Lecture 8: More characteristic classes and the Thom isomorphism

We begin this lecture by carrying out a few of the exercises in Lecture 1. We take advantage of the fact that the Chern classes are *stable characteristic classes*, which you proved in Exercise 7.71 from the Whitney sum formula. We also give a few more computations. Then we turn to the *Euler class*, which is decidedly *unstable*. We approach it via the *Thom class* of an oriented real vector bundle. We introduce the *Thom complex* of a real vector bundle. This construction plays an important role in the course.

In lecture I did not prove the existence of the Thom class of an oriented real vector bundle. Here I do so—and directly prove the basic *Thom isomorphism theorem*—when the base is a CW complex. It follows from Morse theory that a smooth manifold is a CW complex, something we may prove later in the course. I need to assume the theorem that a vector bundle over a contractible base (in this case a closed ball) is trivializable. For a smooth bundle this follows immediately from Proposition 7.3.

The book [BT] is an excellent reference for this lecture, especially Chapter IV.

Elementary computations with Chern classes

(8.1) *Stable tangent bundle of projective space.* We begin with a stronger version of Proposition 7.51. Recall the exact sequence (7.54) of vector bundles over $\mathbb{CP}^n$.

Proposition 8.2. *The tangent bundle of $\mathbb{CP}^n$ is stably equivalent to $(S^*)^{\oplus(n+1)}$.*

Proof. The exact sequence (7.54) shows that $Q \oplus S \cong \underline{\mathbb{C}^{n+1}}$. Tensor with S^* and use (7.57) and the fact that $S \otimes S^*$ is trivializable to deduce that

$$T(CP^n) \oplus \underline{\mathbb{C}} \cong (S^*)^{\oplus(n+1)}. \tag{8.3}$$

□

Exercise 8.4. Construct a canonical orientation of a complex manifold M. This reduces to a canonical orientation of a (finite dimensional) complex vector space. You may want to review the discussion of orientations in Lecture 1.

(8.5) *The L-genus of projective space.* Recall that Hirzebruch's L-class is defined by the power series (7.61). Namely, if a vector bundle $E \to M$ has formal Chern roots $x_1, x_2, \dots, x_k$, then

$$L(E) = \prod_{j=1}^{k} \frac{x_j}{\tanh x_j}. \tag{8.6}$$

Each x_j has degree 2, and the term of order $2i$, which is computed by a finite computation, is a symmetric polynomial of degree i in the variables x_j. It is then a polynomial in the elementary symmetric polynomials $c_1, \dots, c_k$, which are the Chern classes of E. The *L-genus* is the pairing of the L-class (8.6) of the tangent bundle of a complex manifold M with its fundamental class $[M]$.

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Remark 8.7 (L-class of a *real* vector bundle). Since $x/\tanh x$ is a power series in x^2, it follows that the L-class is a power series in the Pontrjagin classes of the underlying real vector bundle. So the L-class is defined for a real vector bundle, and the L-genus for a compact oriented real manifold.

Proposition 8.8. *The L-genus of $\mathbb{CP}^n$ satisfies*

$$\langle L(\mathbb{CP}^n), [\mathbb{CP}^n]\rangle = 1 \tag{8.9}$$

if n is even.

Here $[CP^n] \in H_{2n}(\mathbb{CP}^n)$ is the fundamental class, defined using the canonical orientation of a complex manifold. Also, $L(\mathbb{CP}^n)$ is the L-polynomial of the tangent bundle. The degree of each term in the L-class is divisible by 4, so the left hand side of (8.9) vanishes for degree reasons if n is odd.

Proof. By Proposition 8.2 and the fact that the Chern classes are stable, we can replace $T(\mathbb{CP}^n)$ by $(S^*)^{\oplus(n+1)}$. The Chern roots of the latter are not formal—it *is* a sum of line bundles—and each is equal to the positive generator $y \in H_2(\mathbb{CP}^n)$. Since $\langle y^n, [\mathbb{CP}^n]\rangle = 1$, we conclude that the left hand side of (8.9) is the coefficient of y^n in

$$L\big((S^*)^{\oplus(n+1)}\big) = \left(\frac{y}{\tanh y}\right)^{n+1}. \tag{8.10}$$

