Lecture 7: Characteristic classes

In this lecture we describe some basic techniques in the theory of characteristic classes, mostly focusing on Chern classes of complex vector bundles. There is lots more to say than we can do in a single lecture. Much of what we say follows the last chapter of [BT], which is posted on the web site, and so these notes are terse on some points which you can read in detail there. I highly encourage you to do so!

I will summarize a few results on the computation of the ring of characteristic classes, but we will not attempt to prove them here. Those proof require more algebraic topology than I can safely assume.

Classifying revisited

In Lecture 6 we sloughed over the classification statement, which appeared in passing in the statement of Theorem 6.28. Here is a definitive version.

Theorem 7.1. *Let G be a Lie group and $EG \to BG$ a universal principal G-bundle. Then for any manifold M there is a 1:1 correspondence*

$$[M, BG] \xrightarrow{\cong} \{\text{isomorphism classes of principal } G\text{-bundles over } M\}. \tag{7.2}$$

To a map $f\colon M \to BG$ we associate the bundle $f^*EG \to M$. We gave some ingredients in the proof. For example, Theorem 6.44 proves that (7.2) is surjective. One idea missing is that if $f_0, f_1\colon M \to BG$ are homotopic, then $f_0^*(EG) \to M$ is isomorphic to $f_1^*(EG) \to M$. We give a proof in case all maps are smooth and we use a Hilbert manifold model for the universal bundle, as in (**6.59**).

Proposition 7.3. *Let $P \to \Delta^1 \times M$ be a smooth principal G-bundle. The the restrictions $P\big|_{\{0\}\times M} \to M$ and $P\big|_{\{1\}\times M} \to M$ are isomorphic.*

The assertion about homotopic maps is an immediate corollary: if $F\colon \Delta^1 \times M \to BG$ is a homotopy, consider $F^*(EG) \to \Delta^1 \times M$.

The proof uses the existence of a *connection* and the fundamental existence and uniqueness theorem for ordinary differential equations. Let $\pi\colon P \to N$ be a smooth principal G-bundle. Then at each $p \in P$ there is a short exact sequence

$$0 \longrightarrow \ker(\pi_*)_p \longrightarrow T_pP \longrightarrow T_{\pi(p)}N \longrightarrow 0 \tag{7.4}$$

Definition 7.5. A *horizontal subspace* at p is a splitting of (7.4). A *connection* is a G-invariant splitting of the sequence of vector bundles

$$0 \longrightarrow \ker \pi_* \longrightarrow TP \longrightarrow \pi^*TN \longrightarrow 0 \tag{7.6}$$

over P.

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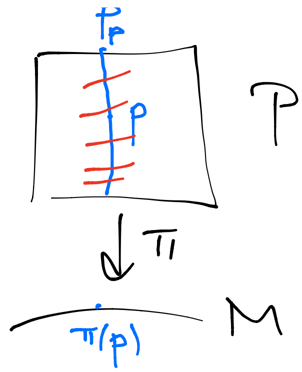

Figure 16. A connection

Recall from Lemma 5.6 that splittings form an affine space. Fix $n \in N$. The G-invariant splittings of (7.4) for $p \in \pi^{-1}(n)$ form a finite dimensional affine space. As n varies these glue together into an affine bundle over N. A partition of unity argument (Exercise 5.7) then shows that connections exist.

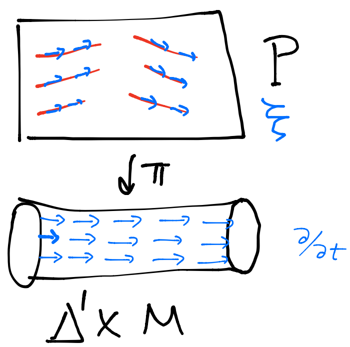

Figure 17. Homotopy invariance

Proof of Proposition 7.3. Let $\partial/\partial t$ denote the vector field on $\Delta^1 \times M$ which is tangent to the $\Delta^1 = [0,1]$ factor. Choose a connection on $\pi\colon P \to \Delta^1 \times M$. The connection determines a G-invariant vector field ξ on P which projects via π to $\partial/\partial t$. The fundamental theorem for ODE gives, for each initial condition $p \in P\big|_{\{0\}\times M}$ an integral curve $\gamma_p\colon [0,1] \to P$ whose composition with $\pi_1 \circ \pi$ is the identity. Here $\pi_1\colon \Delta^1 \times M \to \Delta^1$ is projection onto the first factor. The map $p \mapsto \gamma_p(1)$ is the desired isomorphism of principal bundles. □

Exercise 7.7. Prove Theorem 7.1.

