## Lecture 21: Sheaves on Man

In this lecture and the next we sketch some of the basic ideas which go into the proof of Theorem 20.42. The main references for the proof are the original papers [GMTW] and [MW]. These lectures are an introduction to those papers.

The statement to be proved is a weak homotopy equivalence of two spaces. The main idea is to realize each space as a *moduli space* in $C^\infty$ geometry. Moduli spaces are fundamental throughout geometry. A simple example is to fix a vector space $V$ and construct the parameter space of lines in $V$: the projective space $\mathbb{P}V$. We could then omit the fixed ambient space $V$ and ask for the moduli space of all lines. To formulate that precisely, consider arbitrary smooth families of lines, parametrized by a "test" manifold $M$. The first step is to define a 'smooth family of lines' as a smooth line bundle $L \to M$. The collection $\mathcal{F}(M)$ of line bundles is then a contravariant function of $M$: given a smooth map $f\colon M \to M'$ there is a pullback $f^*\colon \mathcal{F}(M') \to \mathcal{F}(M)$ of line bundles. We seek a universal line bundle $\mathcal{L} \to |\mathcal{F}|$ over a topological space so that any line bundle is pulled back from this family. Of course, we have arrived back at the idea of a classifying space, as discussed in Lecture 6. In this lecture we take up nonlinear versions—families of curved manifolds—and construct the universal space $|\mathcal{F}|$ directly from the map $\mathcal{F}$. For this we isolate certain properties of $\mathcal{F}$: it is a *sheaf*.

We introduce sheaves of *sets* and sheaves of *categories*. The sheaves are functions of an arbitrary smooth manifold, not of open sets on a fixed manifold. The map $\mathcal{F}$ in the previous paragraph is not a sheaf as stated, but is a sheaf of sets if we define $\mathcal{F}(M)$ as the set of line bundles $L \to M$ equipped with an embedding into a vector bundle $M \times \mathcal{H} \to M$ with constant fiber $\mathcal{H}$. In our nonlinear examples we consider fiber bundles of manifolds equipped with an embedding into affine space, as in the definition of the topological bordism category (Definition 20.19). In this lecture we discuss some basics and the construction of a topological space $|\mathcal{F}|$ from a sheaf of sets $\mathcal{F}$. We also introduce sheaves of categories and their classifying spaces.

### Presheaves and sheaves

**(21.1)** *Sheaves on a fixed manifold.* Let $X$ be a smooth manifold. A *presheaf* on $X$ assigns a set $\mathcal{F}(U)$ to each open set $U \subset X$ and a restriction map $i^*\colon \mathcal{F}(U') \to \mathcal{F}(U)$ to each inclusion $i\colon U \hookrightarrow U'$. Formally, then, define the category $\mathrm{Open}(X)$ whose objects are open subsets of $X$ and whose morphisms are inclusions. A *presheaf* is a functor

$$\mathcal{F}\colon \mathrm{Open}(X)^{\mathrm{op}} \longrightarrow \mathrm{Set}\,. \tag{21.2}$$

It is a sheaf if it satisfies a gluing condition, which we specify below in the context we need. A typical example is the structure sheaf $F(U) = C^\infty(U)$ of smooth functions. Other sorts of "functions"—differential forms, sections of a fixed vector bundle—also form sheaves over a fixed manifold.

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**(21.3)** *The category of smooth manifolds.* The sheaves we introduce are defined on all manifolds, not just on open submanifolds of a fixed manifold.

**Definition 21.4.** The category Man has as objects smooth finite dimensional manifolds without boundary and as morphisms smooth maps of manifolds.

This category is quite general, and there are examples of sheaves defined only on interesting subcategories.[1]

**(21.5)** *Presheaves on* Man. Any contravariant function of manifolds is a presheaf.

**Definition 21.6.** A *presheaf on* Man is a functor

$$\mathcal{F}\colon \mathrm{Man}^{\mathrm{op}} \longrightarrow \mathrm{Set}\,. \tag{21.7}$$

We give several examples.

**Example 21.8.** Let $\mathcal{F}(M) = C^\infty(M)$ be the set of smooth functions. A smooth map $(f\colon M \to M')$ of manifolds induces a pullback $\mathcal{F}(f) = f^*\colon \mathcal{F}(M') \to \mathcal{F}(M)$ on functions.

**Example 21.9.** For any $X \in \mathrm{Man}$ define $\mathcal{F}_X(M) = \mathrm{Man}(M, X)$. (Recall that $\mathrm{Man}(M, X)$ is the hom-set in the category Man, so here the set of smooth maps $M \to X$.) The sheaf $\mathcal{F}_X$ is the *representable sheaf* associated to the manifold $X$. Intuitively, we "test" $\mathcal{F}_X$ with the probe $M$.

