Lecture 24: The cobordism hypothesis

In this last lecture we introduce the Baez-Dolan cobordism hypothesis [BD], which has been proved by Hopkins-Lurie in dimension 2 and by Lurie [L1] in all dimensions. We begin by motivating the notion of an *extended* topological quantum field theory. This leads to the idea of higher categories, which are also natural for bordisms. We then state the cobordism hypothesis for framed manifolds. We refer to [F1, Te] for more thorough introductions to the cobordism hypothesis.

In this lecture we extract from the geometry of bordisms an even more elaborate algebraic gadget than before: an (∞, n)-*category*.

We have no pretense of precision, and indeed to define an (∞, n)-category, much less a symmetric monoidal (∞, n)-category, is a nontrivial undertaking. At the same time we discuss some motivating examples which we do not explain in complete detail. The circle of ideas around the cobordism hypothesis is under rapid development as we write. We hope the reader is motivated to explore the references, the references in the references, and the many forthcoming references.

Extended TQFT

(24.1) *Factoring numerical invariants.* Let

$$F\colon \mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle} \to \mathrm{Vect}_k \tag{24.2}$$

be a topological field theory with values in the symmetric monoidal category of vector spaces over k. Thus the theory assigns a number in k to every closed n-manifold X (with $\mathcal{X}(n)$-structure, which we do not mention in the sequel). Suppose X is cut in two by a codimension one submanifold Y, as indicated in Figure 54. We view $X_1\colon \emptyset^{n-1} \to Y$ and $X_2\colon Y \to \emptyset^{n-1}$, so that $F(X_1)\colon k \to F(Y)$ and $F(X_2)\colon F(Y) \to k$. Let $\xi_1, \dots, \xi_k$ be a basis of $F(Y)$ and $\xi^1, \dots, \xi^k$ the dual basis of $F(Y)^\vee$. Write

$$\begin{aligned} F(X_1) &= a^i \xi_i \\ F(X_2) &= b_i \xi^i \end{aligned} \tag{24.3}$$

for some $a^i, b_i \in k$. Then the fact that $F(X) = F(X_2) \circ F(X_1)$ means

$$F(X) = a^i b_i. \tag{24.4}$$

In other words, the TQFT allows us to factorize the numerical invariant of a closed n-manifold into a sum of products of numbers. An n-manifold with boundary has an invariant which is not a single number, but rather a vector of numbers.

Bordism: Old and New (M392C, Fall '12), Dan Freed, December 29, 2012

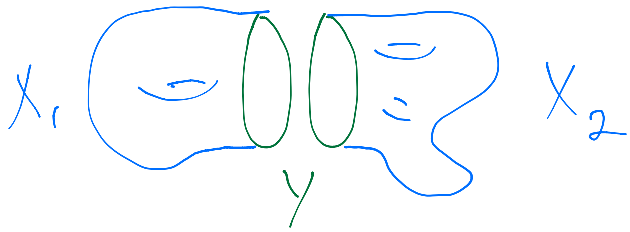


FIGURE 54. Factoring the numerical invariant $F(X)$

(24.5) *Factoring the "quantum Hilbert space".* We ask: can we factor the vector space $F(Y)$? If so, what kind of equation replaces (24.4)? Well, it must be an equation of sets rather than numbers, and more precisely an equation for vector spaces. Our experience teaches us we should not write an *equality* but rather an *isomorphism*, and that isomorphism takes place in the *category* Vect_k. (Compare: the *equation* (24.4) takes place in the *set* k.) So given a decomposition of the closed $(n-1)$-manifold Y, as in Figure 55, we might by analogy with (24.3) write

$$\begin{aligned} F(Y_1) &= V^i c_i \\ F(Y_2) &= W_i c^i \end{aligned} \tag{24.6}$$

for vector spaces $V^i, W_i \in \mathrm{Vect}_k$, and by analogy with (24.4) write

$$F(Y) \cong \bigoplus_i V^i \otimes W_i \tag{24.7}$$

In these expressions $V^i, W_i \in \mathrm{Vect}_k$. But what are c_i, c^i? By analogy they should be dual bases of a Vect_k-*module* $F(Z)$ which is associated to the closed $(n-2)$-manifold Z. Of course, the TQFT (24.2) does not assign anything in[1] codimension 2, so we must *extend* our notion of TQFT to carry out this factorization.