The idea of characteristic classes

Let X be a topological space. There is an associated chain complex

$$C_0 \longleftarrow C_1 \longleftarrow C_2 \longleftarrow \cdots \tag{7.8}$$

of free abelian groups which computes the homology of X. There are several models for the chain complex, depending on the structure of X. A CW structure on X usually leads to the most efficient model, the cellular chain complex. If A is an abelian group, then applying $\mathrm{Hom}(-,A)$ to (7.8) we obtain a cochain complex

$$\mathrm{Hom}(C_0,A)\longrightarrow \mathrm{Hom}(C_1,A)\longrightarrow \mathrm{Hom}(C_2,A)\longrightarrow\cdots \tag{7.9}$$

which computes the *cohomology* groups $H^\bullet(X;A)$. If R is a commutative ring, then the cohomology $H^\bullet(X;R)$ is a $\mathbb{Z}$-graded ring—the multiplication is called the *cup product*—and it is commutative in a graded sense. Just as homology is a homotopy invariant, so to is cohomology. There is an important distinction: if $f\colon X\to X'$ is a continuous map, then the induced map on cohomology is by pullback

$$f^*\colon H^\bullet(X';A)\longrightarrow H^\bullet(X;A). \tag{7.10}$$

As stated, it is unchanged if f undergoes a homotopy.

Suppose $\alpha\in H^\bullet(BG;A)$ is a cohomology class on the classifying space BG. (Recall that there are different, homotopy equivalent, models for BG; see Proposition 6.34. By the homotopy invariance of cohomology, it won't matter which we use.) Then if $P\to M$ is a principal G-bundle over a manifold M, we define $\alpha(P)\in H^\bullet(M;A)$ by

$$\alpha(P)=f_P^*(\alpha) \tag{7.11}$$

where $f_P\colon M\to BG$ is any classifying map. Theorem 7.1 and the homotopy invariance of cohomology guarantee that (7.11) is well-defined. Then $\alpha(P)$ is a *characteristic class* of $P\to M$.

Exercise 7.12. Suppose $g\colon M'\to M$ is smooth and $P\to M$ is a G-bundle. Prove that

$$\alpha(g^*P)=g^*\alpha(P) \tag{7.13}$$

Thus we say that characteristic classes are *natural*.

Cohomology classes in $H^\bullet(BG;A)$ are *universal characteristic classes*, and the problem presents itself to compute the cohomology of BG with various coefficient groups A. We will state a few results at the end of the lecture. First we develop *Chern classes* for complex vector bundles. (Recall from **(6.38)** that this is equivalent to characteristic classes for $G=GL_k(\mathbb{C})$. We will make a contractible choice of a hermitian metric, so may use instead the unitary group $G=U(k)$.)

Complex line bundles

Recall from Corollary 6.51 that a classifying space for complex line bundles is the projective space $\mathbb{P}(\mathcal{H})$ of a complex separable Hilbert space $\mathcal{H}$. To write the chain complex of this space, it is more convenient to use the colimit space $\mathbb{P}(\mathbb{C}^\infty)$, analogous to the discussion in (**6.22**). That space has a cell decomposition with a single cell in each even dimension, so the cellular chain complex is

$$\mathbb{Z} \longleftarrow 0 \longleftarrow \mathbb{Z} \longleftarrow 0 \longleftarrow \mathbb{Z} \longleftarrow 0 \longleftarrow \cdots \tag{7.14}$$

The cochain complex which computes integral cohomology is then

$$\mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots \tag{7.15}$$

With a bit more work we can prove that the integral cohomology ring of the classifying space is

$$H^\bullet\bigl(\mathbb{P}(\mathcal{H});\mathbb{Z}\bigr) \cong \mathbb{Z}[y], \qquad \deg y = 2, \tag{7.16}$$

a polynomial ring on a single generator in degree 2. The generator y is defined by (7.16) only up to sign, and we fix the sign by requiring that

$$\langle y, [\mathbb{P}(V)]\rangle = 1, \tag{7.17}$$

where $[\mathbb{P}(V)] \in H_2(\mathbb{P}(\mathcal{H}))$ is the fundamental class of any projective line ($V \in \mathcal{H}$ two-dimensional).

Recall from (**6.7**) the tautological line bundle $S \to \mathbb{P}(\mathcal{H})$.

