Lecture 3: The Pontrjagin-Thom theorem

In this lecture we give a proof of Theorem 2.35. You can read an alternative exposition in [M3]. We begin by reviewing some definitions and theorems from differential topology.

Neat submanifolds

Recall the local model (**1.8**) of a manifold with boundary. We now define a robust notion of submanifold for manifolds with boundary.

Definition 3.1. Let M be an m-dimensional manifold with boundary. A subset $Y \subset M$ is a *neat submanifold* if about each $y \in Y$ there is a chart (ϕ, U) of M—that is, an open set $U \subset M$ containing y and a homeomorphism $\phi\colon U \to \mathbb{A}^n$ in the atlas defining the smooth structure—such that $\phi(Y) \subset \mathbb{A}^{m-q} \cap \mathbb{A}^m_-$, where $\mathbb{A}^m_-$ is defined in (1.9) and

$$\mathbb{A}^{m-q} = \{(x^1, x^2, \dots, x^m) \in \mathbb{A}^m : x^{m-q+1} = \dots = x^m = 0\}. \tag{3.2}$$

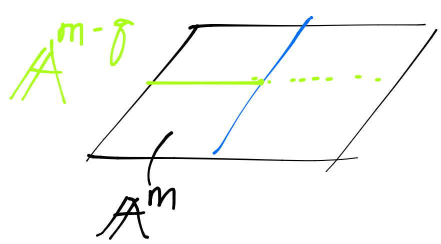

FIGURE 9.

The local model induces a smooth structure on Y, so Y is a manifold with boundary, $\partial Y = \partial M \cap Y$, and Y is transverse to ∂M.

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(3.3) *Normal bundle to neat submanifold.* The neatness condition gives rise to the following diagram of vector bundles over ∂Y:

(3.4)
$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & T(\partial Y) & \hookrightarrow & TY\big|_{\partial Y} & \longrightarrow & \mu^Y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow\cong & & \\
0 & \longrightarrow & T(\partial M) & \hookrightarrow & TM\big|_{\partial Y} & \longrightarrow & \mu^M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \nu^\partial & \xrightarrow{\cong} & \nu\big|_{\partial Y} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

In this diagram the line bundles μ^Y, μ^M, defined as the indicated horizontal quotients, are the normal bundles to the boundaries of the manifolds Y, M, and the diagram determines an isomorphism between them. Similarly, the vector bundles ν^∂, ν, defined as the indicated vertical quotients, are the normal bundles to $\partial Y \subset \partial M$ and $Y \subset M$, respectively; the diagram determines isomorphism between ν^∂ and the restriction of ν to the boundary of ∂Y.

This shows that there is a well-defined normal bundle $\nu \to Y$ to the neat submanifold $Y \subset M$.

(3.5) *Tubular neighborhood of a neat submanifold.* The tubular neighborhood theorem extends to neat submanifolds.

Definition 3.6. Let M be a manifold with boundary, $Y \subset M$ a neat submanifold, and $\nu \to Y$ its normal bundle. A *tubular neighborhood* is a pair (U, φ) where $U \subset M$ is an open set containing Y and $\varphi\colon \nu \to U$ is a diffeomorphism such that $\varphi\big|_Y = \mathrm{id}_Y$, where we identify $Y \subset \nu$ as the image of the zero section.

Theorem 3.7. *Tubular neighborhoods exist.*

The proof is easier if Y is compact. In either case one can use Riemannian geometry. Choose a Riemannian metric on M which is a product metric in a collar neighborhood of ∂M. Use the metric to embed $\nu \subset TM\big|_Y$ as the orthogonal complement of TY. Then for an appropriate function $\epsilon\colon TY \setminus Y \to \mathbb{R}^{>0}$ we define $\varphi(\xi)$ to be the time $\epsilon(\xi)$ position of the geodesic with initial position $\pi(\xi)$ and initial velocity $\xi/|\xi|$. Here $\pi\colon \nu \to Y$ is projection and ξ is presumed nonzero.