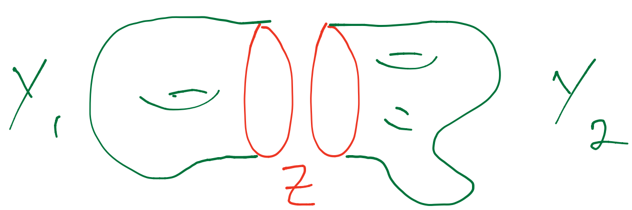


FIGURE 55. Factoring the vector space $F(Y)$

Indeed, one of the main ideas of this lecture is to extend the notion of a TQFT to assign invariants to manifolds of arbitrary codimension—down to points—and thus allow gluing which is completely local.

[1]By 'codimension 2' we mean $(n-2)$-manifolds.

Remark 24.8. In realistic quantum field theories the vector space $F(Y)$ in codimension 1 is usually called the *quantum Hilbert space.* (It *is* a Hilbert space in unitary theories.) The idea that it should be local in the sense that it factors when Y—physically a *spacelike slice* in a Lorentz manifold—is split in two, is an idea which is present in physics. For systems with discrete space, such as statistical mechanical models in which space is a lattice, the quantum Hilbert space is a tensor product of Hilbert spaces attached to each lattice site and obviously obeys a gluing law. For continuous systems one sometimes attaches a *von Neumann algebra* to what corresponds to Z in Figure 55, and then the Hilbert spaces $F(Y_1), F(Y_2)$ are modules over that von Neumann algebra.

Example: $n = 3$ Chern-Simons theory

This topological quantum field theory was introduced[2] in [Wi1]. It was the key example for many of the early mathematical developments in topological quantum field theory; see [F3] for a recent survey. Here we just make some structural remarks which indicate the utility of viewing quantum Chern-Simons as an *extended* TQFT.

(24.9) *Definition using the functional integral.* The data which defines the theory is a compact Lie group G and a class in $H^4(BG;\mathbb{Z})$ called the *level* of the theory. For G a connected simple group, $H^4(BG;\mathbb{Z}) \cong \mathbb{Z}$ and the level can be identified with an integer (usually denoted 'k' in the literature). Let X be a closed oriented 3-manifold. The field in Chern-Simons theory is a connection A on a principal G-bundle over X. The Chern-Simons invariant is a number $\Gamma_X(A) \in \mathbb{C}^\times$, which in fact has unit norm.[3] Suppose $L \subset X$ is a link with components $L_1, \dots, L_\ell$. Let $\rho_1, \dots, \rho_\ell$ be finite dimensional unitary representations of G, which we use to label the components of the link. Then there is an invariant

$$W_{L;\rho_1,\dots,\rho_\ell}(A) \in \mathbb{C} \tag{24.10}$$

defined as the product of the characters of the representations ρ_i applied to the holonomy of the connection A around the various components L_i of the link. Physicists call this the "Wilson line" operator. Formally, the *quantum* Chern-Simons invariant is a *functional integral*

$$F(X, L; \rho_1, \dots, \rho_\ell) = \int DA\, \Gamma_X(A) W_L(A) \tag{24.11}$$

over the infinite dimensional space of G-connections. It is not well-defined mathematically—an appropriate measure $\Gamma_X(A)\, DA$ has not been rigorously constructed—but as a heuristic leads to many predictions which have been borne out, both theoretically and numerically.[4]

[2] We have been lax in not pointing out earlier that the whole notion of a topological quantum field theory was introduced by Witten in an earlier paper [Wi2]

[3] This numerical invariant extends to an invertible quantum field theory which is *not* topological: it is defined on the bordism category of oriented manifolds equipped with a G-connection. Similarly, there is an invertible theory which includes the Wilson line operators (24.10) described below.

[4] One subtlety: in the quantum theory the manifolds have an additional tangential structure—a trivialization of the first Pontrjagin class p_1—which is very close to a 3-framing (Example 9.51).

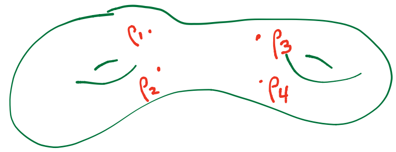


Figure 56. A surface with marked points

(24.12) *Categorical interpretation.* It is natural to make a (topological) bordism category whose objects are oriented 2-manifolds with a p_1-structure and a finite set of marked points; see Figure 56. A bordism between two such surfaces is then a 3-manifold with boundary and a link; see Figure 57. The link is a neat compact 1-dimensional submanifold, and it hits the boundary in the marked points. Each component of the link is labeled by a representation of G. Composition and the symmetric monoidal product (disjoint union) are as usual. Then the Chern-Simons theory is a symmetric monoidal functor from this category to $\mathrm{Vect}_{\mathbb{C}}$.

