Lecture 22: Remarks on the proof of GMTW

Recall that the GMTW Theorem 20.42 asserts the existence of a weak homotopy equivalence

$$B\big({}^t\mathrm{Bord}^{\mathfrak{X}(n)}_{\langle n-1,n\rangle}\big) \simeq \big(\Sigma MT\mathfrak{X}(n)\big)_0. \tag{22.1}$$

The left hand side is the classifying space of the topological bordism category whose morphisms are compact n-manifolds with $\mathfrak{X}(n)$-structure. The right hand side is the 0-space of the suspension of the Madsen-Tillmann spectrum. Both of these pointed spaces were defined in Lecture 20, where we showed that the classical Pontrjagin-Thom theorem is the weak homotopy equivalence (22.1) composed with π_0. Indeed, the ideas of classical Pontrjagin-Thom theory are integral to the proof.

Rather than attempt a direct map between the spaces (22.1), the proof proceeds by constructing sheaves which represent these spaces. More precisely, there is a sheaf of sets $D = D_n^{\mathfrak{X}(n)}$ on Man whose representing space is $\big(\Sigma MT\mathfrak{X}(n)\big)_0$. The Pontrjagin-Thom theory, as well as Phillips' Submersion Theorem [Ph] is used to prove this representing statement. The value $D(M)$ of the sheaf on a test manifold M is a set of submersions over M. Intuitively, it is a set of fiber bundles of compact $(n-1)$-manifolds, but because the Phillips theorem only applies to noncompact manifolds there is a necessary modification. We explain the heuristic idea in the first section below, and then give the technically correct rendition, though not a complete proof. The space on the left hand side of (22.1) is the classifying space of a topological category, and it is fairly easy to construct a sheaf of categories $C = C_n^{\mathfrak{X}(n)}$ on Man which represents this topological category (in the sense of Definition 21.44). Its value on a test manifold M is a category whose objects are fiber bundles over M with fibers closed $(n-1)$-manifolds. The remainder of the proof goes through auxiliary sheaves (of categories) which mediate between C and D. We content ourselves with an overview and refer to the reader to the original papers [GMTW, MW] for a full account.

The main construction: heuristic version

As mentioned in the introduction, this section is a useful false start.

(22.2) *A sheaf of $(n-1)$-manifolds.* Fix a positive integer n and an $\mathfrak{X}(n)$-structure $\mathfrak{X}(n) \to Gr_n(\mathbb{R}^\infty)$. We elaborate on Example 21.14. Let $E\colon \mathrm{Man}^{\mathrm{op}} \to \mathrm{Set}$ be the sheaf whose value on a test manifold M is a pair of maps

$$\begin{array}{ccc} Y & \hookrightarrow & M\times \mathbb{A}^\infty \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi_1} & \\ & M & \end{array} \tag{22.3}$$

in which π is a fiber bundle with fibers closed $(n-1)$-manifolds and the top arrow is an embedding. For simplicity we do not include a tangential structure. Assume for simplicity that M is compact.

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Then for some $m>0$ the embedding factors through an embedding into $\mathbb{A}^m$:

$$\begin{array}{ccc} Y & \hookrightarrow & M\times\mathbb{A}^m \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi_1} & \\ & M & \end{array} \tag{22.4}$$

(22.5) *Relative Gauss map.* The *relative tangent bundle* $T(Y/M)\to Y$ is the kernel of the differential of π, the set of tangent vectors which point along the fibers of π. The embedding gives a Gauss map (see **(20.23)**)

$$\begin{array}{ccc} T(Y/M) & \longrightarrow & S(n-1) \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Gr_{n-1}(\mathbb{R}^m) \end{array} \tag{22.6}$$

We emphasize: A fiber bundle, or proper submersion, has a tangent bundle along the fibers, which is identified with the pullback of the universal subbundle $S(n-1)\to Gr_{n-1}(\mathbb{R}^m)$.