By the Cauchy integral formula, this equals

$$\frac{1}{2\pi i}\int \frac{dy}{y^{n+1}}\left(\frac{y}{\tanh y}\right)^{n+1}, \tag{8.11}$$

where the contour integral is taken over a small circle with center the origin of the complex y-line; the orientation of the circle is counterclockwise. Substitute $z = \tanh y$, and so $dz/(1-z^2) = dy$. Then (8.11) equals

$$\frac{1}{2\pi i}\int \frac{dz}{(1-z^2)z^{n+1}} = \frac{1}{2\pi i}\int dz\, \frac{1+z^2+z^4+\dots}{z^{n+1}} = \begin{cases} 1, & n \text{ even;} \\ 0, & n \text{ odd.} \end{cases} \tag{8.12}$$

□

(8.13) *The Euler characteristic and top Chern class.* We prove the following result at the end of the lecture.

Theorem 8.14. *Let M be a compact complex manifold of dimension n. Then its Euler characteristic is*

$$\chi(M) = \langle c_n(M), [M]\rangle. \tag{8.15}$$

(8.16) *The genus of a plane curve.* Let C be a complex curve, which means a complex manifold of dimension 1. The underlying real manifold is oriented and has dimension 2. Assume that C is compact and connected. Then, say by the classification of surfaces, we deduce that

$$H_0(C) \cong \mathbb{Z}, \qquad \dim H_1(C) = 2g(C), \qquad H_2(C) \cong \mathbb{Z} \tag{8.17}$$

for some integer $g(C) \in \mathbb{Z}^{\geq 0}$ called the *genus* of C. The Euler characteristic is

$$\chi(C) = 2 - 2g(C). \tag{8.18}$$

A *plane curve* is a submanifold $C \subset \mathbb{CP}^2$, and it is cut out by a homogeneous polynomial of degree d for some $d \in \mathbb{Z}^{\geq 1}$. An extension of Exercise 6.9 shows that these polynomials are sections of $(S^*)^{\otimes d} \to \mathbb{CP}^2$, which is the d^{th} power of the hyperplane bundle (and is often denoted $\mathcal{O}(d) \to \mathbb{CP}^2$). We simply assume that C is cut out as the zeros of a transverse section of that bundle.

Proposition 8.19. *The Euler characteristic of a smooth plane curve $C \subset \mathbb{CP}^2$ of degree d is*

$$\chi(C) = \frac{(d-1)(d-2)}{2}. \tag{8.20}$$

Proof. The normal bundle to $C \subset \mathbb{CP}^2$ is canonically the restriction of $(S^*)^{\otimes d} \to \mathbb{CP}^2$ to C, and so we have the exact sequence (see (2.30))

$$0 \longrightarrow TC \longrightarrow T(\mathbb{CP}^2)\big|_C \longrightarrow (S^*)^{\otimes d} \longrightarrow 0. \tag{8.21}$$

Since this sequence splits (in C^∞, not necessarily holomorphically), the Whitney sum formula implies that

$$c(C) = \frac{(1+y)^3}{1+dy} = \frac{1+3y}{1+dy} = 1 + (3-d)y. \tag{8.22}$$

Here we use Proposition 7.51 to obtain the total Chern class of $\mathbb{CP}^2$. Proposition 7.19 together with Corollary 7.22 compute the total Chern class of $(S^*)^{\otimes d}$.

Next, we claim $\langle y, [C] \rangle = d$. One proof is that evaluation of y on a curve in $\mathbb{CP}^2$ is the intersection number of that curve with a generic line, which is the degree of the curve (which is the number of solutions to a polynomial equation of degree d in the complex numbers). Hence by Theorem 8.14 we have

$$\chi(C) = \langle c_1(C), [C] \rangle = (3-d)d, \tag{8.23}$$

to which we apply (8.18) to deduce (8.20). ∎

(8.24) *The Euler characteristic of the $K3$ surface.* A similar computation gives the Euler characteristic of a quartic surface $M \subset \mathbb{CP}^3$ as 24, a fact used in **(5.60)**. Do this computation! You will find

$$c(M) = \frac{(1+y)^4}{1+4y} = 1 + 6y^2. \tag{8.25}$$

Notice that this also proves that $c_1(M) = 0$.

Exercise 8.26. Prove that a degree $(n+1)$ hypersurface $M \subset \mathbb{CP}^n$ has vanishing first Chern class. Such a complex manifold is called *Calabi-Yau.* (In fact, the stronger statement that the complex determinant line bundle $\operatorname{Det} TM \to M$ is holomorphically trivial is true.)

