of the thoM spectrum. The MT spectra are tangential and unstable; the M-spectra are normal and stable. We will see a precise relationship below. For Madsen-Tillmann spectra constructed from reductions of structure group **(10.28)**, we use the notation $MTG(n)$. For example, the Madsen-Tillmann spectrum for oriented bundles is $MTSO(n)$.

Proposition 20.34. *There is a homotopy equivalence* $MTSO(1) \simeq S^{-1} = \Sigma^{-1} S^0$.

Here S^0 is the sphere spectrum.

Proof sketch. First, $BSO(1)$ is contractible, since $SO(1)$ is the trivial group with only the identity element. So the formal Definition 20.28 reduces to $MTSO(1) = BSO(1)^{-\mathbb{R}} \simeq \Sigma^{-1}T_\bullet$ where $T^\bullet$ is the suspension spectrum of a contractible unpointed space, which is the sphere spectrum. (Check that the Thom space $*^{\mathbb{R}}$ of the trivial bundle over a point is the pointed space S^1.) We leave the reader to give the instructive proof based on Definition 20.31. □

(20.35) *The perp map.* Now assume that $\mathfrak{X}$ is a stable tangential structure (Definition 9.45). There is an induced n-dimensional tangential structure $\mathfrak{X}(n)$. Recall from (**9.62**) the perp stable tangential structure $\mathfrak{X}^\perp$. We now construct a map

(20.36) $$\Sigma^n MT\mathfrak{X}(n) \longrightarrow M\mathfrak{X}^\perp$$

from the Madsen-Tillmann spectrum to the Thom spectrum. Namely, the perp map followed by stabilization yields the diagram

(20.37) $$\begin{array}{ccccc} Q(q) & \xrightarrow{\cong} & S(q) & \longrightarrow & S(q) \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{X}(n,n+q) & \xrightarrow{\cong} & \mathfrak{X}^\perp(q,n+q) & \longrightarrow & \mathfrak{X}^\perp(q) \\ \downarrow & & \downarrow & & \downarrow \\ Gr_n(\mathbb{R}^{n+q}) & \xrightarrow[\cong]{\perp} & Gr_q(\mathbb{R}^{n+q}) & \longrightarrow & Gr_q(\mathbb{R}^\infty) \end{array}$$

The induced map on the Thom space of the upper left arrow to the Thom space of the upper right arrow is a map $MT\mathfrak{X}(n)_{n+q} \to M\mathfrak{X}_q^\perp$ on the indicated spaces of the spectra. The maps are compatible with the structure maps of the prespectra as q varies, and so we obtain the map (20.36) of spectra.

(20.38) *The filtration of the Thom spectrum.* The stabilization map

(20.39) $$\begin{array}{ccc} Q(q) & \longrightarrow & Q(q) \\ \downarrow & & \downarrow \\ \mathfrak{X}(n,n+q) & \longrightarrow & \mathfrak{X}(n+q,n+1+q) \end{array}$$

induces a map $\Sigma^n MT\mathfrak{X}(n)_q \to \Sigma^{n+1}MT\mathfrak{X}(n+1)_q$ on Thom spaces, and iterating with n we obtain a sequence of maps

(20.40) $$MT\mathfrak{X}(0) \longrightarrow \Sigma^1 MT\mathfrak{X}(1) \longrightarrow \Sigma^2 MT\mathfrak{X}(2) \longrightarrow \cdots$$

of spectra. Define the colimit to be the stable Madsen-Tillmann spectrum $MT\mathfrak{X}$. The perp maps (20.36) induce a map

(20.41) $$MT\mathfrak{X} \to M\mathfrak{X}^\perp$$

on the colimit. It is clear from the construction (20.37) that (20.41) is an isomorphism. So the (suitably suspended) Madsen-Tillmann spectra (20.40) give a filtration of the Thom spectrum.

The Galatius-Madsen-Tillmann-Weiss theorem

Now we can state the main theorem.

Theorem 20.42 ([GMTW]). *Let n be a nonnegative integer and $\mathcal{X}(n)$ an n-dimensional tangential structure. Then there is a weak homotopy equivalence*

$$B\big({}^t\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}\big) \simeq \big(\Sigma MT\mathcal{X}(n)\big)_0. \tag{20.43}$$

In words: The classifying space of the topological bordism category is the 0-space of the suspension of the Madsen-Tillmann spectrum. We sketch the proof in subsequent lectures. The power of the theorem is that the space on the right hand side is constructed from familiar ingredients in algebraic topology, so its invariants are readily calculable.

(20.44) *The GMTW theorem on π_0.* Theorem 20.42 is a weak homotopy equivalence of (pointed) spaces. It induces an isomorphism of (pointed) sets by applying π_0. Since both sides are infinite loop spaces, π_0 is an abelian group, and so we obtain an isomorphism of abelian groups. A space is a refinement of the set π_0, and so (20.43) is a refinement of this isomorphism of sets. We now compute π_0 of both sides.

For the classifying space of the bordism category, the left hand side of (20.43), we compute π_0 directly from the category. Namely, the morphisms define an equivalence relation on objects: two objects are equivalent if they are connected by a morphism. The construction of the classifying space shows that π_0 is the set of isomorphisms classes. Applied to the bordism category we obtain the bordism group

$$\pi_0 B\big({}^t\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}\big) \cong \Omega^{\mathcal{X}(n)}_{n-1}. \tag{20.45}$$

Assume that $\mathcal{X}(n)$ is induced from a stable tangential structure $\mathcal{X}$. Then for the right hand side of (20.43), we have for q large

$$\begin{aligned} \pi_0\big(\Sigma MT\mathcal{X}(n)\big)_0 &= \pi_{n+q-1}\mathcal{X}(n,n+q)^{Q(q)} \\ &\cong \pi_{n+q-1}\mathcal{X}^{\perp}(q,n+q)^{S(q)} \\ &\cong \pi_{n-1}M\mathcal{X}^{\perp}. \end{aligned} \tag{20.46}$$

Therefore, on the level of π_0 the weak homotopy equivalence (20.43) is an isomorphism

$$\Omega^{\mathcal{X}(n)}_{n-1} \xrightarrow{\ \cong\ } \pi_{n-1}M\mathcal{X}^{\perp}. \tag{20.47}$$

Recall the general Pontrjagin-Thom Theorem 10.33 which is precisely such an isomorphism. So we expect that the weak homotopy equivalence induces the Pontrjagin-Thom collapse map on the level of π_0, and that the GMTW theorem is a generalization of the classical Pontrjagin-Thom theorem.

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