The normal bundle $\nu\to Y$ to the embedding in (22.4) is also the normal bundle to the embedding of each fiber of π in $\mathbb{A}^m$, since π is a submersion, and the embedding induces a classifying map

$$\begin{array}{ccc} \nu & \longrightarrow & Q(m-n+1) \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Gr_{n-1}(\mathbb{R}^m) \end{array} \tag{22.7}$$

(22.8) *Pontrjagin-Thom collapse.* As in Lecture 2 and Lecture 10, choose a tubular neighborhood of $Y\subset M\times\mathbb{A}^m$. Then the Pontrjagin-Thom collapse induced by the embedding, followed by the map on Thom spaces induced from (22.7), is

$$M_+\wedge S^m\longrightarrow Y^\nu\longrightarrow Gr_{n-1}(\mathbb{R}^m)^{Q(m-n+1)}. \tag{22.9}$$

Here M_+ is the union of M and a disjoint basepoint, and the domain is the one-point compactification of $M\times\mathbb{A}^m$. According to Definition 20.31 the last space in (22.9) is the m^{th} space of the Madsen-Tillmann spectrum $MTO(n-1)$. So (22.9) is a pointed map of the m^{h} suspension of M_+ into the m^{th} space of the prespectrum which completes to the spectrum $MTO(n-1)$. Therefore, it represents a map of M into $MTO(n-1)_0$.

In summary, from a fiber bundle (22.4) of $(n-1)$-manifolds with embedding we have produced a map of the base into the 0-space of the spectrum $MTO(n-1)$.

(22.10) *An attempted inverse.* Conversely, a map $M \to MTO(n-1)_0$ is represented, for sufficiently large m, by a pointed map

$$g\colon M_+ \wedge S^m \longrightarrow Gr_{n-1}(\mathbb{R}^m)^{Q(m-n+1)}. \tag{22.11}$$

After a homotopy we can arrange that g be transverse to the zero section of the bundle $Q(m-n+1) \to Gr_{n-1}(\mathbb{R}^m)$. Then the inverse image of the zero section is a submanifold $Y \subset M \times \mathbb{A}^m$ with $\dim Y - \dim M = n-1$. If we assume that M is compact, which we do, then Y is also compact. There is also a classifying map (22.7) of the normal bundle, the restriction of (22.11) to a map $Y \to Gr_{n-1}$. Let $V \to Y$ be the pullback of $S(n-1) \to Gr_{n-1}(\mathbb{R}^m)$.

If—and this is not generally true—the composition $Y \hookrightarrow M \times \mathbb{A}^m \to M$ is a submersion, then since Y is compact it is a fiber bundle. We would deduce that maps into the Madsen-Tillmann spectrum give fiber bundles. But that is not true. Nor is it true, even if the composition is a submersion, that $V \to Y$ can be identified with the relative tangent bundle.

The main construction: real version

The main tool to obtain a submersion is the Phillips Submersion Theorem. It is part of a circle of ideas in differential topology called *immersion theory* [Sp], and one of the main tools used in the proofs is Gromov's *h-principle* [ElMi]. We simply quote the result here.

Theorem 22.12 (Phillips [Ph])**.** *Let X be a smooth manifold with no closed components and M a smooth manifold with* $\dim M \le \dim X$*. Then the differential*

$$\mathrm{Submersion}(X, M) \xrightarrow{\;d\;} \mathrm{Epi}(TX, TM) \tag{22.13}$$

is a weak homotopy equivalence.

Here $\mathrm{Epi}(TX, TM)$ is the space of smooth maps $L\colon TX \to TM$ which sends fibers to fibers and restricts on each fiber to a surjective linear map (epimorphism). The domain of (22.13) is the space of submersions $X \to M$, and the differential maps a submersion to an epimorphism on tangent bundles. Note that a manifold with no closed components is often called an *open manifold*.

Theorem 22.12 is precisely the tool needed to deform the map $Y \to M$ in **(22.10)** to a submersion. But to do so we must replace Y be a noncompact manifold. The simplest choice is $X = \mathbb{R} \times Y$. We indicate the modifications to the previous heuristic section which incorporate this change.

(22.14) *The sheaf D.* We introduce a sheaf $D = D_n^{\mathfrak{X}(n)}\colon \mathrm{Man}^{\mathrm{op}} \to \mathrm{Set}$ which represents $\bigl(\Sigma MT\mathfrak{X}(n)\bigr)_0$.

Definition 22.15. Fix $M \in \mathrm{Man}$. An element of $D(M)$ is a pair (X, θ) consisting of a submanifold $X \subset M \times \mathbb{R} \times \mathbb{A}^\infty$ and an $\mathfrak{X}(n)$-structure θ. The submanifold must satisfy

(i) $\pi_1\colon X \to M$ is a submersion with fibers of dimension n, and
(ii) $\pi_1 \times \pi_2\colon X \to M \times \mathbb{R}$ is proper.