The Thom isomorphism

(8.27) *Relative cell complexes.* Let X be a topological space and $A \subset X$ a closed subspace. We write (X, A) for this pair of spaces. A *cell structure on* (X, A) is a cell decomposition of $X \setminus A$. This means that X is obtained from A by successively attaching 0-cells, 1-cells, etc., starting from the space A. The *relative chain complex* of the cell structure is defined analogously to the absolute chain complex (7.8). Cochain complexes which compute cohomology are obtained algebraically from the chain complex, as in (7.9).

Example: The pair (S^k, ∞) has a cell structure with a single k-cell e^k. The chain complex is

$$\cdots \longleftarrow 0 \longleftarrow \mathbb{Z}\{e^k\} \longleftarrow 0 \longleftarrow \cdots \tag{8.28}$$

where the nonzero entry is in degree k.

If the pair (X, A) satisfies some reasonable point-set conditions, which are satisfied if it admits a cell structure, then the homology/cohomology of the pair are isomorphic (by excision) to the homology/cohomology of the quotient X/A relative to the basepoint A/A.

(8.29) *The cohomology of a real vector space.* Let $\mathbb{V}$ be a real vector space of dimension k. Of course, $\mathbb{V}$ deformation retracts to the origin in $\mathbb{V}$ by scaling, so the cohomology of $\mathbb{V}$ is that of a point. But there is more interesting *relative cohomology*, or *cohomology with compact support.* Suppose $\mathbb{V}$ has an inner product. Let $C_r(\mathbb{V})$ denote the complement of the open ball of radius r about the origin. The pair $\bigl(\mathbb{V}, C_r(\mathbb{V})\bigr)$ has a cell structure with a single k-cell e^k. The chain complex of the pair is then (8.28), and taking $\operatorname{Hom}(-, \mathbb{Z})$ we deduce

$$H^q(\mathbb{V}, C_r(\mathbb{V}); \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & q = k; \\ 0, & \text{otherwise.} \end{cases} \tag{8.30}$$

The result is, of course, independent of the radius (by the excision property of cohomology). Notice that the quotient $\mathbb{V}/C_r(\mathbb{V})$ is homeomorphic to a k-sphere with a basepoint, so (8.30) is consistent with the example (8.28) above.

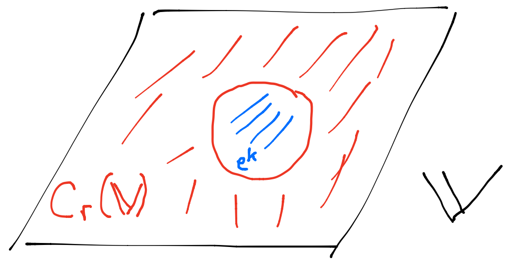

FIGURE 18. The pair $(\mathbb{V}, C_r(\mathbb{V}))$

The isomorphism in (8.30) is determined only up to sign, or rather depends on a precise choice of k-cell e^k. That is, there are two distinguished generators of this cohomology group. These generators form a $\mathbb{Z}/2\mathbb{Z}$-torsor canonically attached to the vector space $\mathbb{V}$.

Lemma 8.31. *This torsor canonically is $\mathfrak{o}(\mathbb{V})$, as defined in (2.7).*

Proof. Recall that the k-cell is defined by the attaching map, which is a homeomorphism (we can take it to be a diffeomorphism) $f\colon S^{k-1} \to S_r(\mathbb{V})$, where $S^{k-1} = \partial D^k$ is the standard $(k-1)$-sphere and $S_r(\mathbb{V})$ is the sphere of radius r in $\mathbb{V}$ centered at the origin. Given an orientation $o \in \mathfrak{o}(\mathbb{V})$ of $\mathbb{V}$, there is an induced orientation of $S_r(\mathbb{V})$ and so a distinguished homotopy class of orientation-preserving diffeomorphisms f. This singles out a generator in (8.30) and proves the lemma.