Definition 7.18. The *first Chern class* of $S \to \mathbb{P}(\mathcal{H})$ is $-y \in H^2\bigl(\mathbb{P}(\mathcal{H})\bigr)$.

Since $S \to \mathbb{P}(\mathcal{H})$ is a universal line bundle, this defines the first Chern class for all line bundles over any base.

Proposition 7.19. *Let $L_1, L_2 \to M$ be complex line bundles. Then*

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2) \in H^2(M). \tag{7.20}$$

Proof. It suffices to prove this universally. Let $\mathcal{H}_1, \mathcal{H}_2$ be infinite dimensional complex separable Hilbert spaces, and $S_i \to \mathbb{P}(\mathcal{H}_i)$ the corresponding tautological line bundles. The *external tensor product* $S_1 \boxtimes S_2$ is classified by the map

$$\begin{array}{ccc} S_1 \boxtimes S_2 & \longrightarrow & S \\ \downarrow & & \downarrow \\ \mathbb{P}(\mathcal{H}_1) \times \mathbb{P}(\mathcal{H}_2) & \xrightarrow{\;f\;} & \mathbb{P}(\mathcal{H}) \end{array} \tag{7.21}$$

where $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and if $L_i \in \mathbb{P}(\mathcal{H}_i)$ contain nonzero vectors ξ_i, the line $f(L_1, L_2)$ is the span of $\xi_1 \otimes \xi_2$. (Note that the fiber of $S_1 \boxtimes S_2$ at (L_2, L_2) is $L_1 \otimes L_2$.) If $V_i \subset \mathcal{H}_i$ is 2-dimensional, and if $L_i \in \mathbb{P}(\mathcal{H}_i)$ is a fixed line, then the image of the projective lines $\mathbb{P}(V_1) \times \{L_2\}$ and $\{L_1\} \times \mathbb{P}(V_2)$ are projective lines in $\mathbb{P}(\mathcal{H})$. It follows that $f^*(y) = y_1 + y_2$ in $H^2\bigl(\mathbb{P}(\mathcal{H}_1) \times \mathbb{P}(\mathcal{H}_2)\bigr)$, where y, y_1, y_2 are the properly oriented generators of $H^\bullet\bigl(\mathcal{H}(\mathbb{P})\bigr), H^\bullet\bigl(\mathcal{H}(\mathbb{P}_1)\bigr), H^\bullet\bigl(\mathcal{H}(\mathbb{P}_2)\bigr)$, respectively. $\square$

Corollary 7.22. *Let $L \to M$ be a complex line bundle. Then*

$$c_1(L^*) = -c_1(L). \tag{7.23}$$

This follows since $L \otimes L^* \to M$ is trivializable.

Higher Chern classes

(7.24) *The Leray-Hirsch theorem.* As a preliminary we quote the following result in the topology of fiber bundles; see [BT] or [H1, §4.D] for a proof.

Theorem 7.25 (Leray-Hirsch)**.** *Let $F \to \mathcal{E} \to B$ be a fiber bundle and R a commutative ring. Suppose $\alpha_1, \dots, \alpha_N \in H^\bullet(\mathcal{E}; R)$ have the property that $i_b^*\alpha_1, \dots, i_b^*\alpha_N$ freely generate the R-module $H^\bullet(\mathcal{E}_b; R)$ for all $b \in B$. Then $H^\bullet(\mathcal{E}; R)$ is isomorphic to the free $H^\bullet(B; R)$-module with basis $\alpha_1, \dots, \alpha_N$.*

Even though the total space $\mathcal{E}$ is not a product $B \times F$, its cohomology behaves as though it is, at least as an R-module. The ring structure is twisted, however, and we will use that to define the higher Chern classes below.

(7.26) *Flag bundles.* Let E be a complex vector space of dimension k with a hermitian metric. There is an associated *flag manifold* $\mathbb{F}(E)$ whose points are orthogonal decompositions

$$E = L_1 \oplus \cdots \oplus L_k \tag{7.27}$$

of E as a sum of lines. If $\dim E = 2$, then $\mathbb{F}(E) = \mathbb{P}(E)$ since L_2 is the orthogonal complement of L_1. In general the flag manifold $\mathbb{F}(E)$ has k tautological line bundles $L_j \to \mathbb{F}(E)$, $j = 1, \dots, k$. This functorial construction can be carried out in families. So to a hermitian vector bundle $E \to M$ of rank k over a smooth manifold M there is an associated fiber bundle—the *flag bundle*

$$\pi\colon \mathbb{F}(E) \to M \tag{7.28}$$

with typical fiber the flag manifold. There are tautological line bundles $L_j \to \mathbb{F}(E)$, $j = 1, \dots, k$.