The relative tangent bundle $T(X/M) \to X$ has rank n and, because of the embedding, comes equipped with a Gauss map

$$\begin{array}{ccc} T(X/M) & \longrightarrow & S(n) \\ \downarrow & & \downarrow \\ X & \longrightarrow & Gr_n(\mathbb{R}\times\mathbb{R}^\infty) \end{array} \tag{22.16}$$

The tangential structure is a fibration $\mathcal{X}(n) \to Gr_n(\mathbb{R}\times\mathbb{R}^\infty)$, fixed once and for all. The relative $\mathcal{X}(n)$-structure θ is a lift

$$\begin{array}{ccc} T(X/M) & \xrightarrow{\theta} & S(n) \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathcal{X}(n) \end{array} \tag{22.17}$$

of the Gauss map, as in (20.25).

We remark that the embedding $X \hookrightarrow M\times\mathbb{R}\times\mathbb{A}^\infty$ is required to satisfy the technical condition in the footnote of Example 21.14. For this exposition we restrict to M compact.

The first condition in Definition 22.15 implies that π_1 is a family of n-manifolds, but it is not a fiber bundle as the fibers may be noncompact, so as we move over the base M the topology of the fibers can change. The second condition implies that each fiber comes with a real-valued function π_2 with compact fibers. The inverse image of a regular value $a\in\mathbb{R}$ is a closed $(n-1)$-manifold, but the topology depends on the regular value. The inverse images of two regular values $a_0 < a_1$ comes with a bordism: the inverse image of $[a_0, a_1]$. See Figure 40.

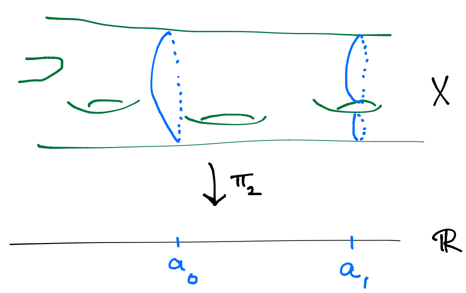

Figure 40. A fiber of $X \to M$

(22.18) *Statement of theorems.* Recall the notion of concordance **(21.32)**.

Theorem 22.19. *For any $M \in$ Man there is a bijection*

$$D[M] \cong \big[M\,,\ \big(\Sigma MT\mathcal{X}(n)\big)_0\big] \tag{22.20}$$

between concordance classes of elements of $D(M)$ and homotopy classes of maps into the 0-space of the suspended Madsen-Tillmann spectrum.

We sketch the construction of the bijection (22.20) in the remainder of this section.

Recall from Theorem 21.38 that the representing space $|D|$ also satisfies

$$D[M] \cong \bigl[M\,,\,|D|\bigr], \tag{22.21}$$

and so the following is not surprising.

Corollary 22.22. *There is a weak homotopy equivalence*

$$|D| \simeq \bigl(\Sigma MT\mathcal{X}(n)\bigr)_0. \tag{22.23}$$

The proof uses an auxiliary sheaves which keep track of the contractible choices (of a tubular neighborhood, of a regular value) which are used below. We refer to [Po, §4.3] for a sketch of how that argument goes.

(22.24) *Sketch of* (22.20) $\longrightarrow$. Given an element $X \subset M \times \mathbb{R} \times \mathbb{A}^\infty$ of $D(M)$, choose $a \in \mathbb{R}$ a regular value of π_2, m a positive integer such that $X \subset M \times \mathbb{R} \times \mathbb{A}^m$. Let $Y \subset M \times \mathbb{A}^m$ be the intersection of X and $M \times \{a\} \times \mathbb{A}^m$. The normal bundle to $Y \subset M \times \mathbb{A}^m$ is the restriction of the normal bundle to $X \subset M \times \mathbb{R} \times \mathbb{A}^m$. Therefore, as in (22.7) but now using the lift (22.17) from the $\mathcal{X}(n)$-structure, we obtain a classifying map

$$\begin{array}{ccc} \nu & \longrightarrow & Q(m-n+1) \\ \big\downarrow & & \big\downarrow \\ Y & \longrightarrow & \mathcal{X}(n,m+1) \end{array} \tag{22.25}$$

where the southeast space is the pullback

$$\begin{array}{ccc} \mathcal{X}(n,m+1) & \longrightarrow & \mathcal{X}(n) \\ \big\downarrow & & \big\downarrow \\ Gr_n(\mathbb{R}\times\mathbb{R}^m) & \longrightarrow & Gr_n(\mathbb{R}\times\mathbb{R}^\infty) \end{array} \tag{22.26}$$

