Lecture 5: More on stabilization

In this lecture we continue the introductory discussion of stable topology. Recall that in Lecture §1 we introduced the *stable stem* $\pi_\bullet^s$, the stable homotopy groups of the sphere. We show that there is a ring structure: $\pi_\bullet^s$ is a $\mathbb{Z}$-graded commutative ring (Definition 1.33). The stable Pontrjagin-Thom theorem identifies it with stably normally framed submanifolds of a sphere. Here we see how stable *normal* framings are equivalent to stable *tangential* framings, and so define a ring $\Omega_\bullet^{\text{fr}}$ of stably tangentially framed manifolds with no reference to an embedding. The image of the *J-homomorphism* gives some easy classes in the stable stem from the stable homotopy groups of the orthogonal group. We describe some low degree classes in terms of *Lie groups*. At the end of the lecture we introduce the concept of a *spectrum*, which is basic to stable homotopy theory.

A reference for this lecture is [DK, Chapter 8].

Ring structure

Recall that elements in the abelian group π_n^s are represented by homotopy classes $\pi_{q+n}S^q$ for q sufficiently large. The multiplication in $\pi_\bullet^s$ is easy to describe. Suppose given classes $a_1 \in \pi_{n_1}^s$ and $a_2 \in \pi_{n_2}^s$, which are represented by maps

$$\begin{aligned} f_1 \colon S^{q_1+n_1} &\longrightarrow S^{q_1} \\ f_2 \colon S^{q_2+n_2} &\longrightarrow S^{q_2} \end{aligned} \tag{5.1}$$

Then the product $a_1 \cdot a_2 \in \pi_{n_1+n_2}^s$ is represented by the smash product (Exercise 4.10)

$$f_1 \wedge f_2 \colon S^{q_1+n_1} \wedge S^{q_2+n_2} \longrightarrow S^{q_1} \wedge S^{q_2}. \tag{5.2}$$

Recall that the smash product of spheres is a sphere (Exercise 4.9), so $f_1 \wedge f_2$ does represent an element of $\pi_{n_1+n_2}^s$.

There is a corresponding ring structure on framed manifolds, which we will construct in the next section.

Tangential framings

(5.3) *Short exact sequences of vector bundles.* Let

$$0 \longrightarrow E' \xrightarrow{\ i\ } E \xrightarrow{\ j\ } E'' \longrightarrow 0 \tag{5.4}$$

be a short exact sequence of vector bundles over a smooth manifold Y.[1] A *splitting* of (5.4) is a linear map $E'' \xrightarrow{\ s\ } E$ such that $j \circ s = \mathrm{id}_{E''}$. A splitting determines an isomorphism

$$E'' \oplus E' \xrightarrow{\ s \oplus i\ } E. \tag{5.5}$$

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[1] These can be real, complex, or quaternionic.

Lemma 5.6. *The space of splittings is a nonempty affine space over the vector space* $\operatorname{Hom}(E'', E')$.

Let's deconstruct that statement, and in the process prove parts of it. First, if s_0, s_1 are splittings, then the difference $\phi = s_1 - s_0$ is a linear map $E'' \to E$ such that $j \circ \phi = 0$. The exactness of (5.4) implies that ϕ factors through a map $\tilde{\phi}\colon E'' \to E'$: in other words, $\phi = i \circ \tilde{\phi}$. This, then, is the affine structure. But we must prove that the space of splittings is nonempty. For that we use a partition of unity argument. Remember that partitions of unity can be used to average sections of a fiber bundle whose fibers are convex subsets of affine spaces. Of course, an affine space is a convex subset of itself.

I outline some details in the following exercise.

Exercise 5.7.

(i) Construct a vector bundle $\underline{Hom}(E'', E') \to Y$ whose sections are homomorphisms $E'' \to E'$. Similarly, construct an *affine bundle* (a fiber bundle whose fibers are affine spaces) whose sections are splittings of (5.4). You will need to use local trivializations of the vector bundles E, E', E'' to construct these fiber bundles.