Proposition 7.29. *Polynomials in the cohomology classes $x_j = c_1(L_j) \in H^2\big(\mathbb{F}(E); \mathbb{Z}\big)$ freely generate the integral cohomology of each fiber $\mathbb{F}(E)_p$, $p \in M$, as an abelian group.*

Corollary 7.30. *The pullback map*

$$\pi^*\colon H^\bullet(M; \mathbb{Z}) \longrightarrow H^\bullet\big(\mathbb{F}(E); \mathbb{Z}\big) \tag{7.31}$$

is injective.

Note the choice of sign for x_j; it is opposite to that for y in (7.17). The image of π^* is the subring of symmetric polynomials in x_j with coefficients in the ring $H^\bullet(M; \mathbb{Z})$.

Sketch proof of Proposition 7.29. This is done in [BT], so we only give a rough outline. Consider first the *projective bundle*

$$\pi_1\colon \mathbb{P}(E) \to M \tag{7.32}$$

whose fiber at $p \in M$ is the projectivization $\mathbb{P}(E_x)$ of the fiber E_p. There is a tautological line bundle $S \to \mathbb{P}(E)$ which restricts on each fiber $\mathbb{P}(E)_p$ of π_1 to the tautological line bundle of that projective space. The chain complex of a finite dimensional projective space is a truncation of (7.14), from which it follows that $y = c_1(S^*)$ and its powers generate the cohomology of the fiber of π_1, in the sense of the Leray-Hirsch Theorem 7.25. So

$$H^\bullet\bigl(\mathbb{P}(E);\mathbb{Z}\bigr) \cong H^\bullet(M;\mathbb{Z})\{1, y, y^2, \dots, y^{k-1}\} \tag{7.33}$$

as abelian groups. Now consider the projective bundle associated to the quotient bundle[1] $Q \to \mathbb{P}(E)$ and keep iterating. □

Exercise 7.34. Work out the details of this proof without consulting [BT]!

(7.35) *Higher chern classes.* Following Grothendieck we define the Chern classes of E using Theorem 7.25. Namely, the class $y^k \in H^{2k}\bigl(\mathbb{P}(E);\mathbb{Z}\bigr)$ must by (7.33) satisfy a polynomial equation of the form

$$y^k \;+\; c_1(E)y^{k-1} \;+\; c_2(E)y^{k-2} \;+\; \cdots + \; c_k(E) = 0 \tag{7.36}$$

for some unique classes $c_i(E) \in H^{2i}(M;\mathbb{Z})$.

Definition 7.37. The class $c_i(E)$ defined by (7.36) is the i^{th} Chern class of $E \to M$.

Proposition 7.38. *The pullback $\pi^* c_i(E)$ to the flag bundle (7.28) is the i^{th} elementary symmetric polynomial in $x_1, \dots, x_k$.*

Proof. Define the submersion

$$\rho_j \colon \mathbb{F}(E) \longrightarrow \mathbb{P}(E) \tag{7.39}$$

to map the flag $E \cong L_1 \oplus \cdots \oplus L_k$ to the line L_j. It is immediate that $\rho_j^*(y) = -x_i$, where $y = c_1(S^*) \in H^2\bigl(\mathbb{P}(E);\mathbb{Z}\bigr)$ as in (7.33), and $x_j = c_1(L_j)$ as in Proposition 7.29. So each x_j is a root of the polynomial equation

$$z^k \;-\; \pi^*c_1(E)z^{k-1} \;+\; \pi^*c_2(E)z^{k-2} \;-\; \cdots + \; (-1)^k\pi^*c_k(E) = 0 \tag{7.40}$$

in the cohomology of $\mathbb{F}(E)$. The conclusion follows. □

Exercise 7.41. Prove that the Chern classes of a trivial vector bundle vanish.

[1]Since E has a metric, we identify Q as the orthogonal complement to S.

(7.42) *The splitting principle.* The pullback $\pi^*E \to \mathbb{F}(E)$ is canonically isomorphic to the sum $L_1 \oplus \cdots \oplus L_k \to \mathbb{F}(E)$ of line bundles. That, combined with Corollary 7.30 and Proposition 7.38, gives a method for computing with Chern classes: one can always assume that a vector bundle is the sum of line bundles. That is not true on the base M, but it is true for the pullback to the flag bundle. Any identity in Chern classes proved there is valid on M, because of the injectivity of the induced map on cohomology. Furthermore, symmetric polynomials in the x_i are polynomials in the Chern classes, by a basic theorem in commutative algebra about polynomial rings, and in particular live on the base M.