There is a category of presheaves.

**Definition 21.10.** Let $\mathcal{F}, \mathcal{F}'\colon \mathrm{Man}^{\mathrm{op}} \to \mathrm{Set}$ be presheaves. A *map*, or morphism, $\varphi\colon \mathcal{F} \to \mathcal{F}'$ of presheaves is a natural transformation of functors.

The construction in Example 21.9 embeds the category of manifolds in the category of presheaves, as expressed by the following general and simple result.

**Lemma 21.11** (Yoneda)**.** *Let $X \in \mathrm{Man}$ and $\mathcal{F}\colon \mathrm{Man}^{\mathrm{op}} \to \mathrm{Set}$. Then there is a bijection*

$$\mathrm{Map}(\mathcal{F}_X, \mathcal{F}) \xrightarrow{\;\cong\;} \mathcal{F}(X). \tag{21.12}$$

*Proof.* We construct maps in each direction and leave the reader to prove they are inverse. First, a natural transformation $\varphi \in \mathrm{Map}(\mathcal{F}_X, \mathcal{F})$ determines $\varphi(X)(\mathrm{id}_X) \in \mathcal{F}(X)$. (Note $\varphi(X)\colon \mathcal{F}_X(X) \to \mathcal{F}(X)$ and $\mathcal{F}_X(X) = \mathrm{Man}(X, X)$.) In the other direction, if $s \in \mathcal{F}(X)$ then define $\varphi \in \mathrm{Map}(\mathcal{F}_X, \mathcal{F})$ by $\varphi(M)(f) = \mathcal{F}(f)(s)$, where $(f\colon M \to X) \in \mathrm{Man}(M, X)$. □

We give two examples of non-representable presheaves.

[1] For example, the sheaf $\mathcal{F}(M) = \{\text{orientations of } M\}$ is defined on the subcategory where maps are required to be local diffeomorphisms. One can further restrict the manifolds to be of a fixed dimension, as for example required by the notion of a 'local field' in theoretical physics [F2].

**Example 21.13.** Let $q \in \mathbb{Z}^{\geq 0}$. Then $\mathcal{F}^q(M) = \Omega^q(M)$ is a presheaf. It is not representable: there is no finite dimensional (or infinite dimensional) smooth manifold $\Omega^q$ such that differential $q$-forms on $M$ correspond to maps $M \to \Omega^q$. But the presheaf $\mathcal{F}^q$ is a stand-in for such a mythical manifold. In that sense presheaves on Man are generalized manifolds. In that regard, an immediate consequence of Lemma 21.11 is $\mathrm{Map}(\mathcal{F}_X, \mathcal{F}^q) = \Omega^q(X)$.

**Example 21.14.** Define $\mathcal{F}(M)$ to be the set of commutative diagrams

$$\begin{array}{ccc} Y & \hookrightarrow & M \times \mathbb{A}^\infty \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi_1} & \\ & M & \end{array} \tag{21.15}$$

in which $\pi$ is a proper submersion and the top arrow is the Cartesian product of $\pi$ and an embedding.[2] So $\mathcal{F}(M)$ is the set of submanifolds of $M \times \mathbb{A}^\infty$ whose projection onto $M$ is a proper submersion. These form a presheaf: morphisms map to pullbacks of subsets and compositions of morphisms map strictly to compositions. (The reader should contemplate what goes wrong with compositions without the embedding.)

*Remark* 21.16. An important theorem of Charles Ehresmann asserts that a proper submersion is a fiber bundle.

**(21.17)** *The sheaf condition.* A sheaf is a presheaf which satisfies a gluing condition; there is no extra data.

**Definition 21.18.** Let $\mathcal{F}\colon \mathrm{Man}^{\mathrm{op}} \to \mathrm{Set}$ be a presheaf. Then $\mathcal{F}$ is a *sheaf* if for every open cover $\{U_\alpha\}$ of a manifold $M$, the diagram

$$\mathcal{F}(M) \longrightarrow \prod_{\alpha_0} \mathcal{F}(U_{\alpha_0}) \rightrightarrows \prod_{\alpha_0,\alpha_1} \mathcal{F}(U_{\alpha_0} \cap U_{\alpha_1}) \tag{21.19}$$

is an equalizer.

This means that if $s_{\alpha_0} \in \mathcal{F}(U_{\alpha_0})$ is a family of elements such that the two compositions in (21.19) agree, then there is a unique $s \in \mathcal{F}(M)$ which maps to $\{s_{\alpha_0}\}$. If we view $\mathcal{F}(U)$ as the space of "sections" of the presheaf $\mathcal{F}$ on the open set $U$, then the condition is that local coherent "sections" of the presheaf glue uniquely to a global section.