(ii) Produce the partition of unity argument. You should prove that if $\mathcal{A} \to Y$ is an affine bundle, and $\mathcal{E} \to Y$ is a fiber subbundle whose fibers are convex subsets of $\mathcal{A}$, then there exist sections of $\mathcal{E} \to Y$. Even better, topologize[2] the space of sections and prove that the space of sections is contractible.

(iii) This is a good time to review the partition of unity argument for the existence of Riemannian metrics. Phrase it in terms of sections of a fiber bundle (which?). More generally, prove that any real vector bundle $\nu \to Y$ admits a positive definite metric, i.e., a smoothly varying inner product on each fiber.

(5.8) *Stable framings.* Let $E \to Y$ be a vector bundle of rank q. A *stable framing* of $E \to Y$ is an isomorphism $\phi\colon \underline{\mathbb{R}^{k+q}} \xrightarrow{\cong} \underline{\mathbb{R}^k} \oplus E$ for some $k \geq 0$. A *homotopy of stable framings* is a homotopy of the isomorphism ϕ. We identify ϕ with

$$\mathrm{id}_{\underline{\mathbb{R}^\ell}} \oplus \phi\colon \underline{\mathbb{R}^{\ell+k+q}} \xrightarrow{\cong} \underline{\mathbb{R}^{\ell+k}} \oplus E \tag{5.9}$$

for any ℓ. With these identifications we define a set of *homotopy classes of stable trivializations.*

(5.10) *The stable tangent bundle of the sphere.* Let $S^m \in \mathbb{A}^{m+1}$ be the standard unit sphere, defined by the equation

$$(x^1)^2 + (x^2)^2 + \cdots + (x^{m+1})^2 = 1. \tag{5.11}$$

Then the vector field

$$\sum_i x^i \frac{\partial}{\partial x^i}, \tag{5.12}$$

[2]The topological space of a smooth manifold is metrizable; one can use the metric space structure induced from a Riemannian metric, for example. Then you can topologize the space of sections using the *topology of uniform convergence on compact sets*. One needn't use the metrizability and can describe this as the *compact-open topology*.

restricted to S^m, gives a trivialization of the normal bundle ν to $S^m \subset \mathbb{A}^{m+1}$. Recall that the tangent bundle to $\mathbb{A}^{m+1}$ is the trivial bundle $\underline{\mathbb{R}^{m+1}} \to \mathbb{A}^{m+1}$. Then a splitting of the short exact sequence

$$0 \longrightarrow TS^m \longrightarrow \underline{\mathbb{R}^{m+1}} \longrightarrow \nu \to 0 \tag{5.13}$$

over S^m gives a stable trivialization

$$\underline{\mathbb{R}} \oplus TS^m \cong \underline{\mathbb{R}^{m+1}} \tag{5.14}$$

of the tangent bundle to the sphere.

(5.15) *Stable normal and tangential framings.* Now suppose $Y \subset S^m$ is a submanifold of dimension n with a normal framing, which we take to be an isomorphism $\underline{\mathbb{R}^q} \xrightarrow{\cong} \mu$, where μ is the rank $q = m - n$ normal bundle defined by the short exact sequence

$$0 \longrightarrow TY \longrightarrow TS^m\big|_Y \longrightarrow \mu \longrightarrow 0 \tag{5.16}$$

This induces a short exact sequence

$$0 \longrightarrow TY \longrightarrow \underline{\mathbb{R}} \oplus TS^m\big|_Y \longrightarrow \underline{\mathbb{R}} \oplus \mu \longrightarrow 0 \tag{5.17}$$

Choose a splitting of (5.17) and use the stable trivialization (5.14) and the trivialization of the normal bundle μ to obtain an isomorphism

$$\underline{\mathbb{R}^{q+1}} \oplus TY \xrightarrow{\cong} \underline{\mathbb{R}^{m+1}} \tag{5.18}$$

of vector bundles over Y. This is a *stable tangential framing* of Y, and is one step in the proof of the following.