As a simple illustration, define the *total Chern class* of $E \to M$ as

$$c(E) = 1 + c_1(E) + c_2(E) + \cdots . \tag{7.43}$$

Then we formally write

$$c(E) = \prod_{j=1}^{k} (1 + x_j); \tag{7.44}$$

The equation is precisely true on $\mathbb{F}(E)$ for $\pi^*c(E)$. Also, for a smooth manifold M we write

$$c(M) = c(TM) \tag{7.45}$$

for the Chern classes of the tangent bundle.

Exercise 7.46. Prove the *Whitney sum formula*: If $E_1, E_2 \to M$ are complex vector bundles, then

$$c(E_1 \oplus E_2) = c(E_1) + c(E_2). \tag{7.47}$$

The formula for the tensor product is more complicated. Find a formula for $c_1(E_1 \otimes E_2)$. Can you find a formula for $c(E_1 \otimes E_2)$ in case one of the bundles is a line bundle?

Exercise 7.48. Define the complex conjugate bundle $\overline{E} \to M$ to a complex vector bundle $E \to M$. Show that a Hermitian metric gives an isomorphism $\overline{E} \xrightarrow{\cong} E^*$. Show that

$$c_i(\overline{E}) = (-1)^i c_i(E). \tag{7.49}$$

Some computations

(7.50) *The total Chern class of complex projective space.* Consider $\mathbb{CP}^n = \mathbb{P}(\mathbb{C}^{n+1})$. As usual we let $y = c_1(S^*)$ for the tautological line bundle $S \to \mathbb{CP}^n$.

Proposition 7.51. *The total Chern class of $\mathbb{CP}^n$ is*

$$c(\mathbb{CP}^n) = (1 + y)^{n+1}. \tag{7.52}$$

This is to be interpreted in the truncated polynomial ring

$$H^\bullet(\mathbb{CP}^n) \cong \mathbb{Z}[y]/(y^{n+1}). \tag{7.53}$$

Proof. We use the exact sequence

$$0 \longrightarrow S \longrightarrow \underline{\mathbb{C}^{n+1}} \longrightarrow Q \longrightarrow 0 \tag{7.54}$$

of vector bundles over $\mathbb{CP}^n$ and the fact (Exercise 7.41) that the Chern classes of a trivial bundle vanish to deduce

$$c(S)c(Q) = 1. \tag{7.55}$$

It follows that

$$c(Q) = \frac{1}{1-y} = 1 + y + \dots + y^n. \tag{7.56}$$

There is a canonical isomorphism

$$TCP^n \cong \operatorname{Hom}(S,Q) \cong Q \otimes S^*, \tag{7.57}$$

which was sketched in lecture and is left as a very worthwhile exercise. Using the splitting principle we write (formally, or precisely up on the flag bundle of Q) $Q = L_1 \oplus \dots \oplus L_n$ and so

$$Q \otimes S^* \cong L_1 \otimes S^* \;\oplus\; \cdots \;\oplus\; L_n \otimes S^*. \tag{7.58}$$

Let $x_j = c_1(L_j)$ be the (formal) *Chern roots* of Q. Then

$$\begin{aligned} c(\mathbb{CP}^n) = c(Q \otimes S^*) &= \prod_{j=1}^{n}(1 + x_i + y) \\ &= \sum_{j=0}^{n} c_j(Q)(1+y)^{n-j} \\ &= \sum_{j=0}^{n} y^j(1+y)^{n-j} \\ &= (1+y)^{n+1} - y^{n+1} \\ &= (1+y)^{n+1}. \end{aligned} \tag{7.59}$$

□

(7.60) *The L-polynomial.* Any symmetric polynomial in $x_1, \dots, x_k$ defines a polynomial in the Chern classes of $E \to M$. So as not to fix the rank or the dimension of the base, we encode these characteristic classes by formal power series in a variable x. For example,

$$L = \frac{x}{\tanh x} \tag{7.61}$$

is Hirzebruch's "L-polynomial", introduced in his classic book [Hir], which explains in more detail the yoga for dealing with characteristic classes by "multiplicative sequences". In this case the L-polynomial is actually a power series in x^2, not just in x. This means that L is a characteristic class of *real* vector bundles, as we will see later.