Choose a tubular neighborhood of $Y \subset M \times \mathbb{A}^m$. The Pontrjagin-Thom collapse, as in (22.9), is a map

$$M_+ \wedge S^m \longrightarrow \mathcal{X}(n,m+1)^{Q(m-n+1)}. \tag{22.27}$$

The last space is the $(m+1)$-space of the Madsen-Tillmann spectrum $MT\mathcal{X}(n)$, so that (22.27) represents a (non-pointed) map of M into the 0-space of the shifted spectrum $\Sigma MT\mathcal{X}(n)$.

(22.28) *Sketch of* (22.20) $\longleftarrow$. We proceed as in **(22.10)**, but now mapping into the last space in (22.27), to obtain as there

(i) a submanifold $Y \subset M \times \mathbb{A}^m$ with $\dim Y - \dim M = n-1$,
(ii) a classifying map of the normal bundle

$$\begin{array}{ccc} \nu & \longrightarrow & Q(m-n+1) \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\;g\;} & \mathcal{X}(n,m+1) \end{array} \tag{22.29}$$

and

(iii) a rank n vector bundle $W \to Y$, defined as $g^*\bigl(S(n) \to \mathcal{X}(n,m+1)\bigr)$.

These bundles over Y come equipped with isomorphisms

$$\begin{aligned} \nu \oplus W &\xrightarrow{\cong} \underline{\mathbb{R}^{m+1}} \\ \nu \oplus TY &\xrightarrow{\cong} TM \oplus \underline{\mathbb{R}^m} \end{aligned} \tag{22.30}$$

The first is induced from the tautological exact sequence (6.8) after choosing once and for all a splitting over $\mathcal{X}(n,m+1)$. The second comes from splitting the usual exact sequence for a submanifold. Combining these isomorphisms we obtain an isomorphism

$$\underline{\mathbb{R}^{m+1}} \oplus TY \xrightarrow{\cong} W \oplus \nu \oplus TY \xrightarrow{\cong} W \oplus TM \oplus \underline{\mathbb{R}^m}. \tag{22.31}$$

The next step is to "strip off" the trivial bundle of rank m in the isomorphism (22.31). This is possible for m sufficiently large. The proof is an application of the following general principle, which can be proved by obstruction theory. Recall that for $k \in \mathbb{Z}^{\ge 0}$ a space is *k-connected* if it is connected and all homotopy groups π_q, $q \le k$ vanish. A map is k-connected if its mapping cylinder is k-connected.

Proposition 22.32.

(i) *Let $E \to Y$ be a (continuous) fiber bundle with k-connected fiber and base Y a CW complex of dimension ℓ. Then the space $\Gamma(Y;E)$ of sections is $(k-\ell)$-connected.*
(ii) *Let $(E_1 \to E_2) \longrightarrow Y$ be a map of fiber bundles. Assume the map on each fiber is k-connected and $\dim Y = \ell$. Then the induced map of sections $\Gamma(Y;E_1) \to \Gamma(Y;E_2)$ is $(k-\ell)$-connected.*

Our application is to the map

$$\mathrm{Iso}\bigl(\underline{\mathbb{R}} \oplus TY\,,\, W \oplus TM\bigr) \longrightarrow \mathrm{Iso}\bigl(\underline{\mathbb{R}^{m+1}} \oplus TY\,,\, W \oplus TM \oplus \underline{\mathbb{R}^m}\bigr) \tag{22.33}$$

of fiber bundles of isomorphisms of vector bundles over Y. On fibers this is the standard embedding of general linear groups $GL_{n+d}(\mathbb{R}) \hookrightarrow GL_{n+d+m}$, where $d = \dim M$. This map is $(n+d-1)$-connected, so the induced map on sections is $(n-1)$-connected. Since $n-1>0$, this implies that the isomorphism (22.31) is isotopic to an isomorphism which is the stabilization of an isomorphism

$$\underline{\mathbb{R}} \oplus TY \xrightarrow{\cong} W \oplus TM. \tag{22.34}$$

Now compose the isomorphism (22.34) with projection onto TM to obtain an epimorphism

$$T(\mathbb{R}\times Y)\cong \underline{\mathbb{R}}\oplus TY \xrightarrow{\;\cong\;} TM. \tag{22.35}$$