Proposition 5.19. *Let $Y \subset S^m$ be a submanifold. Then there is a 1:1 correspondence between homotopy classes of stable normal framings of Y and stable tangential framings of Y.*

Proof. The argument before the proposition defines a map from (stable) normal framings to stable tangential framings. Conversely, if $\underline{\mathbb{R}^k} \oplus TY \xrightarrow{\cong} \underline{\mathbb{R}^{k+n}}$ is a stable tangential framing, with $k \geq 1$, then from a splitting of the short exact sequence

$$0 \longrightarrow \underline{\mathbb{R}^k} \oplus TY \longrightarrow \underline{\mathbb{R}^k} \oplus TS^m\big|_Y \longrightarrow \mu \longrightarrow 0 \tag{5.20}$$

we obtain a stable normal framing $\mu \oplus \underline{\mathbb{R}^{k+n}} \xrightarrow{\cong} \underline{\mathbb{R}^{k+m}}$. I leave it to you to check that homotopies of one framing induce homotopies of the other, and that the two maps of homotopy classes are inverse. $\square$

Application to framed bordism

Recall the stabilization sequence (4.42) of normally framed submanifolds $Y \subset S^m$. The stabilization sits $S^m \subset S^{m+1}$ as the equator and prepends the standard normal vector field $\partial/\partial x^1$ to the framing. By Proposition 5.19 the normal framing induces a stable tangential framing of Y, and the homotopy class of the stable tangential framing is unchanged under the stabilization map σ in the sequence (4.42). Conversely, if Y^n has a stable tangential framing, then by the Whitney embedding theorem we realize $Y \subset S^m$ as a submanifold for some m, and then by Proposition 5.19 there is a stable framing $\underline{\mathbb{R}^{q+k}} \xrightarrow{\cong} \mu$ of the normal bundle. This is then a framing of the normal bundle to $Y \subset S^{m+k}$, which defines an element of $\Omega^{\text{fr}}_{n;S^{m+k}}$. This argument proves

Proposition 5.21. *The colimit of* (4.42) *is the bordism group* Ω^{fr}_n *of n-manifolds with a stable tangential framing.*

A bordism between two stably framed manifolds Y_0, Y_1 is, informally, a compact $(n+1)$-manifold X with boundary $Y_0 \amalg Y_1$ and a stable tangential framing of X which restricts on the boundary to the given stable tangential framings of Y_i. The formal definition follows Definition 1.19.

The following is a corollary to Theorem 3.9.

Corollary 5.22 (stable Pontrjagin-Thom)**.** *There is an isomorphism*

$$\phi\colon \pi^s_n \longrightarrow \Omega^{\text{fr}}_n \tag{5.23}$$

for each $n \in \mathbb{Z}^{\geq 0}$.

(5.24) *Ring structure.* Letting n vary we obtain an isomorphism $\phi\colon \pi^s_\bullet \to \Omega^{\text{fr}}_\bullet$ of $\mathbb{Z}$-graded abelian groups. We saw at the beginning of this lecture that the domain is a $\mathbb{Z}$-graded *ring*. So there is a corresponding ring structure on codomain. It is given by *Cartesian product*. For recall that we may assume that the representatives f_1, f_2 of two classes a_1, a_2 in the stable stem (see (5.1)) are *pointed*, in the sense they map the basepoint ∞ to ∞, and then these map under ϕ to the submanifolds Y_1, Y_2 defined as the inverse images of $p_i \in S^{q_i}$, where $p_i \neq \infty$. Then $Y_1 \times Y_2$ is the inverse image of $(p_1, p_2) \in S^{q_1} \wedge S^{q_2}$.