To illustrate, let's write the L-polynomial for a rank two complex vector bundle $E \to M$ where M has dimension four. Let the formal Chern roots of E be x_1, x_2. First, we expand

$$\frac{x}{\tanh x} = \frac{x \cosh x}{\sinh x} = \frac{x(1 + x^2/2! + \dots)}{x + x^3/6! + \dots} = 1 + \frac{x^2}{3} + \dots. \tag{7.62}$$

Thus

$$L = (1 + \frac{x_1^2}{3})(1 + \frac{x_2^2}{3}) = 1 + \frac{x_1^2 + x_2^2}{3} = 1 + \frac{(x_1 + x_2)^2 - 2x_1x_2}{3} = 1 + \frac{c_1^2 - 2c_2}{3}. \tag{7.63}$$

For example, for $M = \mathbb{CP}^2$ we computed in Proposition 7.51 that $c_1(\mathbb{CP}^2) = 3y$ and $c_2(\mathbb{CP}^2) = 3y^2$, so

$$L(\mathbb{CP}^2) = 1 + \frac{9y^2 - 6y^2}{3} = 1 + y^2. \tag{7.64}$$

The pairing with the fundamental class $[\mathbb{CP}^2] \in H_4(\mathbb{CP}^2)$ gives 1.

Exercise 7.65. Compute the L-polynomial up to degree 8 for any vector bundle of any rank.

Exercise 7.66. Prove that $\langle L(\mathbb{CP}^n), [\mathbb{CP}^n] \rangle = 1$ for all n.

Exercise 7.67. Recall the K3 surface $X \subset \mathbb{CP}^3$ defined by a homogeneous quartic polynomial. Compute the total Chern class of X. (There are some hints at the end of Chapter IV of [BT].)

Real vector bundles

We can leverage the Chern classes of a complex bundle to define *Pontrjagin classes* of a real vector bundle. Let $V \to M$ be a real vector bundle of rank k. Define its *complexification* $E = V \otimes_{\mathbb{R}} \mathbb{C} \to M$. Since $E \cong \overline{E}$ we deduce from Exercise 7.48 that the odd Chern classes $c_{2h+1}(E)$ are torsion of order 2. We use the even Chern classes to define the Pontrjagin classes of V:

$$p_i(V) = (-1)^i c_{2i}(V \otimes_{\mathbb{R}} \mathbb{C}) \ \in H^{4i}(M; \mathbb{Z}). \tag{7.68}$$

The sign convention is not totally standard, but this is more prevalent. The formal Chern roots of E come in opposite pairs $x, -x$, and taking just one element in each pair we have the formal expression

$$p(V) = \prod_j (1 + x_j^2) \tag{7.69}$$

which we usually write simply as $\prod(1 + x^2)$.

Exercise 7.70. Prove that the total Pontrjagin class of a sphere is trivial: $p(S^n) = 1$.

Exercise 7.71. Prove that both Chern classes and Pontrjagin classes are *stable* in the sense that they don't change under stabilization of vector bundles (by adding trivial bundles).

Characteristic classes of principal G-bundles

There is much to say about the computation of the cohomology of BG. If G is a finite group, this is reduces to group cohomology à la Eilenberg-MacLane. For a connected compact Lie group G one can use a maximal torus of G to formulate a generalized splitting principle and make computations in terms of Lie theory. The beautiful classic papers of Borel and Hirzebruch [BH1, BH2, BH3] are a fount of useful information derived from this strategy. We just quote one general theorem in this area which determines the *real* cohomology in terms of invariant polynomials on the Lie algebra. For simplicity I state it in terms of compact Lie groups.

Theorem 7.72. *Let G be a compact Lie group and $\mathfrak{g}$ its Lie algebra. Then there is a canonical isomorphism of $H^\bullet(BG;\mathbb{R})$ with the ring of Ad-invariant polynomials on $\mathfrak{g}$, where a polynomial of degree i gives a cohomology class of degree $2i$.*

In particular, this is a polynomial ring.

As a special case, we have the following.

Theorem 7.73. *The real cohomology of the classifying space of the orthogonal group is a polynomial ring on the Pontrjagin classes:*

$$H^\bullet(BO(k);\mathbb{R}) \cong \mathbb{R}[p_1,\dots,p_i], \qquad \deg p_i = 4i, \tag{7.74}$$

where i is the greatest positive integer such that $2i \le k$.

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