Lecture 23: An application of Morse-Cerf theory

We review quickly the idea of a Morse function and recall the basic theorems of Morse theory. Passing through a single critical point gives an *elementary bordism*; a very nice Morse function—an *excellent* function—decomposes an arbitrary bordism as a sequence of elementary bordisms. The space of excellent functions is not connected, but is if we relax the excellence standard slightly. This basic idea of Cerf theory relates different decompositions. We use it to classify 2-dimensional oriented TQFTs with values in the category of vector spaces. This is one of the earliest theorems in the subject, dating at least from Dijkgraaf's thesis [Dij].

Morse functions

(23.1) *Critical points and the hessian.* Let M be a smooth manifold and $f\colon M \to \mathbb{R}$ a smooth function. Recall that $p \in M$ is a *critical point* if $df_p = 0$. A number $c \in \mathbb{R}$ is a *critical value* if $f^{-1}(c)$ contains a critical point. At a critical point p the second differential, or Hessian,

$$d^2 f_p\colon T_pM \times T_pM \longrightarrow \mathbb{R} \tag{23.2}$$

is a well-defined symmetric bilinear form. To evaluate it on $\xi_1, \xi_2 \in T_pM$ extend ξ_2 to a vector field to near p, and set $d^2 f_p(\xi_1, \xi_2) = \xi_1 \xi_2 f(p)$, the iterated directional derivative. We say p is a *nondegenerate critical point* if the Hessian (23.2) is a nondegenerate symmetric bilinear form.

Lemma 23.3 (Morse)**.** *If p is a nondegenerate critical point of the function $f\colon M \to \mathbb{R}$, then there exists a local coordinate system $x^1, \dots, x^n$ about p such that*

$$f = (x^1)^2 + \cdots + (x^r)^2 - (x^{r+1})^2 - \cdots - (x^n)^2 + c \tag{23.4}$$

for some p.

The number $n - r$ of minus signs in (23.4) is the *index* of the critical point p.

An application of Sard's theorem proves that Morse functions exist, and in fact are open and dense in the space of C^∞ functions (in the Whitney topology **(20.1)**).

(23.5) *Morse functions on bordisms.* If X is a manifold with boundary we consider smooth functions which are constant on ∂X and have no critical points on ∂X. The following terminology is apparently due to Thom.

Definition 23.6. Let $X\colon Y_0 \to Y_1$ be a bordism. An *excellent* function $f\colon X \to \mathbb{R}$ satisfies

(i) $f(Y_0) = a_0$ is constant;
(ii) $f(Y_1) = a_1$ is constant; and
(iii) The critical points $x_1, \dots, x_N$ have distinct critical values $c_1, \dots, c_N$ which satisfy

$$a_0 < c_1 < \cdots < c_N < a_1. \tag{23.7}$$

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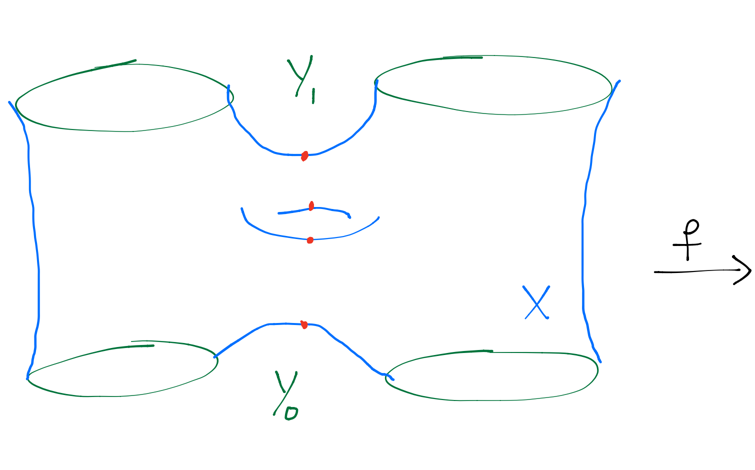

FIGURE 41. An excellent function on a bordism

We depict an excellent function on a bordism in Figure 41.