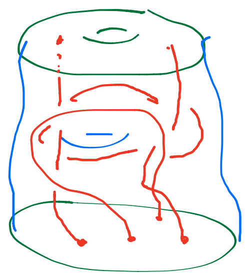


Figure 57. A bordism with a link/braid

(24.13) *Cutting out the links.* The idea now is to convert to a standard bordism category by cutting out a tubular neighborhood of the marked points and links. Already from an object (Figure 56) we obtain a 2-manifold with boundary in a 3-dimensional theory. Thus codimension 2 manifolds (1-manifolds) are immediately in the game. If we cut out a tubular neighborhood of the link in Figure 57, then we obtain a 3-manifold with *corners*.

Consider a closed component of a link. A tubular neighborhood is a diffeomorphic to a solid torus, but not canonically so: the isotopy classes form a $\mathbb{Z}$-torsor where the generator of $\mathbb{Z}$ acts by a Dehn twist. To fix this indeterminacy the links are given a normal framing. Then, up to isotopy, there is a unique identification of a tubular neighborhood of each closed component with $D^2 \times S^1$, and in the 3-manifold with the tubular neighborhood removed there is a contribution of a standard $S^1 \times S^1$ to the boundary. Now the labels ρ_i can be interpreted as a basis for the vector space $F(S^1 \times S^1)$. In fact, there is a finite set of labels in the quantum theory.[5]

For a component of the link with boundary, the normal framing fixes up to isotopy a diffeomorphism of a tubular neighborhood with a solid cylinder, and the intersection with the incoming or outgoing 2-manifold is a disk, as in Figure 58. This can be re-drawn as in Figure 59, which suggests that ρ_i be interpreted as an object in the linear category $F(S^1)$. This is indeed what happens in the extended TQFT.

[5]For a connected and simply connected group G the vector space is a quotient of the representation ring of G; the story is more complicated for a general compact Lie group.

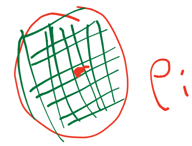


FIGURE 58. Tubular neighborhood of marked point

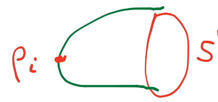


FIGURE 59. An object $\rho_i \in F(S^1)$

Morse functions revisited

(24.14) *Multi-cuttings and locality.* In Lecture 23 we used a single Morse function—in fact, an excellent function—to decompose a bordism into a composition of elementary bordisms (Figure 41). But the elementary bordisms (e.g. Figure 42) are not completely local; they contain more than a local neighborhood of the critical point. To achieve something entirely local we must slice again in the other direction, say by a second Morse function. For a 2-dimensional manifold this is enough to achieve locality (Figure 60). For an n-dimensional manifold we need n functions.

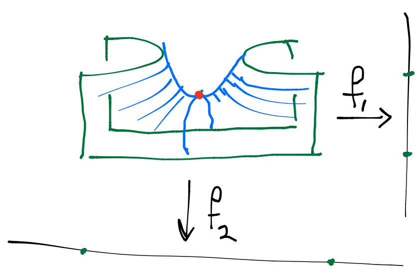


FIGURE 60. Cutting a surface with 2 Morse functions near a critical point

The multi-categorical nature of multi-cuttings is already evident in Figure 60. Recall from Figure 38 that a single function on a manifold, thought of as "time", gives rise to a composition law for bordisms. Hence n time functions induce n composition laws. These should be thought of as "internal" to an n-category; there is still disjoint union which induces a symmetric monoidal structure.

(24.15) *Collapsing identity maps.* The standard Morse picture collapses the four vertical lines in Figure 60. The resulting manifold with corners is a (closed) square $D^1 \times D^1$. As time (f_1) flows from bottom to top two of the boundary edges ($S^0 \times D^1$) flow to the other two boundary edges ($D^1 \times S^0$) through the square. The four corner points ($S^0 \times S^0$) remain inert through the flow. In this interpretation the square is a map

$$D^1 \times D^1 \colon S^0 \times D^1 \Longrightarrow D^1 \times S^0 \tag{24.16}$$

and the two pairs of boundary edges are maps

$$S^0 \times S^0 \to S^0 \times S^0. \tag{24.17}$$

We combine (24.16) and (24.17) into a single diagram:

$$S^0 \times S^0 \;\underset{S^0\times D^1}{\overset{D^1\times S^0}{\rightrightarrows}}\; S^0 \times S^0 \qquad \Uparrow D^1\times D^1 \tag{24.18}$$

The general n-dimensional handle of index q is depicted as

$$S^{p-1} \times S^{q-1} \;\underset{S^{p-1}\times D^q}{\overset{D^p\times S^{q-1}}{\rightrightarrows}}\; S^{p-1} \times S^{q-1} \qquad \Uparrow D^p\times D^q \tag{24.19}$$

where $p = n - q$.