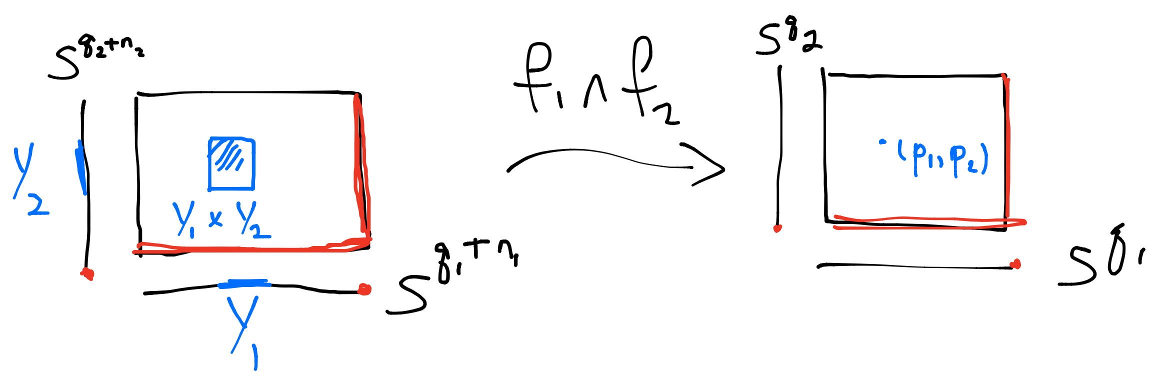

FIGURE 15. Ring structure on $\Omega^{\text{fr}}_\bullet$

J-homomorphism

(5.25) *Twists of framing.* Let $Y \subset M$ be a normally framed submanifold of a smooth manifold M, and suppose its codimension is q. Denote the framing as $\phi\colon \underline{\mathbb{R}^q} \to \nu$, where ν is the normal bundle. Let $g\colon Y \to GL_q\mathbb{R} = GL(\mathbb{R}^q)$ be a smooth map. Then $\phi \circ g$ is a new framing of ν, the *g-twist of ϕ*.

Remark 5.26. As stated in Exercise 5.7(iii), there is a positive definite metric on the normal bundle $\nu \to Y$, and the metric is a contractible choice. Furthermore, the Gram-Schmidt process gives a deformation retraction of all framings onto the space of *orthonormal* framings. Let

$$O(q) = \{g\colon \mathbb{R}^q \to \mathbb{R}^q : g \text{ is an isometry}\} \tag{5.27}$$

denote the *orthogonal group* of $q \times q$ orthogonal matrices. Then we can twist orthonormal framings by a map $g\colon Y \to O(q)$.

Exercise 5.28. Construct a deformation retraction of $GL_q(\mathbb{R})$ onto $O(q)$. Start with the case $q = 1$ to see what is going on, and you might try $q = 2$ as well. For the general case, you might consider the Gram-Schmidt process.

Exercise 5.29. Is the space of maps $Y \to O(q)$ contractible? Proof or counterexample.

(5.30) *The unstable J-homomorphism.* Specialize to $M = S^m$ and let $Y = S^n \subset S^m$ be an equatorial n-sphere with the canonical normal framing. Explicitly, write $S^m = \mathbb{A}^m \cup \{\infty\}$ as usual, introduce standard affine coordinates $x^1, \dots, x^m$, and let S^n be the unit n-sphere

$$S^n = \{(x^1, \dots, x^m) : x^1 = \dots = x^{q-1} = 0, \quad (x^q)^2 + \dots + (x^m)^2 = 1\}. \tag{5.31}$$

We use the framing $\partial/\partial r, \partial/\partial x^1, \dots, \partial/\partial x^{q-1}$, where $\partial/\partial r$ is the outward normal to S^n in the affine subspace $\mathbb{A}^{n+1}$ defined by $x^1 = \dots = x^{q-1} = 0$. Then restricting to pointed maps $g\colon S^n \to O(q)$ we obtain a homomorphism

$$J\colon [S^n, O(q)]_* \longrightarrow \Omega^{\mathrm{fr}}_{n;S^m}. \tag{5.32}$$