Proposition 23.8. *Let $X\colon Y_0 \to Y_1$ be a bordism. Then the space of excellent functions on X is open and dense.*

(23.9) *Passing a critical level.* The basic theorems of Morse theory tell the structure of $X_{a',a''} = f^{-1}([a',a''])$ if a',a'' are regular values. If there are no critical values in $[a',a'']$, then $X_{a',a''}$ is diffeomorphic to the Cartesian product of $[a',a'']$ and $Y = f^{-1}(a)$ for any $a \in [a',a'']$. If there is a single critical value $c \in [a',a'']$ and $f^{-1}(c)$ contains a single critical point of index q, then $X_{a',a''}$ is obtained from $X_{a',c-\epsilon}$ by attaching an *n-dimensional q-handle*. We defer to standard books [M4, PT] for a detailed treatment of Morse theory.

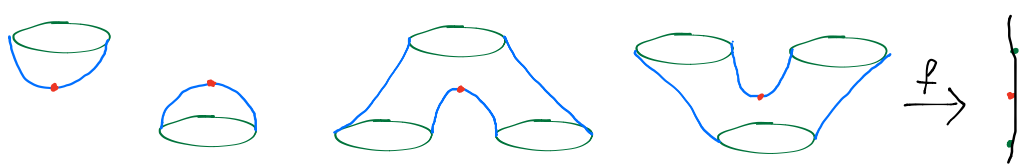

FIGURE 42. Some elementary 2-dimensional bordisms

Definition 23.10. A bordism $X\colon Y_0 \to Y_1$ is an *elementary bordism* if it admits an excellent function with a single critical point.

The elementary 2-dimensional bordisms are depicted in Figure 42.

(23.11) *Decomposition into elementary bordisms.* An excellent function on *any* bordism $X\colon Y_0 \to Y_1$ expresses it as a composition of elementary bordisms

$$X = X_N \circ \cdots \circ X_1 \tag{23.12}$$

where $X_1 = f^{-1}([a_0, c_1 + \epsilon])$, $X_2 = f^{-1}([c_1 + \epsilon, c_2 + \epsilon])$, $\dots$,$X_N = f^{-1}([c_{N-1} + \epsilon, a_1])$. Excellent functions connected by a path of excellent functions lead to an equivalent decomposition: corresponding elementary bordisms are diffeomorphic. We can track the equivalence class by a *Kirby graphic* (Figure 43) which indicates the distribution of critical points and their indices. The space of excellent functions is not connected; a bordism has (infinitely) many decompositions with different Kirby graphic.

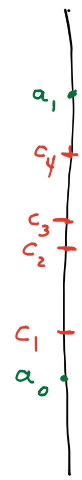

FIGURE 43. The Kirby graphic of Figure 41

Elementary Cerf theory

Jean Cerf [C] studied a filtration on the space of smooth functions. The subleading part of the filtration connects different components of excellent functions.

Definition 23.13. A smooth function $f\colon M \to \mathbb{R}$ on an n-manifold M has a *birth-death singularity* at $p \in M$ if there exist local coordinates $x^1, \dots, x^n$ in which

$$f = (x^1)^3 + (x^1)^2 + \cdots + (x^r)^2 - (x^{r+1})^2 - \cdots - (x^n)^2 + c \tag{23.14}$$

We say the index of p is $n - r$.

There is an intrinsic definition: p is a degenerate critical point, the null space $N_p \subset T_pM$ of $d^2 f_p$ has dimension one, and the third differential $d^3 f_p$ is nonzero on N_p.

Definition 23.15. Let $X\colon Y_0 \to Y_1$ be a bordism and $f\colon X \to \mathbb{R}$ a smooth function.

(i) f is *good of Type α* if f is excellent except at a single point at which f has a birth-death singularity.

(ii) f is *good of Type β* if f is excellent except that there exist exactly two critical points x_i, x_{i+1} with the same critical value $f(x_i) = f(x_{i+1})$.