The Phillips Submersion Theorem 22.12 implies that there is a submersion $\pi_1\colon X=\mathbb{R}\times Y\to M$ whose differential is isotopic to (22.35). The isomorphism (22.34) induces an isomorphism

$$W \xrightarrow{\;\cong\;} T(X/M)=\ker d\pi_1. \tag{22.36}$$

Projection gives a function $\pi_2\colon X\to\mathbb{R}$, and we can use the Whitney embedding theorem to construct $\pi_3\colon X\hookrightarrow \mathbb{A}^{m'}$ for m' sufficiently large. The product $\pi_1\times\pi_2\times\pi_3\colon X\hookrightarrow M\times\mathbb{R}\times\mathbb{A}^{m'}$ is the desired element of $D(M)$. (A more delicate argument produces the $\mathcal{X}(n)$-structure.)

Remark 22.37. This completes the sketch construction of the two maps in (22.20). The proof that they are inverse is based on [MW, Lemma 2.5.2].

A sheaf model of the topological bordism category

It is fairly straightforward to construct a sheaf of (discrete) categories whose representing space is the topological bordism category ${}^t\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}$. This is a more elaborate version of Example 21.30, where there is a "representing category in the category of smooth infinite dimensional manifolds".

Definition 22.38. The sheaf of categories $C=C^{\mathcal{X}(n)}_n\colon \mathrm{Man}^{\mathrm{op}}\to\mathrm{Cat}$ is defined on a test manifold $M\in\mathrm{Man}$ as follows. The objects of $C(M)$ are triples (a,Y,θ) consisting of a smooth function $a\colon M\to\mathbb{R}$, an embedding $Y\hookrightarrow M\times\mathbb{A}^\infty$ such that $\pi_1\colon Y\to M$ is a proper submersion, and an $\mathcal{X}(n)$-structure θ on the relative tangent bundle. A morphism $(a_0,Y_0,\theta_0)\to(a_1,Y_1,\theta_1)$ is a pair (X,Θ) consisting of a neat submanifold $X\hookrightarrow M\times[a_0,a_1]\times\mathbb{A}^\infty$ with $\mathcal{X}(n)$-structure Θ such that $\pi_1\colon X\to M$ is a proper submersion and, for some $\delta\colon M\to\mathbb{R}^{>0}$

$$\begin{aligned} X\cap\big(M\times[a_0,a_0+\delta)\times\mathbb{A}^\infty\big)&=Y_0\times[a_0,a_0+\delta)\\ X\cap\big(M\times(a_1-\delta,a_1]\times\mathbb{A}^\infty\big)&=Y_1\times(a_1-\delta,a_1]\end{aligned} \tag{22.39}$$

as $\mathcal{X}(n)$-manifolds.

Here $M\times[a_0,a_1]\subset M\times\mathbb{R}$ is the subset of pairs (m,t) such that $a_0(m)\le t\le a_1(t)$. Composition is by union, as usual when we have embeddings.

As indicated, $C(M)$ is the space of smooth maps $M\to {}^t\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}$, with the appropriate smooth structure on ${}^t\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}$, and this gives a map of topological categories

$$\eta\colon |C|\longrightarrow {}^t\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}\,. \tag{22.40}$$

We have not defined a weak equivalence of topological categories, which is what we'd like to say (22.40) is, but in any case the definition would amount to the following.

Theorem 22.41. *η induces a weak homotopy equivalence of classifying spaces*

$$B|C| \longrightarrow B\bigl({}^t\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}\bigr). \tag{22.42}$$

Sketch of proof. The space of q-simplices $N_q|C|$ in the nerve of $|C|$ is the geometric realization of the extended, smooth singular simplices on the space $N_q{}^t\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}$, so by Theorem 21.28 the map

$$N_q|C| \longrightarrow N_q{}^t\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle} \tag{22.43}$$

induced by η is a weak homotopy equivalence. Thus $B\eta$ is also a weak homotopy equivalence. □