Here's an alternative proof. Let $a \in H^k(\mathbb{V}, C_r(\mathbb{V}); \mathbb{Z})$ be a generator. Its image in $H^k(\mathbb{V}, C_r(\mathbb{V}); \mathbb{R})$ can, by the de Rham theorem, be represented by a k-form ω_a on $\mathbb{V}$ whose support is contained in the open ball $B_r(\mathbb{V})$ of radius r centered at the origin. There is a unique orientation of $\mathbb{V}$—a point $o \in \mathfrak{o}(\mathbb{V})$—such that

$$\int_{(\mathbb{V},o)} \omega_a = 1. \tag{8.32}$$

(The integral in the opposite orientation is -1.) The isomorphism of the lemma maps $a \mapsto o$. ☐

(8.33) *Thom classes.* Let $\pi\colon V \to M$ be a real vector bundle of rank k. Assume it carries an

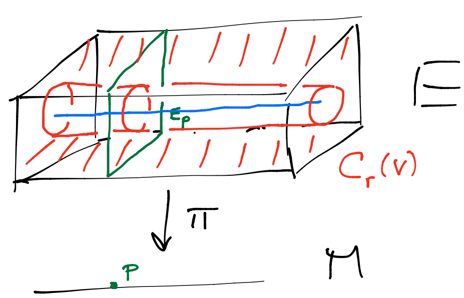

FIGURE 19. The pair $(V, C_r(V))$

inner product. Consider the pair $(V, C_r(V))$, where $C_r(V) \subset V$ is the set of all vectors of norm at least r. Recall also the notion of an *orientation* of a real vector bundle (Definition 2.14), which is a section of the double cover $\mathfrak{o}(V) \to M$.

Definition 8.34. A *Thom class* for $\pi\colon V \to M$ is a cohomology class $U_V \in H^k\bigl(V, C_r(V); \mathbb{Z}\bigr)$ such that $i_p^*U_V$ is a generator of $H^k\bigl(V_p, C_r(V_p); \mathbb{Z}\bigr)$ for all $p \in M$.

It is clear that a Thom class induces an orientation of $V \to M$. The converse is also true.

Proposition 8.35. *Let $\pi\colon V \to M$ be an oriented real vector bundle. Then there exists a Thom class $U_V \in H^k\bigl(V, C_r(V); \mathbb{Z}\bigr)$.*

We sketch a proof below.

(8.36) *Thom isomorphism theorem.* Given the Thom class, we apply the Leray-Hirsch theorem (Theorem 7.25) to the pair $\bigl(V, C_r(V)\bigr)$, which is a fiber bundle over M with typical fiber $\bigl(\mathbb{V}, C_r(\mathbb{V})\bigr)$.

Corollary 8.37. *The integral cohomology of $\bigl(V, C_r(V)\bigr)$ is a free $H^\bullet(M;\mathbb{Z})$-module with a single generator U_V.*

Put differently, the map

$$H^\bullet(M;\mathbb{Z}) \xrightarrow{U_V \smile \pi^*(-)} H^{k+\bullet}\bigl(V, C_r(V); \mathbb{Z}\bigr) \tag{8.38}$$

is an isomorphism of abelian groups. This map—the *Thom isomorphism*— is pullback from the base followed by multiplication by the Thom class.

It follows immediately from (7.39) that there is a *unique* Thom class compatible with a given orientation.

Proof of Proposition 8.35. As stated earlier a smooth manifold M admits a CW structure, which means it is constructed by iteratively attaching cells, starting with the empty set. Let $\{e_\alpha\}_{\alpha\in A}$ denote the set of cells. For convenience, denote $\mathcal{V} = \bigl(V, C_r(V)\bigr)$. We prove that $\mathcal{V}$ has a cell decomposition with cells $\{f_\alpha\}_{\alpha\in A}$ indexed by the same set A, and $\dim f_\alpha = \dim e_\alpha + k$. Furthermore, the cellular chain complex of $\mathcal{V}$ is the shift of the cellular chain complex of M by k units to the right. The same is then true of cochain complexes derived from these chain complexes. In particular, there is an *isomorphism*

$$H^0(M;\mathbb{Z}) \xrightarrow{\cong} H^k(\mathcal{V};\mathbb{Z}). \tag{8.39}$$

The image of $1 \in H^0(M;\mathbb{Z})$ is the desired Thom class U_V. A bit more argument (using properties of the cup product) shows that the map (8.39) is the map (8.38), and so this gives a proof of the Thom isomorphism.