We say f is *good* if it is either excellent or good of Type α or good of Type β.

Theorem 23.16 (Cerf [C])**.** *Let $X\colon Y_0\to Y_1$ be a bordism. Then the space of good functions is connected. More precisely, if f_0,f_1 are excellent, then there exists a path f_t of good functions such that f_t is excellent except at finitely many values of t.*

There is an even more precise statement. The space of good functions is an infinite dimensional manifold, the space of good functions which are not excellent is a codimension one submanifold, and the path $t\mapsto f_t$ crosses this submanifold transversely at finitely many values of t.

A path of good functions has an associated Kirby graphic which encodes the excellent chambers and wall crossings of the path. The horizontal variable it t and the vertical is the critical value. The curves in the graphic are labeled by the index of the critical point in the preimage. Birth-death singularities occur with critical points of neighboring indices. Kirby uses these graphics in his calculus [Ki]. Figure 44 shows some simple Kirby graphics.

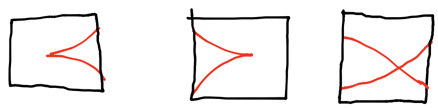

Figure 44. Kirby graphics of a birth, death, and exchange

Example 23.17. The prototype for crossing a wall of Type α is the path of functions

$$f_t(x)=\frac{x^3}{3}-tx \tag{23.18}$$

defined for $x\in\mathbb{R}$. Then f_t is Morse for $t\neq 0$, has no critical points if $t<0$, and has two critical points $x=\pm\sqrt{t}$ for $t>0$. As t increases through $t=0$ the two critical points are born; as t decreases through $t=0$ they die. The critical values are $\pm t^{3/2}$, up to a multiplicative constant, which explains the shape of the Kirby graphic.

These Cerf wall crossings relate different decompositions (23.12) of a bordism into elementary bordisms. In the next section we apply this to construct a 2-dimensional TQFT by "generators and relations": we define it on elementary bordisms and use the Cerf moves to check consistency.

Application to TQFT

(23.19) *Frobenius algebras.* Before proceeding to 2-dimensional field theories, we need some algebra.

Definition 23.20. Let k be a field. A *commutative Frobenius algebra* (A,τ) over k is a finite dimensional unital commutative associative algebra A over k an a linear map $\tau\colon A\to k$ such that

$$\begin{aligned} A\times A&\longrightarrow k\\ x,y&\longmapsto \tau(xy)\end{aligned} \tag{23.21}$$

is a nondegenerate pairing.

Example 23.22 (Frobenius)**.** Let G be a finite group. Let A be the vector space of functions $f\colon G \to \mathbb{C}$ which are *central*: $f(gxg^{-1}) = f(x)$ for all $x, g \in G$. Define multiplication as convolution:

$$f_1 * f_2(x) = \sum_{x_1x_2=x} f_1(x_1)f_2(x_2). \tag{23.23}$$

A straightforward check shows $*$ is commutative and associative and the unit is the "δ-function", which is 1 at the identity $e \in G$ and 0 elsewhere. The trace is

$$\tau(f) = \frac{f(e)}{\#G}. \tag{23.24}$$

If we remove the central condition, then we obtain the noncommutative Frobenius algebra of all complex-valued functions on G.

Example 23.25. Let M be a closed oriented n-manifold. Then $H^\bullet(M;\mathbb{C})$ is a super commutative Frobenius algebra. Multiplication is by cup product and the trace is evaluation on the fundamental class. The 'super' reflects the sign in the cup product. For $M = S^2$ we obtain an ordinary commutative Frobenius algebra since there is no odd cohomology. This is a key ingredient in the original construction of Khovanov homology [Kh].

(23.26) *2-dimensional oriented TQFT.* The following basic result was well-known by the late 1980s. It appears in Dijkgraaf's thesis [Dij]. More mathematical treatments can be found in [Ab, Ko]. The Morse theory proof we give below is taken from [MoSe, Appendix].