Applying Pontrjagin-Thom we can rewrite this as

$$J\colon \pi_n O(q) \longrightarrow \pi_{n+q} S^q. \tag{5.33}$$

This is the *unstable J-homomorphism.*

Exercise 5.34. Show that the normally framed S^n in (5.31) is null bordant: it bounds the unit ball D^{n+1} in $\mathbb{A}^{n+1}$ with normal framing $\partial/\partial x^1, \dots, \partial/\partial x^{q-1}$. Then show that (5.32) is indeed a homomorphism.

Exercise 5.35. Work out some special cases of (5.32) and (5.33) explicitly. Try $n = 0$ first. Then try $n = 1$ and $m = 2, 3$. You should discover that there is a nontrivial map $S^3 \to S^2$, "nontrivial" in the sense that it is not homotopic to a constant map. Here is one explicit, geometric construction. Consider the complex vector space $\mathbb{C}^2$, and restrict scalars to the real numbers $\mathbb{R} \subset \mathbb{R}$. Show that the underlying vector space is isomorphic to $\mathbb{R}^4$. Each unit vector $\xi \in S^3 \subset \mathbb{R}^4 \cong \mathbb{C}^2$ spans a complex line $\ell(\xi) = \mathbb{C} \cdot \xi \subset \mathbb{C}^2$. The resulting map $S^3 \to \mathbb{P}(\mathbb{C}^2) = \mathbb{CP}^1 \simeq S^2$ is not null homotopic. Prove this by considering the inverse image of a regular value.

(5.36) *The stable orthogonal group.* There is a natural sequence of inclusions

$$O(1) \xrightarrow{\sigma} O(2) \xrightarrow{\sigma} O(3) \xrightarrow{\sigma} \cdots \tag{5.37}$$

At the end of this lecture we construct the limiting space

$$O = \operatorname*{colim}_{q\to\infty} O(q). \tag{5.38}$$

As a set it is the union of the $O(q)$. Its homotopy groups are the (co)limit of the homotopy groups of the finite $O(q)$. More precisely, for each $n \in \mathbb{Z}^{\ge 0}$ the sequence

$$\pi_n O(1) \longrightarrow \pi_n O(2) \longrightarrow \pi_n O(3) \longrightarrow \cdots \tag{5.39}$$

stabilizes.

Exercise 5.40. Prove this. One method is to use the transitive action of $O(q)$ on the sphere S^{q-1}. Check that the stabilizer of a point (which?) is $O(q-1) \subset O(q)$. Use this to construct a fiber bundle with total space $O(q)$ and base S^{q-1}. In fact, this is a principal bundle with structure group $O(q-1)$. Now apply the long exact sequence of homotopy groups for a fiber bundle (more generally, fibration), as explained for example in [H, §4.2].

The stable homotopy groups of the orthogonal group (as well as the unitary and symplectic groups) were computed by Bott in the late 1950's using Morse theory. The following is known as the *Bott periodicity theorem*.

Theorem 5.41 (Bott)**.** *For all $n \in \mathbb{Z}^{\ge 0}$ there is an isomorphism $\pi_{n+8} O \cong \pi_n O$. The first few homotopy groups are*

$$\pi_{\{0,1,2,3,4,5,6,7\}} O \cong \{\mathbb{Z}/2\mathbb{Z}\,, \mathbb{Z}/2\mathbb{Z}\,, 0\,, \mathbb{Z}\,, 0\,, 0\,, 0\,, \mathbb{Z}\}. \tag{5.42}$$

A vocal rendition of the right hand side of (5.42) is known as the *Bott song*.