*Remark* 21.20. An open cover expresses $M$ as a colimit of the diagram

$$\coprod_{\alpha_0,\alpha_1} U_{\alpha_0} \cap U_{\alpha_1} \rightrightarrows \coprod_{\alpha_0} U_{\alpha_0}. \tag{21.21}$$

The sheaf condition asserts that $\mathcal{F}(M)$ is the limit of $\mathcal{F}$ applied to (21.21).

[2] We also want to add the condition that for any compact $K \subset M$ the embedding $\pi^{-1}(K) \hookrightarrow \mathbb{A}^\infty$ factors through a smooth embedding $\pi^{-1}(K) \hookrightarrow \mathbb{A}^m$ for some finite $m$. This condition applies to all similar subsequent examples.

**(21.22)** *Intuition.* We can often regard $\mathcal{F}(M)$ as a smooth family of elements of $\mathcal{F}(\mathrm{pt})$ parametrized by $M$. So for a representable sheaf $\mathcal{F}_X$ we have $\mathcal{F}(\mathrm{pt}) = X$ and $\mathcal{F}_X(M)$ is a smooth family of points of $X$ parametrized by $M$. Similarly, for the sheaf in Example 21.14, $\mathcal{F}(\mathrm{pt})$ is the set of submanifolds of $\mathbb{A}^\infty$ and $\mathcal{F}(M)$ is a smooth family of such submanifolds. In other examples, e.g. Example 21.13, the intuition must be refined: for $q > 0$ there are no nonzero $q$-forms on a point. In this case an element of $\mathcal{F}(M)$ is a smooth coherent family of elements of $\mathcal{F}(U)$ for arbitrarily small open sets $U$. That is exactly what the sheaf condition asserts.

### The representing space of a sheaf

**(21.23)** *Extended simplices.* Recall (18.4) that a nonempty finite ordered set $S$ determines a simplex $\Sigma(S)$ whose vertex set is $S$. The simplex $\Sigma(S)$ is a subspace of the abstract affine space $\Sigma_e(S)$ spanned by $S$. Whereas $\Sigma(S)$ is not a smooth manifold—it is a manifold with corners—the affine space $\Sigma_e(S)$ is. So

$$\Sigma_e\colon \Delta \longrightarrow \mathrm{Man} \tag{21.24}$$

is a functor whose image consists of affine spaces and (very special) affine maps.

**(21.25)** *The space attached to a sheaf on* Man. The following definition allows us to represent topological spaces by sheaves.

**Definition 21.26.** Let $\mathcal{F}\,\mathrm{Man}^{\mathrm{op}} \to \mathrm{Set}$ be a sheaf. The *representing space* $|\mathcal{F}|$ is the geometric realization of the simplicial set

$$\Delta^{\mathrm{op}} \xrightarrow{\Sigma_e^{\mathrm{op}}} \mathrm{Man}^{\mathrm{op}} \xrightarrow{\mathcal{F}} \mathrm{Set}\,. \tag{21.27}$$

For example, if $\mathcal{F} = \mathcal{F}_X$ is the representable sheaf attached to a smooth manifold $X$, then $S \mapsto \mathcal{F}\big(\Sigma_e(S)\big)$ is the (extended, smooth) singular simplicial set associated to $X$, a manifold analog of Example 18.17. The Milnor theorem quoted after (18.20) holds for extended smooth simplices.

**Theorem 21.28** (Milnor)**.** *The canonical map* $|\mathcal{F}_X| \to X$ *is a weak homotopy equivalence.*

The canonical map is induced from the evaluation

$$\mathcal{F}\big((\Sigma_e(S))\big) \times \Sigma(S) = \mathrm{Man}\big(\Sigma_e(S), X\big) \times \Sigma(S) \xrightarrow{\mathrm{ev}} X. \tag{21.29}$$