Theorem 23.27. *Let $F\colon \mathrm{Bord}^{SO}_{\langle 1,2\rangle} \to \mathrm{Vect}_k$ be a TQFT. Then $F(S^1)$ is a commutative Frobenius algebra. Conversely, if A is a commutative Frobenius algebra, then there exists a TQFT $F_A\colon \mathrm{Bord}^{SO}_{\langle 1,2\rangle} \to \mathrm{Vect}_k$ such that $F_A(S^1) = A$.*

Remark 23.28. The 2-dimensional field theory constructed from the Frobenius algebra in Example 23.22 has a "classical" description: it counts principal G-bundles, which for a finite group G are regular covering spaces with Galois group G. The invariant $F(X)$ of a closed surface of genus g is given by a classical formula of Frobenius. The TQFT provides a proof of that formula by cutting a surface of genus g into elementary pieces.

We give the proof of Theorem 23.27 which is in [MS].

Proof. Given $F\colon \mathrm{Bord}^{SO}_{\langle 1,2\rangle} \to \mathrm{Vect}_k$ define the vector space $A = F(S^1)$. The elementary bordisms in Figure 45 define a unit $u\colon k \to A$, a trace $\tau\colon A \to k$, and a multiplication $m\colon A \otimes A \to A$. (We read "time" as flowing up in these bordisms; the bottom boundaries are incoming and the top boundaries are outgoing.) The bilinear form (23.21) is the composition in Figure 46, and it has an inverse given by the cylinder with both boundary components outgoing, as is proved by the S-diagram argument. Therefore, it is nondegenerate. This proves that (A, u, m, τ) is a commutative Frobenius algebra.

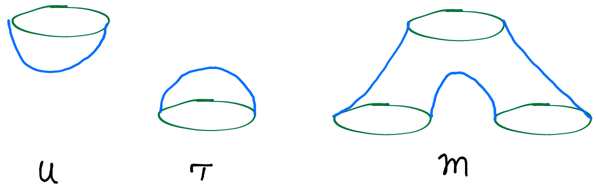

FIGURE 45. Elementary bordisms which define the Frobenius structure

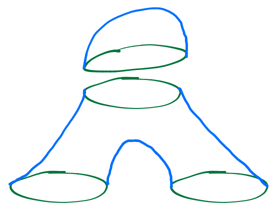

FIGURE 46. The bilinear form

Next we compute the map defined by the time-reversal of the multiplication (Figure 47). Let $x_1, \dots, x_n$ and $x^1, \dots, x^n$ be dual bases of A relative to (22.42): $\tau(x^i x_j) = \delta^i_j$. Then

$$\begin{aligned} m^*\colon A &\longrightarrow A\otimes A \\ x &\longmapsto x x_i \otimes x^i \end{aligned} \tag{23.29}$$

This is the adjoint of multiplication relative to the pairing (23.21). Similarly, note that the unit $u = \tau^*$ is adjoint to the trace. In fact, these adjunctions follow from general duality in symmetric monoidal categories. The time-reversal is the dual in the bordism category (Definition 1.22, **(2.20)**, Theorem 15.29),[1] and the dual in the category of vector spaces is the usual dual. A symmetric monoidal functor, such as F_A, maps duals to duals (Proposition 15.34).

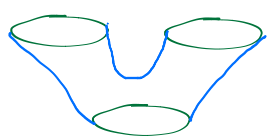

FIGURE 47. The adjoint m^*

For the converse, suppose A is a commutative Frobenius algebra. We construct a 2-dimensional TQFT F_A.