(5.43) *The stable J-homomorphism.* The (co)limit $q \to \infty$ in (5.33) gives the *stable J-homomorphism*

$$J\colon \pi_n O \longrightarrow \pi_n^s. \tag{5.44}$$

Lie groups

Definition 5.45. A *Lie group* is a quartet (G, e, μ, ι) consisting of a smooth manifold G, a basepoint $e \in G$, and smooth maps $\mu\colon G \times G \to G$ and $\iota\colon G \to G$ such that the underlying set G and the map μ define a group with identity element e and inverse map ι.

It is often fruitful in mathematics to combine concepts from two different areas. A Lie group, the marriage of a group and a smooth manifold, is one of the most fruitful instances.

If you have never encountered Lie groups before, I recommend [War, §3] for an introduction to some basics. We will not review these here, but just give some examples of *compact* Lie groups.

(5.46) *Orthogonal groups.* We already introduced the orthogonal group $O(q)$ in (5.27). The identity element e is the identity $q \times q$ matrix. The multiplication μ is matrix multiplication. The inverse $\iota(A)$ of an orthogonal matrix is its transpose. You can check by explicit formulas that μ and ι are smooth. The orthogonal group has two components, distinguished by the determinant homomorphism

$$\det\colon O(q) \longrightarrow \{\pm 1\}. \tag{5.47}$$

The identity component—the kernel of (5.47)—is the *special orthogonal group* $SO(q)$.

(5.48) *Unitary groups.* There is an analogous story over the complex numbers. The *unitary group* is

$$U(q) = \{g\colon \mathbb{C}^q \to \mathbb{C}^q : g \text{ is an isometry}\}, \tag{5.49}$$

where we use the standard hermitian metric on $\mathbb{C}^q$. Again μ is matrix multiplication and now ι is the transpose conjugate. The unitary group $U(1)$ is the group of unit norm complex numbers, which we denote '$\mathbb{T}$'. The kernel of the determinant homomorphism

$$\det\colon U(q) \longrightarrow \mathbb{T} \tag{5.50}$$

is the *special unitary group* $SU(n)$.

Exercise 5.51. Work out the analogous story for the quaternions $\mathbb{H}$. Define a metric on $\mathbb{H}^q$ using quaternionic conjugation. Define the group $Sp(q)$ of isometries of $\mathbb{H}^q$. Now there is no determinant homomorphism. Show that the underlying smooth manifold of the Lie group $Sp(1)$ of unit norm quaternions is diffeomorphic to S^3. Note, then, that $O(1), U(1), Sp(1)$ are diffeomorphic to S^0, S^1, S^3, respectively.

(5.52) *Parallelization of Lie groups.* Let G be a Lie group. Then any $g \in G$ determines left multiplication

$$\begin{aligned} L_g\colon G &\longrightarrow G \\ x &\longmapsto gx \end{aligned} \tag{5.53}$$

which is a diffeomorphism that maps e to g. Its differential is then an isomorphism

$$d(L_g)_e\colon T_eG \xrightarrow{\cong} T_gG. \tag{5.54}$$

This defines a *parallelism* $\underline{T_eG} \xrightarrow{\cong} TG$, a trivialization of the tangent bundle of G. There is a similar, but if G is nonabelian different, parallelism using right translation. A parallelism determines a homotopy class of stable tangential framings. Thus we have shown

Proposition 5.55. *The left invariant parallelism of a compact Lie group G determine a class $[G] \in \Omega^{\mathrm{fr}}_\bullet \cong \pi^s_\bullet$ in the stable stem.*

Low dimensions

The first several stable homotopy groups of spheres are

$$\pi^s_{\{0,1,2,3,4,5,6,7,8\}} \cong \{\mathbb{Z}\,,\ \mathbb{Z}/2\mathbb{Z}\,,\ \mathbb{Z}/2\mathbb{Z}\,,\ \mathbb{Z}/24\mathbb{Z}\,,\ 0\,,\ 0\,,\ \mathbb{Z}/2\mathbb{Z}\,,\ \mathbb{Z}/240\mathbb{Z}\,,\ \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\}. \tag{5.56}$$

It is interesting to ask what part of this is in the image of the stable J-homomorphism (5.44). Not much: compare (5.42) and (5.56). Throwing out π^s_0 we have that J is surjective on π^s_n for $n = 1, 3, 7$. The first class which fails to be in the image of J is the generator of π^s_2.

