Lecture 6: Classifying spaces

A vector bundle $E \to M$ is a family of vector spaces parametrized by a smooth manifold M. We ask: Is there a *universal* such family? In other words, is there a vector bundle $E^{\text{univ}} \to B$ such that any vector bundle $E \to M$ is obtained from $E^{\text{univ}} \to B$ by *pullback*? If so, what is this universal parameter space B for vector spaces? This is an example of a *moduli problem*. In geometry there are many interesting spaces which are universal parameter spaces for geometric objects. In this lecture we study universal parameter spaces for linear algebraic objects: *Grassmannians*, named after the 19$^{\text{th}}$ century mathematician Hermann Grassmann. We will see that there is no finite dimensional manifold which is a universal parameter space B. This is typical: to solve a moduli problem we often have to expand the notion of "space" with which we begin. Here there are several choices, one of which is to use an infinite dimensional manifold. Another is to use a colimit of finite dimensional manifolds, as in (**4.32**). Yet another is to pass to *simplicial sheaves*, but we do not pursue that here.

The universal parameter space B is called a *classifying space*: it classifies vector bundles. Classifying spaces are important in bordism theory. We use them to define *tangential structures*, which are important in both the classical and modern contexts.

For much of this lecture we do not specify whether the vector bundles are real, complex, or quaternion. All are allowed. In the last part of the lecture we discuss classifying spaces for *principal bundles*, a more general notion.

One excellent reference for some of this and the following lecture is [BT, Chapter IV].

Grassmannians

Let V be a finite dimensional vector space and k an integer such that $0 \le k \le \dim V$.

Definition 6.1. The *Grassmannian* $Gr_k(V)$ is the collection

$$Gr_k(V) = \{W \subset V : \dim W = k\} \tag{6.2}$$

of all linear subspaces of V of dimension k. Similarly, we define the Grassmannian

$$Gr_{-k}(V) = \{W \subset V : \dim W + k = \dim V\} \tag{6.3}$$

of codimension k linear subspaces of V.

We remark that the notation in (6.3) is nonstandard. The Grassmannian is more than a set: it can be given the structure of a smooth manifold. The following exercise is a guide to defining this.

Exercise 6.4.

(i) Introduce a locally Euclidean topology on $Gr_k(V)$. Here is one way to do so: Suppose $W \in Gr_k(V)$ is a k-dimensional subspace and C an $(n-k)$-dimensional subspace such that

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$W \oplus C = V$. (We say that C is a complement to W in V.) Then define a subset $\mathcal{O}_{W,C} \subset Gr_k(V)$ by

$$\mathcal{O}_{W,C} = \{W' \subset V : W' \text{ is the graph of a linear map } W \to C\}.$$

Show that $\mathcal{O}_{W,C}$ is a vector space, so has a natural topology. Prove that it is consistent to define a subset $U \subset Gr_k(V)$ to be open if and only if $U \cap \mathcal{O}_{W,C}$ is open for all W, C. Note that $\{\mathcal{O}_{W,C}\}$ is a cover of $Gr_k(V)$. (For example, show that $W \in \mathcal{O}_{W,C}$.)

(ii) Use the open sets $\mathcal{O}_{W,C}$ to construct an atlas on $Gr_k(V)$. That is, check that the transition functions are smooth. (Hint: You may first want to check it for two charts with the same W but different complements. Then it suffices to check for two different W which are transverse, using the same complement for both.)

(iii) Prove that $GL(V)$ acts smoothly and transitively on $Gr_k(V)$. What is the subgroup which fixes $W \in Gr_k(V)$?

Exercise 6.5. Introduce an inner product on V and construct a diffeomorphism $Gr_k(V) \to Gr_{-k}(V)$.

Exercise 6.6. Be sure you are familiar with the projective spaces $Gr_1(V) = \mathbb{P}V$ for $\dim V = 2$. (What about $\dim V = 1$?) Do this over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$.