**Example 21.30.** Fix a (separable) complex Hilbert space $\mathcal{H}$. Define a sheaf $\mathcal{F}$ by letting $\mathcal{F}(M)$ be the set of commutative diagrams

$$\begin{array}{ccc} L & \hookrightarrow & M \times \mathcal{H} \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi_1} & \\ & M & \end{array} \tag{21.31}$$

in which $\pi$ is a complex line bundle and the horizontal embedding composed with projection onto $\mathcal{H}$ is linear on each fiber of $\pi$. (So it is an embedding of the line bundle $L \to M$ into the bundle with constant fiber $\mathcal{H}$.) In this case we claim there is a natural map $|\mathcal{F}| \to \mathbb{P}\mathcal{H}$ which is a weak homotopy equivalence. In essence $\mathcal{F}(M)$ is the space of *smooth* maps $M \to \mathbb{P}\mathcal{H}$, where we introduce an appropriate infinite dimensional smooth structure on $\mathbb{P}\mathcal{H}$. (As a simple special case, for which we do not need the smooth structure, consider $\mathcal{F}(\text{pt})$.) So while $\mathcal{F}$ is not representable in Man, it is in a larger category which includes infinite dimensional smooth manifolds, and then the proof of Theorem 21.28 applies. We do not attempt details here.

**(21.32)** *Concordance.* We introduce an equivalence relation on sections of a sheaf. It is an adaptation of homotopy equivalence of functions to the sheaf world.

**Definition 21.33.** Let $\mathcal{F}\colon \text{Man}^{\text{op}} \to \text{Set}$ be a sheaf, $M \in \text{Man}$, and $s_0, s_1 \in \mathcal{F}(M)$. Then $s_0$ and $s_1$ are *concordant* if there exists $s \in \mathcal{F}(\mathbb{R} \times M)$ such that

$$\begin{aligned} i_-^* s &= \pi_2^* s_0 \qquad \text{on } (-\infty, \epsilon) \times M, \\ i_+^* s &= \pi_2^* s_1 \qquad \text{on } (1-\epsilon, \infty) \times M \end{aligned} \tag{21.34}$$

for some $\epsilon > 0$.

The maps in (21.34) are the inclusions and projections

$$M \xleftarrow{\pi_2} (-\infty, \epsilon) \times M \xrightarrow{i_-} \mathbb{R} \times M \xleftarrow{i_+} (1-\epsilon, \infty) \times M \xrightarrow{\pi_2} M \,. \tag{21.35}$$

This is just a smooth version of a homotopy, which would normally be expressed on the manifold-with-boundary $[0,1] \times M$, which is not in the category Man. See Figure 39.

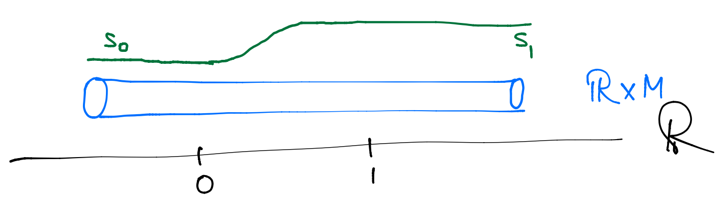


FIGURE 39. A concordance

Concordance is an equivalence relation. We denote the set of concordance classes of elements of $\mathcal{F}(M)$ as $\mathcal{F}[M]$. The map $M \mapsto \mathcal{F}[M]$ is not usually a sheaf: equivalence classes do not glue.

**Example 21.36.** For the sheaf $\mathcal{F}$ of Example 21.30 the set $\mathcal{F}[M]$ is the set of equivalence classes of complex line bundles $L \to M$. For the standard cover $\{S^2 \setminus \{p_1\}\,,\, S^2 \setminus \{p_2\}\}$ of $M = S^2$ by two open sets, the diagram (21.19) fails to be an equalizer.

**(21.37)** *The meaning of the representing space.* The representation space represents concordance classes.

**Theorem 21.38** ([MW]). *There is a bijection*

$$\mathcal{F}[M] \longrightarrow \bigl[M, |\mathcal{F}|\bigr], \tag{21.39}$$

*where the codomain is the set of homotopy classes of continuous maps $M \to |\mathcal{F}|$.*

*Sketch of proof.* A detailed proof may be found in [MW, Appendix]. We content ourselves here with describing the map (21.39), which is formal, and its inverse, which is less formal. We use the Yoneda Lemma 21.11, the Milnor Theorem 21.28, and the fact that the representing space $|-|$ is a functor to construct (21.39) as the composition

$$\mathcal{F}(M) \cong \mathrm{Map}(\mathcal{F}_M, \mathcal{F}) \xrightarrow{\;|-|\;} \bigl[|\mathcal{F}_M|, \mathcal{F}|\bigr] \xleftarrow{\;\cong\;} \bigl[M, |\mathcal{F}|\bigr]. \tag{21.40}$$

To see that this passes to concordance classes, note that a concordance is a map $\mathcal{F}_{\mathbb{R}\times M} \to \mathcal{F}$, by Yoneda, and so induces[3]

$$|\mathcal{F}_{\mathbb{R}\times M}| \simeq |\mathcal{F}_{\mathbb{R}}| \times |\mathcal{F}_M| \simeq \mathbb{R} \times |\mathcal{F}_M| \longrightarrow |\mathcal{F}|, \tag{21.41}$$

a homotopy of maps $M \to |\mathcal{F}|$.