It is easy to prove that the topological group $\mathrm{Diff}^{SO}(S^1)$ of orientation-preserving diffeomorphisms retracts onto the group of rotations, which is connected. Since diffeomorphisms act on A through their isotopy class, the action is trivial. Thus is Y is any oriented manifold diffeomorphic to a circle, there is up to isotopy a unique orientation-preserving diffeomorphism $Y \to S^1$. For any

[1] We also note that an oriented surface admits an orientation-reversing involution, so is diffeomorphic to the same underlying manifold with the opposite orientation.

closed oriented 1-manifold Y define $F_A(Y) = A^{\otimes(\#\pi_0 Y)}$; orientation-preserving diffeomorphisms of closed 1-manifolds act as the identity.

The value of F_A on elementary 2-dimensional bordisms (Figure 42) are given by the structure maps $u = \tau^*, \tau, m, m^*$ of the Frobenius algebra. An arbitrary bordism is a composition of elementary bordisms (tensor identity maps) via an excellent Morse function, and we use such a decomposition to define F_A. However we must check that the value is independent of the excellent Morse function. For that we use Cerf's Theorem 23.16. It suffices to check what happens when we cross a wall of Type α or of Type β.

First, a simplification. Since time-reversal implements duality, if an equality of maps holds for a wall-crossing it also holds for its time-reversal. This cuts down the number of diagrams one needs to consider.

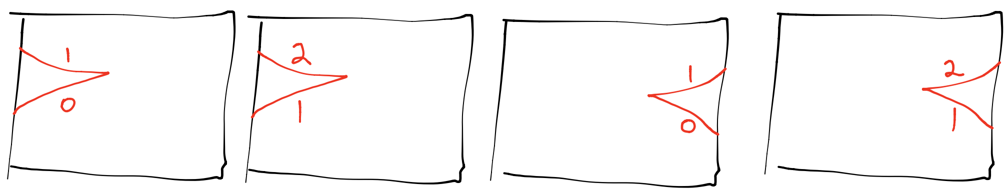

FIGURE 48. The four Type α wall-crossings

There are four Type α wall-crossings, as indicated by their Kirby graphics in Figure 48. The numbers indicate the index of the critical point. If f_t is a path of Morse functions with the first Kirby graphic, then the three subsequent ones may be realized by $-f_t$, f_{1-t}, and $-f_{1-t}$, respectively. (Here $0 \le t \le 1$.) It follows that we need only check the first. The corresponding transition of bordisms is indicated in Figure 49. These bordisms both map to $\mathrm{id}_A \colon A \to A$: for the first this expresses that u is an identity for the multiplication m.

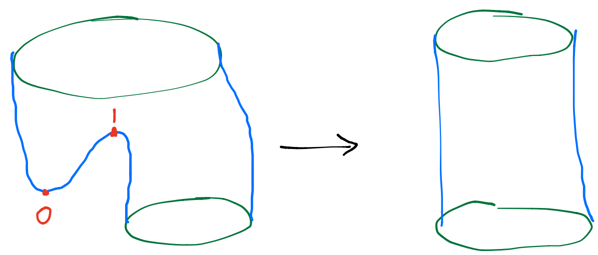

FIGURE 49. Crossing a birth-death singularity

In a Type β wall-crossing there are two critical points and the critical levels cross. So on either side of the wall the bordism X is a composition of two elementary bordisms. We assume X is connected or there is nothing to prove. Furthermore, if the indices of the critical points are q_1, q_2, then the Euler characteristic of the bordism is $(-1)^{q_1} + (-1)^{q_2}$, by elementary Morse theory. Let C denote the critical contour at the critical time t_{crit}, when the two critical levels cross. Since the bordism is connected there are two possibilities: either C is connected or it consists of two components, each with a single critical point. In the latter case there would have to be another critical point in the bordism to connect the two components, else the bordism would not be connected. Therefore, C is connected and it follows easily that both critical points have index 1, whence X has Euler characteristic -2.

Now in each elementary bordism (Figure 42) the number of incoming and outgoing circles differs by one, so in a composition of two elementary bordisms the number of circles changes by two or does not change at all. This leads to four possibilities for the number of circles: $1 \to 1$, $2 \to 2$, $3 \to 1$, or $1 \to 3$. The last is the time-reversal of the penultimate, so we have three cases to consider.