(6.7) *Universal vector bundles over the Grassmannian.* There is a tautological exact sequence

$$0 \longrightarrow S \longrightarrow \underline{V} \longrightarrow Q \longrightarrow 0 \tag{6.8}$$

of vector bundles over the Grassmannian $Gr_k(V)$. The fiber of the *universal subbundle* S at $W \in Gr_k(V)$ is W, and the fiber of the *universal quotient bundle* Q at $W \in Gr_k(V)$ is the quotient V/W. The points of $Gr_k(V)$ are vector spaces—subspaces of V—and the universal subbundle is the family of vector spaces parametrized by $Gr_k(V)$.

Exercise 6.9. For $k = 1$ we denote $Gr_k(V)$ as $\mathbb{P}V$; it is called the *projective space* of V. Construct a tautological linear map

$$V^* \longrightarrow \Gamma(\mathbb{P}V; S^*) \tag{6.10}$$

where the codomain is the space of sections of the *hyperplane bundle* $S^* \to \mathbb{P}V$. This bundle is often denoted $\mathcal{O}(1) \to \mathbb{P}V$.

Pullbacks and classifying maps

(6.11) *Pullbacks of vector bundles.* Just as functions and differential forms pullback under smooth maps—they are *contravariant objects* on a smooth manifold—so too do vector bundles.

Definition 6.12. Let $f\colon M'\to M$ be a smooth map and $\pi\colon E\to M$ a smooth vector bundle. The *pullback* $\pi'\colon f^*E\to M'$ is the vector bundle whose total space is

$$f^*E=\{(m',e)\in M'\times E : f(m')=\pi(e)\}; \tag{6.13}$$

the projection $\pi'\colon f^*E\to M'$ is the restriction of projection $M'\times E\to M'$ onto the first factor. So we have a canonical isomorphism of fibers

$$(f^*E)_{p'}=E_{f(p')},\qquad p'\in M'. \tag{6.14}$$

Projection $M'\times E\to E$ onto the second factor restricts to the map $\tilde f$ in the *pullback diagram*

$$\begin{array}{ccc} f^*E & \xrightarrow{\ \tilde f\ } & E \\ {\scriptstyle\pi'}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ M' & \xrightarrow{\ f\ } & M \end{array} \tag{6.15}$$

Quite generally, if $E'\to M'$ is any vector bundle, then a commutative diagram of the form

$$\begin{array}{ccc} E' & \xrightarrow{\ \tilde f\ } & E \\ {\scriptstyle\pi'}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ M' & \xrightarrow{\ f\ } & M \end{array} \tag{6.16}$$

in which $\tilde f$ is a linear isomorphism on each fiber expresses $E'\to M'$ as the pullback of $E\to M$ via f: it defines an isomorphism $E'\to f^*E$.

Vector bundles may simplify under pullback; they can't become more "twisted".

Exercise 6.17. Consider the Hopf map $f\colon S^3\to S^2$, which you constructed in Exercise 5.35. Identify S^2 as the complex projective line $\mathbb{CP}^1=\mathbb{P}(\mathbb{C}^2)$. Let $\pi\colon S\to\mathbb{P}(\mathbb{C}^2)$ be the universal subbundle. It is nontrivial—it does not admit a global trivialization—though we have not yet proved that. Construct a trivialization of the pullback $f^*S\to S^3$. This illustrates the general principle that bundles may untwist under pullback.

(6.18) *Classifying maps.* Now we show that any vector bundle $\pi\colon E\to M$ may be expressed as a pullback of the universal quotient bundle[1] over a Grassmannian, at least in case M is compact.

Theorem 6.19. *Let $\pi\colon E\to M$ be a vector bundle of rank k over a compact manifold M. Then there is a finite dimensional vector space V and a smooth maps $f,\tilde f$ which express π as the pullback*

$$\begin{array}{ccc} E & \xrightarrow{\ \tilde f\ } & Q \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow \\ M & \xrightarrow{\ f\ } & Gr_{-k}(V) \end{array} \tag{6.20}$$

[1]We can use the universal subbundle instead, but the construction we give makes the universal quotient bundle more natural.