The inverse construction is a bit more intricate. One begins with a map $g\colon M \to |\mathcal{F}|$, a representative of a homotopy class, and then must construct an element of $\mathcal{F}(M)$. This is accomplished using the sheaf property, which allows to construct a coherent family of elements of $\mathcal{F}(U)$ for a covering of $M$ by open sets $U$. The first step is a simplicial approximation theorem, which realizes $g$ up to homotopy as the geometric realization of a map $g'\colon sC \to s\mathcal{F}$ of simplicial sets, where $sC$ is the simplicial set associated to an ordered simplicial complex $C$ together with a homeomorphism $|C| \to M$—in fact, a smooth triangulation of $M$—and $s\mathcal{F}$ is the simplicial set (21.27). The second step is to construct a vector field on $M$ from the triangulation $C$, a vector field which pushes towards lower dimensional simplices. This induces a map $h\colon M \to M$ homotopic to the identity such that each simplex $\Delta$ in $C$ has an open neighborhood $U_\Delta$ which retracts onto $\Delta$ under $h$. Then $h^*g'(\Delta) \in \mathcal{F}(U_\Delta)$ is a coherent family of elements, so glues to the desired element of $\mathcal{F}(M)$, whose concordance class is independent of the choices. We refer to [MW, Appendix] for details. □

**Example 21.42.** The application of Theorem 21.38 to Example 21.30 produces the theorem that $\mathbb{P}\mathcal{H}$ classifies equivalence classes of line bundles over a smooth manifold $M$, something we discussed in Lecture 6.

[3] We assume the geometric realization commutes with products; see Remark 19.31.

## Sheaves of categories

Let Cat be the category whose objects are categories $C_\bullet = (C_0, C_1)$ and whose morphisms are functors. We use the formulation (**13.7**) of categories as pairs of sets with various structure maps. A functor is a pair of maps (one on objects, one on morphisms), and composition of functors is associative on the nose.[4]

**Definition 21.43.** A *sheaf of categories* $\mathcal{F}_\bullet\colon \mathrm{Man}^{\mathrm{op}} \to \mathrm{Cat}$ is a pair of set-valued functors $(\mathcal{F}_0, \mathcal{F}_1)\colon \mathrm{Man}^{\mathrm{op}} \to \mathrm{Set}$ togther with structure maps (13.8) which satisfy the defining relations of a category.

So $\mathcal{F}_0$ and $\mathcal{F}_1$ separately satisfy the sheaf condition. For any test manifold $M$ the category $\mathcal{F}_\bullet(M)$ is discrete: $\mathcal{F}_0(M)$ and $\mathcal{F}_1(M)$ are sets.

**Definition 21.44.** Let $\mathcal{F}_\bullet\colon \mathrm{Man}^{\mathrm{op}} \to \mathrm{Cat}$ be a sheaf of categories. The *representing category* is the topological category

$$|\mathcal{F}_\bullet| = \bigl(|\mathcal{F}_0|\,,\,|\mathcal{F}_1|\bigr). \tag{21.45}$$

A topological category has a classifying space, so there is a space $B|\mathcal{F}_\bullet|$ associated to a sheaf $\mathcal{F}_\bullet$ of categories. One of the constructions used in the proof, which we will not recount here, is a sheaf $\beta(\mathcal{F}_\bullet)$ of sets associated to a sheaf $\mathcal{F}_\bullet$ of categories with the property

$$|\beta(\mathcal{F}_\bullet)| \simeq B|\mathcal{F}_\bullet|. \tag{21.46}$$

See [GMTW, §2.4], [MW, §4.1] for the construction of the *cocycle sheaf*.

## References

[F2] D. S. Freed, *Lectures on twisted K-theory and orientifolds*, http://www.ma.utexas.edu/users/dafr/ESI.pdf.

[GMTW] Søren Galatius, Ulrike Tillmann, Ib Madsen, and Michael Weiss, *The homotopy type of the cobordism category*, Acta Math. **202** (2009), no. 2, 195–239, arXiv:math/0605249.

[MW] Ib Madsen and Michael Weiss, *The stable moduli space of Riemann surfaces: Mumford's conjecture*, Ann. of Math. (2) **165** (2007), no. 3, 843–941.

[4] There is a "weaker" notion involving natural transformations on functors: categories are objects of a 2-category. We will discuss higher categories, at least heuristically, in the last two lectures.