For each cell e_α there is a continuous map $\Phi_\alpha\colon D_\alpha \to M$, where D_α is a closed ball. Its restriction to the open ball is a homeomorphism onto its image $e_\alpha \subset M$ and M is the disjoint union of these images. The pullback $\Phi_\alpha^*V \to D_\alpha$ is trivializable. Fix a trivialization. This induces a homeomorphism $\Phi_\alpha^*\mathcal{V} \approx D_\alpha \times \bigl(\mathbb{V}, C_r(\mathbb{V})\bigr) \approx \bigl(D_\alpha \times \mathbb{V}, D_\alpha \times C_r(\mathbb{V})\bigr)$. This pair has a cell structure with a single cell, which is the Cartesian product of D_α and the k-cell described in **(8.29)**. Now the orientation of $V \to M$ induces an orientation of $\Phi_\alpha^* V \to D_\alpha$, and so picks out the k-cell e^k, as in the proof of Lemma 8.31. Define $f_\alpha = e_\alpha \times e^k$. These cells make up a cell decomposition of $\mathcal{V}$. Furthermore, $\partial(f_\alpha) = \partial(e_\alpha) \times e^k$, since $\partial(e^k) = 0$. ☐

Possession of a cell structure for a space is far more valuable than knowledge of its homology or cohomology; the latter can be derived from the former. So you should keep the picture of the cell structure used in the proof.

Exercise 8.40. Think through the argument in the proof without the assumption that $V \to M$ is oriented. Now there is a sign ambiguity in the definition of e^k. Can you see how to deal with that and what kind of statement you can make?

(8.41) *The Thom complex.* As mentioned above, the cohomology of a pair (X, A) is the reduced cohomology of the quotient space X/A with basepoint A/A, at least if certain point-set conditions are satisfied. The quotient $V/C_r(V)$ is called the *Thom complex* of $V \to M$ and is denoted M^V. Figure 19 provides a convenient illustration: imagine the red region collapsed to a point. Note there is no projection from M^V to M: there is no basepoint in M and no distinguished image of the basepoint in M^V. Also, note that the zero section (depicted in blue) induces an inclusion

$$i\colon M \longrightarrow M^V. \tag{8.42}$$

Exercise 8.43. What is the Thom complex of the trivial vector bundle $M \times \mathbb{R}^k \to M$?

Exercise 8.44. There is a nontrivial real line bundle $V \to S^1$, often called the *Möbius bundle*. What is its Thom complex?

The Euler class

Definition 8.45. Let $\pi\colon V \to M$ be an oriented real vector bundle of rank k with Thom class U_V. The *Euler class* $e(V) \in H^k(M; \mathbb{Z})$ is defined as

$$e(V) = i^*(U_V), \tag{8.46}$$

where i is the zero section (8.42).

Proposition 8.47. *If $\pi\colon V \to M$ is an oriented real vector bundle which admits a nonzero section, then $e(V) = 0$.*

Proof. First, if M is compact, then the norm of the section $s\colon M \to V$ achieves a minimum, and taking r less than that minimum produces a Thom class whose pullback $s^*(U_V)$ by the section vanishes. Since the section is homotopic to the zero section i, the result follows. If M is not compact and the norm of the section does not achieve a minimum, then let $r\colon M \to \mathbb{R}^{>0}$ be a variable function whose value at $p \in M$ is less than $\|s(p)\|$. ☐

Proposition 8.48. *Let $L\to M$ be a complex line bundle and $L_{\mathbb{R}}\to M$ the underlying oriented rank 2 real vector bundle. Then*

$$e(L_{\mathbb{R}})=c_1(L)\;\in H^2(M;\mathbb{Z}). \tag{8.49}$$

Proof. Consider the fiber bundle

$$\mathbb{P}(L^*\oplus\underline{\mathbb{C}})\longrightarrow M \tag{8.50}$$

with typical fiber a projective line, or 2-sphere. The dual tautological line bundle $S^*\to\mathbb{P}(L^*\oplus\underline{\mathbb{C}})$ has a first Chern class $\widetilde{U}=c_1(S^*)$ which restricts on each fiber to the positive generator of the cohomology of the projective line. There are two canonical sections of (8.50). The first $j\colon M\to\mathbb{P}(L^*\oplus\underline{\mathbb{C}})$ maps each point of M to the trivial line $\mathbb{C}$; the second $i\colon M\to\mathbb{P}(L^*\oplus\underline{\mathbb{C}})$ maps each point to the line L^*. Note that the complement of the image of j may be identified with L: every line in $L^*_p\oplus\mathbb{C}$ not equal to $\mathbb{C}$ is the graph of a linear functional $L^*_p\to\mathbb{C}$, which can be identified with an element of L_p. Now $j^*(S^*)\to M$ is the trivial line bundle and $i^*(S^*)\to M$ is canonically the line bundle $L\to M$. It follows that $\widetilde{U}$ lifts to a relative class[1] $U\in H^2\bigl(\mathbb{P}(L^*\oplus\mathbb{C}),j(M);\mathbb{Z}\bigr)$, the Thom class of $L_{\mathbb{R}}\to M$. Then