Proof. Since $E \to M$ is locally trivializable and M is compact, there is a finite cover $\{U^\alpha\}_{\alpha\in A}$ of M and a basis $s_1^\alpha,\dots,s_k^\alpha\colon U^\alpha\to E$ of local sections over each U^α. Let $\{\rho^\alpha\}$ be a partition of unity subbordinate to the cover $\{U^\alpha\}$. Then $\tilde{s}_i^\alpha = \rho^\alpha s_i^\alpha$ extend to global sections of E which vanish outside U^α. Define V to be the linear span of the finite set $\{\tilde{s}_i^\alpha\}_{\alpha\in A,\, i=1,\dots,k}$ over the ground field. Then for each $p\in M$ the linear map

$$\begin{aligned} ev_p\colon V &\longrightarrow E_p \\ \tilde{s}_i^\alpha &\longmapsto \tilde{s}_i^\alpha(p) \end{aligned} \tag{6.21}$$

is surjective and induces an isomorphism $V/\ker ev_p \xrightarrow{\cong} E_p$. The inverses of these isomorphisms fit together to form the map $\tilde{f}$ in the diagram (6.20), where f is defined by $f(p)=\ker ev_p$. □

Classifying spaces

Theorem 6.19 shows that every vector bundle $\pi\colon E\to M$ over a smooth compact manifold is pulled back from the Grassmannian, but it does not provide a single classifying space for *all* vector bundles; the vector space V depends on π. Furthermore, we might like to drop the assumption that M is compact (and even generalize further to continuous vector bundles over nice topological spaces). There are several approaches, and we outline three of them here. For definiteness we work over $\mathbb{R}$; the same arguments apply to $\mathbb{C}$ and $\mathbb{H}$.

(6.22) *The infinite Grassmannian as a colimit.* Fix $k\in\mathbb{Z}^{>0}$ and consider the sequence of closed inclusions

$$\mathbb{R}^q \longrightarrow \mathbb{R}^{q+1} \longrightarrow \mathbb{R}^{q+1} \longrightarrow \cdots, \tag{6.23}$$

where at each stage the map is $(\xi^1,\xi^2,\dots)\mapsto(0,\xi^1,\xi^2,\dots)$. There is an induced sequence of closed inclusions

$$Gr_k(\mathbb{R}^q) \longrightarrow Gr_k(\mathbb{R}^{q+1}) \longrightarrow Gr_k(\mathbb{R}^{q+2}) \longrightarrow \cdots \tag{6.24}$$

where at each stage the map is $W\mapsto 0\oplus W$. Similarly, there is an induced sequence of closed inclusions

$$Gr_{-k}(\mathbb{R}^q) \longrightarrow Gr_{-k}(\mathbb{R}^{q+1}) \longrightarrow Gr_{-k}(\mathbb{R}^{q+2}) \longrightarrow \cdots \tag{6.25}$$

where at each stage the map is $K\mapsto \mathbb{R}\oplus K$. These maps fit together to a lift of (6.25) to pullback maps of the universal quotient bundles:

$$\begin{array}{ccccccc} Q_q & \longrightarrow & Q_{q+1} & \longrightarrow & Q_{q+2} & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ Gr_{-k}(\mathbb{R}^q) & \longrightarrow & Gr_{-k}(\mathbb{R}^{q+1}) & \longrightarrow & Gr_{-k}(\mathbb{R}^{q+2}) & \longrightarrow & \cdots \end{array} \tag{6.26}$$

We take the colimit (see (**4.32**)) of this diagram to obtain a vector bundle[2]

(6.27) $$\pi\colon Q^{\mathrm{univ}} \longrightarrow B_k.$$

Now B_k is a topological space but not a manifold in any sense, and π is a continous vector bundle. Any classifying map (6.20) for a vector bundle over a compact smooth manifold induces a classifying map into $Q^{\mathrm{univ}} \to B_k$. More is true, but we will not prove this here; see [H2, Theorem 1.16], for example.