Proof of Pontrjagin-Thom

Definition 3.8. Let $f_0, f_1\colon M \to N$ be smooth maps of manifolds. A *smooth homotopy* $F\colon f_0 \to f_1$ is a smooth map $F\colon \Delta^1 \times M \to N$ such that for all $x \in M$ we have $F(0,x) = f_0(x)$ and $F(1,x) = f_1(x)$.

Here $\Delta^1 = [0,1]$ is the 1-simplex. Smooth homotopy is an equivalence relation; the set of equivalence classes is denoted $[M,N]$. This is also the set of homotopy classes of *continuous* maps under continuous homotopy, which can be proved by approximation theorems which show that C^∞ maps are dense in the space of continuous maps.

Recall the definition of $\Omega^{\mathrm{fr}}_{n;M}$ from (2.32); it is the set of framed bordism classes of normally framed n-dimensional closed submanifolds of a smooth manifold M.

Theorem 3.9 (Pontrjagin-Thom)**.** *For any smooth compact m-manifold M there is an isomorphism*

$$\phi\colon [M,S^q] \longrightarrow \Omega^{\mathrm{fr}}_{n;M}, \qquad n = m-q. \tag{3.10}$$

The forward map is the inverse image of a regular value; the inverse map is the Pontrjagin-Thom collapse, as illustrated in Figure 8.

Proof. Write $S^q = \mathbb{A}^q \cup \{\infty\}$ (stereographic projection) and fix $p \in \mathbb{A}^q$. Given $f\colon M \to S^q$ use the transversality theorems from differential topology to perturb to a smoothly homotopy $f_0\colon M \to S^q$ such that p is a regular value. Define $\phi\bigl([f]\bigr) = \bigl[(f_0)^{-1}(p)\bigr]$, where $[f]$ is the smooth homotopy class of f and $\bigl[(f_0)^{-1}(p)\bigr]$ is the framed bordism class of the inverse image. Note that $(f_0)^{-1}(p)$ is compact since M is. To see that ϕ is well-defined, suppose $F\colon \Delta^1 \times M \to S^q$ is a smooth homotopy from f_0 to f_1, where p is a simultaneous regular value of f_0, f_1. The transversality theorems imply there is a perturbation F' of F which is transverse to $\{p\}$ and which equals F in a neighborhood of $\{0,1\} \times M \subset \Delta^1 \times M$. Then[1] $(F')^{-1}(p)$ is a framed bordism from $(f_0)^{-1}(p)$ to $(f_1)^{-1}(p)$.

The inverse map

$$\psi\colon \Omega^{\mathrm{fr}}_{n;M} \longrightarrow [M,S^q] \tag{3.11}$$

is described in (2.34). That construction depends on a choice of tubular neighborhood (U,φ) and cutoff function (Figure 7). To see it is well-defined, suppose $X \subset \Delta^1 \times M$ is a framed bordism, which in particular is a neat submanifold. We use the existence of tubular neighborhoods (Theorem 3.7) to construct a Pontrjagin-Thom collapse map $\Delta^1 \times M \to S^q$, which is then a smooth homotopy between the Pontrjagin-Thom collapse maps on the boundaries. (We need to know that if we have a tubular neighborhood of $\partial X \subset \partial M$ we can extend that particular tubular neighborhood to one of $X \subset M$. If we construct tubular neighborhoods using geodesics, as indicated in **(3.5)**, then this is a simple matter of extending a Riemannian metric on ∂M to a Riemannian metric on M.)

The composition $\phi \circ \psi$ is clearly the identity. To show that $\psi \circ \phi$ is also the identity, note that if $f_0\colon M \to S^q$ has p as a regular value and we set $Y = (f_0)^{-1}(p)$, then the map $f_1\colon M \to S^q$ representing $(\psi \circ \phi)(f_0)$ also has p as a regular value and $(f_1)^{-1}(p) = Y$. Furthermore, by construction $df_0\big|_Y = df_1\big|_Y$. The desired statement follows from the following lemma.