Higher categories

(24.20) *(m, n)-categories.* Intuitively, a higher category has objects, 1-morphisms which map between objects, 2-morphisms which map between 1-morphisms, etc. The diagrams (24.18) and (24.19) are 2-morphisms (double arrows) which map between 1-morphisms (single arrows). There are k composition laws for k-morphisms, and the composition laws are no longer required to be associative. We allow ∞-categories which have morphisms of all orders. An *(∞, n)-category* is an ∞-category in which all k-morphisms are invertible for $k > n$. In this notation a $(1, 1)$-category is an ordinary category and a $(1, 0)$-category is a groupoid.

What follows are two examples of 2-categories. Together with the multi-bordism category indicated in the previous section, these give some of the most important ways in which multi-categories arise.

Example 24.21 (Higher groupoids from a topological space)**.** This generalizes Example 13.14. Let Y be a topological space. The simplest invariant $\pi_0 Y$ is the *set* of path components. The next simplest is $\pi_{\le 1} Y$, the fundamental *groupoid* of Y. Its objects are the points of Y and a morphism $y_0 \to y_1$ is a *homotopy class* of continuous paths $\gamma\colon [0, 1] \to Y$ with $\gamma(0) = y_0$ and $\gamma(1) = y_1$. It is clear how to go further. We construct a *2-groupoid* $\pi_{\le 2} Y$ as follows. (A 2-groupoid is a $(2, 0)$-category, i.e., a 2-category in which all morphisms are invertible.) An object is a point of Y as before. A 1-morphism in $\pi_{\le 2} Y$ is a continuous path—there is no identification of homotopic paths.

Let $y_0, y_1 \in Y$ and $\gamma, \gamma' \colon [0,1] \to Y$ two continuous paths from y_0 to y_1. A 2-morphism $\Gamma\colon \gamma \Rightarrow \gamma'$ is a *homotopy class* of continuous maps $\Gamma\colon [0,1] \times [0,1] \to Y$ such that

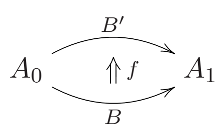
$$
\begin{aligned}
\Gamma(t_1, 0) &= \gamma(t_1) \\
\Gamma(t_1, 1) &= \gamma'(t_1) \\
\Gamma(0, t_2) &= y_0 \\
\Gamma(1, t_2) &= y_1
\end{aligned}
\tag{24.22}
$$

for all $t_1, t_2 \in [0,1]$. The last two equations allow us to factor Γ through the lune obtained by collapsing the vertical boundary edges of the square $[0,1] \times [0,1]$. Thus the domain has the shape of the diagram (24.18), as befits a 2-morphism. We identify homotopic maps Γ, where the map on the boundary is static during the homotopy. Vertical composition of 2-morphisms is associative on the nose, but other compositions are only associative up to homotopy.

It should be clear how to define the *fundamental* m*-groupoid* $\pi_{\le m} Y$ of the topological space Y for any $m \in \mathbb{Z}^{\ge 0}$. There is an assertion (either a definition or theorem, depending on the approach, though I don't know a reference in which it is a theorem) in higher category theory that an $(\infty, 0)$-category *is* a topologial space.

Example 24.23 (The Morita 2-category of algebras)**.** Let k be a field. We construct a 2-category $C = \mathrm{Alg}_k$ which is not a groupoid. (In the above nomenclature it is a $(2,2)$-category.) The objects are algebras over k. For algebras A_0, A_1 a morphism $B\colon A_0 \to A_1$ is an (A_1, A_0)-bimodule. That is, B is a k-vector space which is simultaneously a left module for A_1 and a right module for A_0. The actions commute, so equivalently B is a left $(A_1 \otimes A_0^{\mathrm{op}})$-module. The collection $C(A_0, A_1)$ of these bimodules is a 1-category: a morphism $f\colon B \Rightarrow B'$ is a linear map $f\colon B \to B'$ which intertwines the (A_1, A_0)-action. So f is a 2-morphism in C:

(24.24)



Composition of bimodules (1-morphisms) is by tensor product over an algebra. Thus if A_0, A_1, A_2 are k-algebras, $B_1\colon A_0 \to A_1$ an (A_1, A_0)-bimodule, and $B_2\colon A_1 \to A_2$ an (A_2, A_1)-bimodule, then $B_2 \circ B_1 \colon A_0 \to A_2$ is the (A_2, A_0)-bimodule $B_2 \otimes_{A_1} B_1$. This composition is only associative up to isomorphism.