Comments on the rest

We hope to have given a "reader's guide" to much of the proof in [GMTW]. At this point we have a sheaf of categories C which represents the space $B\bigl({}^t\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}\bigr)$ and a sheaf of spaces D which represents the space $\bigl(\Sigma MT\mathcal{X}(n)\bigr)_0$. To put them on equal footing we regard D as a sheaf of categories with only identity morphisms. The goal now is to construct a weak homotopy equivalence of these sheaves. This is not done directly, but by means of two intermediate sheaves of categories. There are two main discrepancies between $C(\mathrm{pt})$ and $D(\mathrm{pt})$, and so between $C(M)$ and $D(M)$ which are parametrized families. First, objects in D are to be thought of as fibers at an unspecified regular value $a\in\mathbb{R}$ of a proper map $X\to\mathbb{R}$, whereas objects in C have a specified value of a. Second, morphisms in D are manifolds without boundary ($X\to\mathbb{R}$) whereas morphisms in C are manifolds with boundary. The intermediate sheaves $D^{\pitchfork}$, $C^{\pitchfork}$ mediate these discrepancies. We sketch the definitions below. The main work is in proving that straightforwardly defined maps

$$D \xleftarrow{\ \alpha\ } D^{\pitchfork} \xrightarrow{\ \gamma\ } C^{\pitchfork} \xleftarrow{\ \delta\ } C \tag{22.44}$$

are weak homotopy equivalences.

(22.45) *The sheaf $D^{\pitchfork}$ and the map α.* For convenience we omit the $\mathcal{X}(n)$-structures from the notation: they are just carried along.

The objects are a subsheaf of $\mathcal{F}_{\mathbb{R}}\times D$ where $\mathcal{F}_{\mathbb{R}}$ is the representable sheaf of real-valued functions (see Example 21.9). An object in $D^{\pitchfork}(M)$ is a pair (a,X) where $X\subset M\times\mathbb{R}\times\mathbb{A}^\infty$ is an object of $D(M)$ and $a\colon M\to\mathbb{R}$ has the property that $a(m)$ is a regular value of $\pi_2\big|_{\pi_1^{-1}(M)}$. It is a category of partially ordered sets: there is a unique morphisms $(a_0,X_0)\to(a_1,X_1)$ if $(a_0,X_0)\le(a_1,X_1)$, and the latter is true if and only if $X_0=X_1$, the functions satisfy $a_0\le a_1$ and $a_0=a_1$ on a union of components of M.

The map α is the forgetful map which forgets a.

(22.46) *The sheaf $C^{\pitchfork}$ and the maps δ, γ.* This is very similar to C, but the objects and morphisms are not "sharply cut off" at points $a \in \mathbb{R}$. So objects have a bicollaring and morphisms are open with collars at a_0, a_1. We refer to [GMTW, §2] for details.

The map δ puts product bicollars and collars on the objects and morphisms of C.

To define $\gamma(a, X)$ we use the fact that a consists of regular values to find a function $\epsilon\colon M \to \mathbb{R}^{>0}$ so that $(a-\epsilon, a+\epsilon)$ also consists of regular values. (The notation is as in Definition 22.38.) Then $Y = (\pi_1 \times \pi_2)^{-1}\bigl(M \times (a-\epsilon, a+\epsilon)\bigr)$ is an object of $C^{\pitchfork}$. There is a similar construction on morphisms.

(22.47) *Proofs of equivalences.* The techniques to prove that the maps α, γ, δ are weak equivalences are presented in [GMTW] with technical details in [MW].

References

[ElMi] Y. Eliashberg and N. Mishachev, *Introduction to the h-principle*, Graduate Studies in Mathematics, vol. 48, American Mathematical Society, Providence, RI, 2002. 3

[GMTW] Søren Galatius, Ulrike Tillmann, Ib Madsen, and Michael Weiss, *The homotopy type of the cobordism category*, Acta Math. **202** (2009), no. 2, 195–239, `arXiv:math/0605249`. 1, 8, 9

[MW] Ib Madsen and Michael Weiss, *The stable moduli space of Riemann surfaces: Mumford's conjecture*, Ann. of Math. (2) **165** (2007), no. 3, 843–941. 1, 7, 9

[Ph] Anthony Phillips, *Submersions of open manifolds*, Topology **6** (1967), 171–206. 1, 3

[Po] Geoffrey Powell, *The Mumford conjecture (after Madsen and Weiss)*, Astérisque (2006), no. 307, Exp. No. 944, viii, 247–282. Séminaire Bourbaki. Vol. 2004/2005. 5

[Sp] David Spring, *The golden age of immersion theory in topology: 1959–1973. A mathematical survey from a historical perspective*, Bull. Amer. Math. Soc. (N.S.) **42** (2005), no. 2, 163–180 (electronic). 3