We have more luck looking for classes represented by compact Lie groups in the left invariant framing. Lie groups do represent the generators of the first several groups, starting in degree one:

$$\mathbb{T}\ ,\ \mathbb{T}\times\mathbb{T}\ ,\ Sp(1)\ ,\ -\ ,\ -\ ,\ Sp(1)\times Sp(1) \tag{5.57}$$

There is no compact Lie group which represents the generator of $\pi^s_7 \cong \mathbb{Z}/240\mathbb{Z}$, but that class is represented by a Hopf map

$$S^{15} \longrightarrow S^8, \tag{5.58}$$

analogous to the Hopf map $S^3 \to S^2$ described in Exercise 5.35.

Exercise 5.59. As an intermediary construct the Hopf map $S^7 \to S^4$ by realizing S^7 as the unit sphere in $\mathbb{C}^4 \cong \mathbb{H}^2$ and S^4 as the quaternionic projective line $\mathbb{HP}^1$. Now use the quaternions and octonions to construct (5.58).

Returning to the stable stem, the 8-dimensional Lie group $SU(3)$ represents the generator of π^s_8. For more discussion of the stable stem in low degrees, see [Ho].

(5.60) *π^s_3 and the K3 surface.* As stated in (5.57) the generator of $\pi^s_3 \cong \mathbb{Z}/24\mathbb{Z}$ is represented by $Sp(1) \cong SU(2)$ in the left invariant framing. Recall that the underlying manifold is the 3-sphere S^3. The following argument, often attributed to Atiyah, proves that 24 times the class of S^3 vanishes. It does not prove that any smaller multiple does not vanish, but perhaps we will prove that later in the course by constructing a bordism invariant

$$\Omega^{\mathrm{fr}}_3 \longrightarrow \mathbb{Z}/24\mathbb{Z} \tag{5.61}$$

which is an isomorphism. To prove that 24 times this class vanishes we construct a compact 4-manifold X with a parallelism (framing of the tangent bundle) whose boundary has 24 components, each diffeomorphic to S^3, and such that the framing restricts to a stabilization of the Lie group framing. The argument combines ideas from algebraic geometry, geometric PDE, and algebraic and differential topology. We will only give a brief sketch.

First, let $W \subset \mathbb{CP}^3$ be the smooth complex surface cut out by the quartic equation

$$(z^0)^4 + (z^1)^4 + (z^2)^4 + (z^3)^4 = 0, \tag{5.62}$$

where z^0, z^1, z^2, z^3 are homogeneous coordinates on $\mathbb{CP}^3$. Then W is a compact (real) 4-manifold. Characteristic class computations, which we will learn in a few lectures, can be used to prove that the Euler class of W is 24. Further computation and theorems of Lefschetz prove that W is simply connected and has vanishing first Chern class. Now a deep theorem of Yau—his proof of the Calabi conjecture—constructs a *hyperkähler metric* on W. This in particular gives a quaternionic structure on each tangent space. In other words, there are global endomorphisms

$$I, J, K \colon TW \longrightarrow TW \tag{5.63}$$

which satisfy the algebraic relations $I^2 = J^2 = K^2 = -\operatorname{id}_{TW}$, $IJ = -JI$, etc.

Let $\xi \colon W \to TW$ be a smooth vector field on W which is transverse to the zero section and has exactly 24 simple zeros. Let X be the manifold W with open balls excised about the zeros of ξ, and deform ξ so that it is the outward normal vector field at the boundary ∂X. Then the global vector fields $\xi, I\xi, J\xi, K\xi$ provide the desired parallelism.

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