Theorem 6.28. *Let $\pi\colon E \to X$ be a vector bundle over a metrizable space X. Then there is a classifying diagram*

(6.29) $$\begin{array}{ccc} E & \xrightarrow{\tilde f} & Q^{\mathrm{univ}} \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ M & \xrightarrow{f} & B_k \end{array}$$

and the map f is unique up to homotopy. Furthermore, the set of homotopy classes of maps $M \to B_k$ is in 1:1 correspondence with the set of isomorphism classes of vector bundles $E \to M$.

(6.30) *The infinite Grassmannian as an infinite dimensional manifold.* Let $\mathcal{H}$ be a separable (real, complex, or quaternionic) Hilbert space. Fix $k \in \mathbb{Z}^{>0}$. Define the Grassmannian

(6.31) $$Gr_k(\mathcal{H}) = \{W \subset \mathcal{H} : \dim W = k\}.$$

We can use the technique of Exercise 6.4 to introduce charts and a manifold structure on $Gr_k(\mathcal{H})$, but now the local model is an infinite dimensional Hilbert space.

Digression: Calculus in finite dimensions is developed on (affine spaces over) finite dimensional vector spaces. A topology on the vector space is needed to take the limits necessary to compute derivatives, and there is a unique topology compatible with the vector space structure. It is usually described by a Euclidean metric, i.e., by an inner product on the vector space. In infinite dimensions one also needs a topology compatible with the linear structure, but now there are many different species of topological vector space. By far the easiest, and the closest to the finite dimensional situation, is the topology induced from a Hilbert space structure: a *complete* inner product. That is the topology we use here, and then the main theorems of differential calculus go through almost without change.

We call $Gr_{-k}(\mathcal{H})$ a *Hilbert manifold.*

Choose an orthonormal basis $e_1, e_2, \dots$ of $\mathcal{H}$ and so define the subspace $\mathbb{R}^q \subset \mathcal{H}$ as the span of $e_1, e_2, \dots, e_q$. This induces a commutative diagram

(6.32) $$\begin{array}{ccccccccc} \cdots & \longrightarrow & Gr_k(\mathbb{R}^{q-1}) & \longrightarrow & Gr_k(\mathbb{R}^q) & \longrightarrow & Gr_k(\mathbb{R}^{q+1}) & \longrightarrow & \cdots \\ & & & \searrow & \downarrow & \swarrow & & & \\ & & & & Gr_k(\mathcal{H}) & & & & \end{array}$$

[2]Using the standard inner product, as in Exercise 6.5, we can take orthogonal complements to replace codimension k subspaces with dimension k subspaces and the universal quotient bundle with the universal subbundle.

of inclusions, and so an inclusion of the colimit

$$i\colon B_k \longrightarrow Gr_k(\mathcal{H}). \tag{6.33}$$

Proposition 6.34. *The map i in (6.33) is a homotopy equivalence.*

One way to prove Proposition 6.34 is to first show that i is a *weak* homotopy equivalence, that is, the induced map $i_*\colon \pi_n B_k \to Gr_k(\mathcal{H})$ is an isomorphism for all n. (We must do this for all basepoints $p \in B_k$ and the corresponding $i(p) \in Gr_k(\mathcal{H})$.) Then we would show that the spaces in (6.33) have the homotopy type of CW complexes. For much more general theorems along these lines, see [Pa1]. In any case I include Proposition 6.34 to show that there are different models for the classifying space which are homotopy equivalent.

(6.35) *Classifying space as a simplicial sheaf.* We began with the problem of classifying finite rank vector bundles over a compact smooth manifold. We found that the classifying space is not a compact smooth manifold, nor even a finite dimensional manifold. We have constructed two models: a topological space B_k and a smooth manifold $Gr_k(\mathcal{H})$. There is a third possibility which expands the idea of "space" in a more radical way: to a simplicial sheaf on the category of smooth manifolds. This is too much of a digression at this stage, so we will not pursue it. The manuscript [FH] in progress contains expository material along these lines.