$$e(L_{\mathbb{R}})=i^*(U)=i^*\bigl(c_1(S^*)\bigr)=c_1(i^*S^*)=c_1(L). \tag{8.51}$$

∎

I leave the proof of the next assertion as an exercise.

Proposition 8.52. *Let $V_1,V_2\to M$ be oriented real vector bundles. Then*

$$e(V_1\oplus V_2)=e(V_1)e(V_2). \tag{8.53}$$

Exercise 8.54. Prove Proposition 8.52.

Corollary 8.55. *Let $E\to M$ be a rank k complex vector bundle. Then*

$$c_k(E)=e(E_{\mathbb{R}}). \tag{8.56}$$

Exercise 8.57. Prove Corollary 8.55. Use Proposition 8.48 and the Whitney sum formulas Proposition 8.52 and Exercise 7.46.

[1] Use excision to push to the pair $\bigl(L,C_r(L)\bigr)$ considered above.

(8.58) *The Euler characteristic.*

Proposition 8.59. *Let M be a compact oriented n-manifold. Then its Euler characteristic is*

$$\chi(M) = \langle e(M), [M]\rangle. \tag{8.60}$$

Proof. I will sketch a proof which relies on a relative version of the de Rham theorem: If M is a smooth manifold and $A \subset M$ a closed subset, then the de Rham complex of smooth differential forms on M supported in $M \setminus A$ computes the real relative cohomology $H^\bullet(M, A; \mathbb{R})$. We also use the fact that the integer on the right hand side of (8.60) can be computed from the pairing of $e_{\mathbb{R}}(M) \in H^n(M; \mathbb{R})$ with the fundamental class, and that—again, by the de Rham theorem—if ω is a closed n-form which represents $e_{\mathbb{R}}(M)$, then that pairing is $\int_M \omega$. Here $e_{\mathbb{R}}$ is the image of the (integer) Euler class in real cohomology by extension of scalars $\mathbb{Z} \to \mathbb{R}$.

Now for the proof: Recall that the Euler characteristic of M is the self-intersection number of the diagonal in $M \times M$, or equivalently the self-intersection number of the zero section of $TM \to M$. It is computed by choosing a section $\xi\colon M \to TM$—that is, a vector field—which is transverse to the zero section. The intersection number is the sum of local intersection numbers at the zeros of ξ, and each local intersection number is ± 1. Choose a local framing of M on a neighborhood N_i about each zero $p_i \in M$ of ξ—that is, a local trivialization of $TM \to M$ restricted to N_i. By transversality and the inverse function theorem we can cut down the neighborhoods N_i so that $\xi\colon N_i \to \mathbb{R}^n$ (relative to the trivialization) is a diffeomorphism onto its image. Fix a Riemannian metric on M and suppose $\|\xi\| > r$ on the complement of the union of the N_i. Let $\omega \in \Omega^n(TM)$ be a closed differential form with support in $TM \setminus C_r(TM)$ which represents the real Thom class $U_{M;\mathbb{R}} \in H^n\bigl(TM, C_r(TM); \mathbb{R}\bigr)$. Since the section $\xi\colon M \to TM$ is homotopic to the zero section i, we have

$$\chi(M) = \int_M \xi^*\omega. \tag{8.61}$$

Because of the support condition on ω, the integral is equal to the sum of integrals over the neighborhoods N_i. Under the local trivialization ω represents the integral generator of $H^n\bigl(\mathbb{R}^n, C_r(\mathbb{R}^n); \mathbb{R}\bigr)$—this by the definition (Definition 8.34) of the Thom class—and so $\int_{N_i} \xi^*\omega = \pm 1$. I leave you to check that the sign is the local intersection number. □

References

[BT] Raoul Bott and Loring W. Tu, *Differential forms in algebraic topology*, Graduate Texts in Mathematics, vol. 82, Springer-Verlag, New York, 1982.