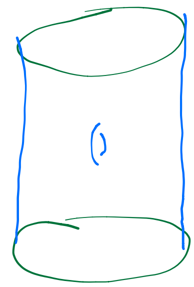

FIGURE 50. $1 \to 1$

The first, $1 \to 1$, is a torus with two disks removed. Figure 50 is not at the critical time—the two critical levels are distinct. Note that at a regular value between the two critical values, the level curve has two components, by the classification of elementary bordisms (Figure 42). So the composition is

$$A \xrightarrow{m^*} A \otimes A \xrightarrow{m} A \tag{23.30}$$

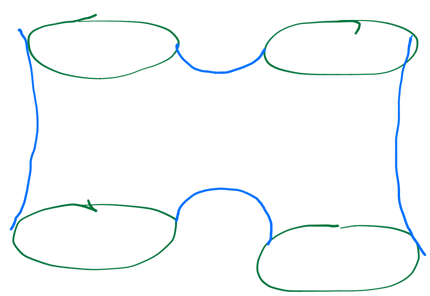

FIGURE 51. $2 \to 2$

The second case, $2 \to 2$, is somewhat more complicated than the others. The number of circles in the composition is either $2 \to 1 \to 2$ or $2 \to 3 \to 2$. The $2 \to 1 \to 2$ composition, depicted in Figure 51, is $m^* \circ m$, which is the map

$$x \otimes y \longmapsto xy \longmapsto xyx_i \otimes x^i, \tag{23.31}$$

using the dual bases introduced above. The $2 \to 3 \to 2$ composition, depicted in Figure 52, is either $(m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes m^*)$ or $(\mathrm{id} \otimes m) \circ (m^* \otimes \mathrm{id})$, so either

$$x \otimes y \longmapsto x \otimes yx_i \otimes x^i \longmapsto xyx_i \otimes x^i \tag{23.32}$$

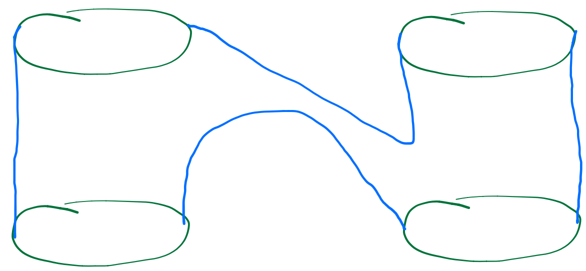

Figure 52. $2 \to 2$

or

$$x \otimes y \longmapsto xx_i \otimes x^i \otimes y \longmapsto xx_i \otimes x^i y. \tag{23.33}$$

To see that these are equal, use the identity $z = \tau(zx_j)x^j$ for all $z \in A$. Thus

$$xx_i \otimes x^i y = \tau(x^i yx_j)xx_i \otimes x^j = yx_j x \otimes x^j = xyx_i \otimes x^i. \tag{23.34}$$

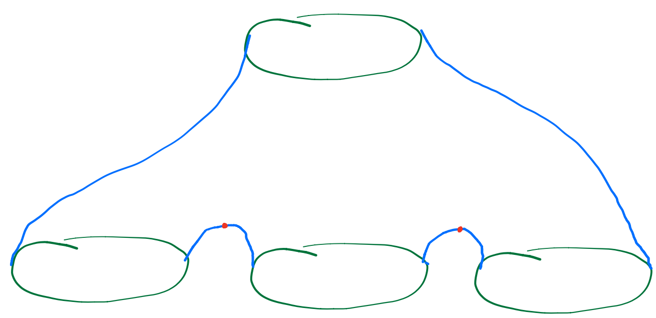

Figure 53. $3 \to 1$

The last case, $3 \to 1$, is depicted in Figure 52 at the critical time. On either side of the wall we have a composition $3 \to 2 \to 1$, and the two different compositions $A^{\otimes 3} \longrightarrow A^{\otimes 2} \longrightarrow A$ are equal by the associative law for m. □

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