Classifying spaces for principal bundles

Recall first the definition.

Definition 6.36. Let G be a Lie group. A *principal G bundle* is a fiber bundle $\pi\colon P \to M$ over a smooth manifold M equipped with a right G-action $P \times G \to P$ which is simply transitive on each fiber.

The hypothesis that π is a fiber bundle means it it admits local trivializations. For a *principal* bundle a local trivialization is equivalent to a local section. In one direction, if $U \subset M$ and $s\colon U \to P$ is a section of $\pi\big|_U\colon P\big|_U \to U$, then there is an induced local trivialization

$$\begin{aligned} \varphi\colon U \times G &\longrightarrow P \\ x, g &\longmapsto s(x) \cdot g \end{aligned} \tag{6.37}$$

where '$\cdot$' denotes the G-action on P.

(6.38) *From vector bundles to principal bundles and back.* Let $\pi\colon E \to M$ be a vector bundle of rank k. Assume for definiteness that π is a *real* vector bundle. There is an associated principal $GL_k(\mathbb{R})$-bundle $\mathcal{B}(E) \to M$ whose fiber at $x \in M$ is the spaces of bases $b\colon \mathbb{R}^k \xrightarrow{\cong} E_x$. These fit together into a principal bundle which admits local sections: a local section of the principal bundle $\mathcal{B}(E) \to M$ is a local trivialization of the vector bundle $E \to M$. Conversely, if $P \to M$ is a principal $G = GL_k(\mathbb{R})$-bundle, then there is an *associated* rank k vector bundle $E \to M$ defined as

$$E = P \times \mathbb{R}^k / G, \tag{6.39}$$

where the right G-action on $P \times \mathbb{R}^k$ is

$$(p,\xi)\cdot g = (p\cdot g, g^{-1}\xi), \qquad p \in P, \quad \xi \in \mathbb{R}^k, \quad g \in G, \tag{6.40}$$

and we use the standard action of $GL_k(\mathbb{R})$ on $\mathbb{R}^k$ to define $g^{-1}\xi$.

(6.41) *Fiber bundles with contractible fiber.* We quote the following general proposition in the theory of fiber bundles.

Proposition 6.42. *Let $\pi\colon \mathcal{E} \to M$ be a fiber bundle whose fiber F is contractible and a metrizable topological manifold, possibly infinite dimensional. Assume that the base M is metrizable. Then π admits a section. Furthermore, if $\mathcal{E}, M, F$ all have the homotopy type of a CW complex, then π is a homotopy equivalence.*

See [Pa1] for a proof of the first assertion. The last assertion follows from the long exact sequence of homotopy groups and Whitehead's theorem (**6.52**).

(6.43) *Classifying maps for principal bundles.* Now we characterize universal principal bundles.

Theorem 6.44. *Let G be a Lie group. Suppose $\pi^{\text{univ}}\colon P^{\text{univ}} \to B$ is a principal G-bundle and P^{univ} is contractible. Then for any continuous principal G-bundle $P \to M$ with M metrizable, there is a classifying diagram*

$$\begin{array}{ccc} P & \xrightarrow{\tilde{\varphi}} & P^{\text{univ}} \\ \downarrow & & \downarrow \\ M & \xrightarrow{\varphi} & B \end{array} \tag{6.45}$$

In the commutative diagram (6.45) the map $\tilde{\varphi}$ commutes with the G-actions on P, P^{univ}, i.e., it is a map of principal G-bundles.