Lemma 3.12. *Let M be a closed manifold, $Y \subset M$ a normally framed submanifold, and $f_0, f_2\colon M \to S^q$ such that $(f_0)^{-1}(p) = (f_2)^{-1}(p) = Y$ and $df_0\big|_Y = df_2\big|_Y$, where $p \in \mathbb{A}^q \subset S^q$. Then f_0 is smoothly homotopic to f_2.*

[1] This relies on the following theorem: If W is a compact manifold with boundary, $F\colon W \to S$ a smooth map to a manifold S, and $p \in S$ is a regular value of both F and $F\big|_{\partial W}$, then $F^{-1}(p) \subset W$ is a *neat* submanifold.

Proof. We first make a homotopy of f_0 localized in a neighborhood of Y to make f_0 and f_2 agree in a neighborhood of Y. For that choose a tubular neighborhood (U, φ) of Y such that neither f_0 nor f_2 hits $\infty \in S^q$ in U. The framing identifies $U \approx Y \times \mathbb{R}^q$, and under the identification f_0, f_2 correspond to maps $g_0, g_2 \colon Y \times \mathbb{R}^q \to \mathbb{A}^q$. For a cutoff function ρ of the shape of Figure 7 define the homotopy

$$(t, y, \xi) \longmapsto g_0(y, \xi) \;+\; t\lambda(|\xi|)\big(g_2(y, \xi) - g_0(y, \xi)\big). \tag{3.13}$$

Let g_1 be the time-one map; it glues to f_0 on the complement of U to give a smooth map $f_1 \colon M \to S^q$. Then $f_1 = f_2$ in a neighborhood $V \subset U$ of Y, and $f_1 = f_0$ on the complement of U. I leave as a calculus exercise to prove that we can adjust the cutoff function (sending it to zero quickly) so that f_1 does not take the value p in $U \setminus V$. This uses the fact that $(dg_0)_{(y,0)} = (dg_1)_{(y,0)}$ for all $y \in Y$.

The second step is to construct a homotopy from f_1 to f_2. For this write $S^q = \mathbb{A}^q \cup \{p\}$, use the fact that both f_1 and f_2 map to the affine part of this decomposition on the complement of V, and then average in that affine space to make the homotopy, as in (3.13). □

□

Exercise 3.14. Fill in the two missing details in the proof of Lemma 3.12. Namely, first show how to construct a cutoff function ρ so that $f_1(x) \neq p$ for all $x \in U \setminus V$. Construct an example (think low dimensions!) to show that this fails if the normal framings do not agree up to homotopy on Y. Then construct the homotopy in the second step of the proof.

Exercise 3.15. Show by example that Theorem 3.9 can fail for M noncompact.

Exercise 3.16. A *framed link* in S^3 is a closed normally framed 1-dimensional submanifold $L \subset S^3$. What can you say about these up to framed bordism, i.e., can you compute $\Omega^{\text{fr}}_{1;S^3}$? Is the framed bordism class of a link an interesting link invariant? How can you compute it?

Exercise 3.17. A *Lie group* G is a smooth manifold equipped with a point $e \in G$ and smooth maps $\mu \colon G \times G \to G$ and $\iota \colon G \to G$ such that (G, e, μ, ι) is a group. In other words, it is the marriage of a smooth manifold and a group, with compatible structures. Prove that every Lie group is *parallelizable*, i.e., that there exists a trivialization of the tangent bundle $TG \to G$. In fact, construct a canonical trivialization.

Exercise 3.18. Show that the complex numbers of unit norm form a Lie group $\mathbb{T} \subset \mathbb{C}$. What is the underlying smooth manifold? Do the analogous exercise for the unit quaternions $Sp(1) \subset \mathbb{H}$. The notation $Sp(1)$ suggests that there is also a Lie group $Sp(n)$ for any positive integer n. There is! Construct it.

References

[M3] John W. Milnor, *Topology from the differentiable viewpoint*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1997. Based on notes by David W. Weaver, Revised reprint of the 1965 original.