The cobordism hypothesis

(24.25) *The* (∞, n)*-category of bordisms.* We motivated above the idea that using multiple Morse functions we can make out of n-manifolds an n-category: n-manifolds with corners of all codimensions form the n-morphisms in that category. This is an (n,n)-category in the nomenclature of **(24.20)**. This is already a huge step above what we had before, an n-categorical generalization of Definition 14.3. Now we want to generalize Definition 20.19 in the sense that we will consider a *topological space* of n-morphisms. Now an n-morphism is an n-manifold with corners, together with partitions of the various corners telling which are incoming and which are outgoing. (There are also collar neighborhoods.) The discussion in Lecture 20 indicates how that can be done. However, using the assertion at the end of Example 24.21 we can replace that space by its fundamental ∞-groupoid, which amounts to saying that an $(n+1)$-morphism is a diffeomorphism of n-dimensional bordisms, an $(n+2)$-morphism is an isotopy of such diffeomorphisms, etc. In this way we obtain an (∞, n)-bordism category which we denote Bord_n. Of course, we can include a tangential structure as well. The relevant example for us is n-framings (Example 9.51), which we denote as $\mathcal{X}(n) = EO(n)$, and thus denote the resulting bordism category $\mathrm{Bord}_n^{EO(n)}$.

(24.26) *Fully extended TQFT.* Following Definition 14.20 we define a (fully) extended topological quantum field theory to be a homomorphism of symmetric monoidal (∞, n)-categories

$$F\colon\ \mathrm{Bord}_n^{EO(n)} \longrightarrow C \tag{24.27}$$

into an arbitrary symmetric monoidal (∞, n)-category C.

(24.28) *Finiteness.* Recall Theorem 15.36 which asserts that the objects which appear in the image of an ordinary TQFT are dualizable. The corresponding finiteness condition in an (∞, n)-category is *n-dualizability*, or *full dualizability*. We do not elaborate here, but defer to [L1, §2.3].

(24.29) *The cobordism hypothesis.* The cobordism hypothesis is the next in a sequence of theorems in the course. The first is stated in (**2.28**): the oriented bordism group Ω_0^{SO} is the free abelian group on one generator. It may be accurate to attribute this to Brouwer as it is the basis of oriented intersection theory. This statement only uses 0- and 1-manifolds, and on such manifolds an orientation is equivalent to a 1-framing. This result was restated in Theorem 16.8. The second result in this line is Theorem 16.10. It roughly asserts that $\mathrm{Bord}_{\langle 0,1\rangle}^{SO} = \mathrm{Bord}_{\langle 0,1\rangle}^{EO(1)}$ is the free 1-category with duals[6] with a single generator pt_+. But it is much easier to formulate in terms of homomorphisms out of $\mathrm{Bord}_{\langle 0,1\rangle}$, and that is how Theorem 16.10 is stated. Still, it is a theorem about the structure of the bordism category, a statement about 0- and 1-manifolds. The cobordism hypothesis is a similar statement, but about the bordism (∞, n)-category.

Theorem 24.30 (cobordism hypothesis)**.** *Let C be a symmetric monoidal (∞, n)-category. Then the map*

$$\begin{aligned} \Phi\colon\ \mathrm{TQFT}_n^{EO(n)}(C) &\longrightarrow (C^{\mathrm{fd}})^{\sim} \\ F &\longmapsto F(\mathrm{pt}_+) \end{aligned} \tag{24.31}$$

is an equivalence of ∞-groupoids.

[6] i.e., every object has a dual

The domain is the multi-category of homomorphisms $\mathrm{Bord}_n^{EO(n)} \to C$. The multi-category analog of Proposition 15.34(ii) implies that the domain is an ∞-groupoid: all morphisms are invertible. The notation in the codomain follows Definition 16.4 and Definition 16.5: it is the maximal ∞-groupoid underlying the subcategory of fully dualizable objects.

The cobordism hypothesis is a statement about the n-framed bordism category. There are many variations. We will stop here and not comment on the proof nor on the applications.

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