Proof. A G-map $\tilde{\varphi}$ is equivalently a section of the associated fiber bundle

$$(P \times P^{\text{univ}})/G \to M \tag{6.46}$$

formed by taking the quotient by the diagonal right G-action. The fiber of the bundle (6.46) is P^{univ}. Sections exist by Proposition 6.42, since P^{univ} is contractible. □

(6.47) *Back to Grassmannians.* The construction in (**6.38**) defines a principal $GL_k(\mathbb{R})$-bundle over the universal Grassmannian, but we can construct it directly and it has a nice geometric meaning. We work in the infinite dimensional manifold model (**6.30**). Thus let $\mathcal{H}$ be a separable (real) Hilbert space. Introduce the infinite dimensional *Stiefel manifold*

$$St_k(\mathcal{H}) = \{b\colon \mathbb{R}^k \to \mathcal{H} : b \text{ is injective}\}. \tag{6.48}$$

It is an open subset of the linear space $\mathrm{Hom}(\mathbb{R}^k,\mathcal{H}) \cong \mathcal{H}\oplus\cdots\oplus\mathcal{H}$, which we give the topology of a Hilbert space. Then the open subset $St_k(\mathcal{H})$ is a Hilbert manifold. There is an obvious projection

$$\pi\colon St_k(\mathcal{H}) \longrightarrow Gr_k(\mathcal{H}) \tag{6.49}$$

which maps b to its image $b(\mathbb{R}^k) \subset \mathcal{H}$. We leave the reader to check that π is smooth. In fact, π is a principal bundle with structure group $GL_k(\mathbb{R})$.

Theorem 6.50. *$St_k(\mathcal{H})$ is contractible.*

Corollary 6.51. *The bundle (6.49) is a universal $GL_k(\mathbb{R})$-bundle.*

The corollary is an immediate consequence of Theorem 6.50 and Theorem 6.44. We give the proof of Theorem 6.50 below.

(6.52) *Remark on contractibility.* A fundamental theorem of Whitehead asserts that if X, Y are connected[3] pointed topological spaces which have the homotopy type of a CW complex, and $f\colon X \to Y$ is a continuous map which induces an isomorphism $f_*\colon \pi_n X \to \pi_n Y$ for all $n \in \mathbb{Z}^{\ge 0}$, then f is a homotopy equivalence. A map which satisfies the hypothesis of the theorem is called a *weak homotopy equivalence*. An immediate corollary is that if X satisfies the hypotheses and all homotopy groups of X vanish, then X is contractible. For "infinite spaces" with a colimit topology, weak contractibility can often be verified by an inductive argument. That is the case for the Stiefel space $St_k(\mathbb{R}^\infty)$ with a colimit topology, analogous to that for the Grassmannian in (6.24). We prefer instead a more beautiful geometric argument using the Hilbert manifold $St_k(\mathcal{H})$, which is homotopy equivalent (as in Proposition 6.34).

Exercise 6.53. Carry out this argument. You will want to consider submersions $St_k(\mathbb{R}^q) \to St_{k-1}(\mathbb{R}^q)$, as we do below. Then you will need the long exact sequence of homotopy groups for a fibration.

(6.54) *The unit sphere in Hilbert space.* The Stiefel manifold $St_1(\mathcal{H})$ is the unit sphere $S(\mathcal{H}) \subset \mathcal{H}$, the space of unit norm vectors. As a first case of Theorem 6.50 we prove that this infinite dimensional sphere with the induced topology is contractible, summarizing an elegant argument of Richard Palais [Pa2].

Lemma 6.55. *Let X be a normal topological space and $A \subset X$ a closed subspace homeomorphic to $\mathbb{R}$. Then there exists a fixed point free continuous map $f\colon X \to X$.*

Proof. The map $x \mapsto x+1$ on $\mathbb{R}$ induced a map $g\colon A \to A$ with no fixed points. By the Tietze extension theorem g extends to a map $\tilde{g}\colon X \to A$. Let f be the extension g followed by the inclusion $A \hookrightarrow X$. □

Theorem 6.56. *$S(\mathcal{H})$ is contractible.*

[3] Whitehead's theorem easily extends to nonconnected spaces.

Proof. Let $\{e_n\}_{n\in\mathbb{Z}}$ be an orthonormal basis of $\mathcal{H}$, set $S = S(\mathcal{H})$ and let $D = \{\xi \in \mathcal{H} : \|\xi\| \le 1\}$ be the closed unit ball in $\mathcal{H}$. Define $i\colon \mathbb{R} \hookrightarrow D$ by letting $i\big|_{[n,n+1]}$ be a curve on S which connects e_n and e_{n+1}, $n \in \mathbb{Z}$. Explicitly, for $t \in [n, n+1]$,

$$i\colon t \longmapsto \cos\big[(t-n)\pi/2\big]e_n + \sin\big[(t-n)\pi/2\big]e_{n+1}. \tag{6.57}$$

Then by the lemma there is a continous map $f\colon D \to D$ with no fixed points. We use it, as in Hirsh's beautiful proof of the Brouwer fixed point theorem, to construct a deformation retraction $g\colon D \to S$: namely, $g(\xi)$ is the intersection of S with the ray emanating from $\xi \in D$ in the direction $\xi - f(\xi)$. Then g is a homotopy equivalence. On the other hand, there is an easy radial deformation retraction of D to $0 \in D$, and so D is contractible. □

Proof of Theorem 6.50. Let $\pi\colon St_k(\mathcal{H}) \to St_{k-1}(\mathcal{H})$ map $b\colon \mathbb{R}^k \to \mathcal{H}$ to the restriction of b to $\mathbb{R}^{k-1} \subset \mathbb{R}^k$. In terms of bases, if b maps the standard basis of $\mathbb{R}^k$ to $\xi_1, \xi_2, \dots, \xi_k$, then $\bar{b} = \pi(b)$ gives the independent vectors $\xi_2, \dots, \xi_k$. The fiber over $\bar{b}$ is the set of nonzero vectors in the orthogonal complement $\mathcal{H}'$ of the span of $\xi_2, \dots, \xi_k$, which is a closed subspace of $\mathcal{H}$, hence a Hilbert space. Now the set of nonzero vectors in a Hilbert space deformation retracts onto the unit sphere, which by Theorem 6.56 is contractible. Then Proposition 6.42 implies that π is a homotopy equivalence. Now proceed by induction beginning with the statement that $St_1(\mathcal{H})$ is contractible. □

Remark 6.58. An alternative proof of Theorem 6.50 is based on *Kuiper's theorem*, which states that the Banach Lie group $GL(\mathcal{H})$ of all invertible linear operators $\mathcal{H} \to \mathcal{H}$ in the norm topology is contractible. This group acts transitively on $St_k(\mathcal{H})$ with stabilizer a contractible group. It follows that the quotient is also contractible.

(6.59) *Other Lie groups.* Let G be a *compact* Lie group. (Note G need not be connected.) The *Peter-Weyl theorem* asserts that there is an embedding $G \subset U(k) \subset GL_k(\mathbb{C})$ for some $k > 0$. Let $EG = St_k(\mathcal{H})$ be the Stiefel manifold for a *complex* separable Hilbert space $\mathcal{H}$. Then the restriction of the free $GL_k(\mathbb{C})$-action to G is also free; let BG be the quotient. It is a Hilbert manifold, and

$$EG \longrightarrow BG \tag{6.60}$$

is a universal principal G-bundle, by Theorem 6.44.

This gives Hilbert manifold models for the classifying space of any compact Lie group.

Exercise 6.61. What is the classifying Hilbert manifold of $O(1) = \mathbb{Z}/2\mathbb{Z}$? What about $\mathbb{T} = U(1)$? What about the unit quaternions $Sp(1)$? Show that the classifying Hilbert manifold of a finite cyclic group is an infinite dimensional *lens space*.

Exercise 6.62. Let G be a connected compact Lie group and $T \subset G$ a maximal torus. Then T acts freely on EG, and there is an induced fiber bundle $BT \to BG$. What is the fiber? Describe both manifolds explicitly for the classical groups $G = O(k)$, $U(k)$, and $Sp(k)$.

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