

Lecture 9: Tangential structures

We begin with some examples of *tangential structures* on a smooth manifold. In fact, despite the name—which is appropriate to our application to bordism—these are structures on arbitrary real vector bundles over topological spaces; the name comes from the application to the tangent bundle of a smooth manifold. Common examples may be phrased as a reduction of structure group of the tangent bundle. The general definition allows for more exotic possibilities. We move from a geometric description—and an extensive discussion of orientations and spin structures—to a more abstract topological definition. Note there are both stable and unstable tangential structures. The stable version is what is usually studied in classical bordism theory; the unstable version is relevant to the modern developments, such as the cobordism hypothesis.

I suggest you think through this lecture first for a single tangential structure: orientations.

Orientations revisited

(9.1) Existence and uniqueness. Let $V \rightarrow M$ be a real vector bundle of rank n over a manifold M . (In this whole discussion you can replace a manifold by a metrizable topological space.) In Lecture 2 we constructed an associated double cover $\mathfrak{o}(V) \rightarrow M$, the orientation double cover of the vector bundle $V \rightarrow M$. An *orientation* of the vector bundle is a section of $\mathfrak{o}(V) \rightarrow M$. There is an existence and uniqueness exercise.

Exercise 9.2. The obstruction to existence is the isomorphism class of the orientation double cover: orientations exists if and only if $\mathfrak{o}(V) \rightarrow M$ is trivializable. Show that this isomorphism class is an element of $H^1(M; \mathbb{Z}/2\mathbb{Z})$. If this class vanishes, show that the set of orientations is a torsor for $H^0(M; \mathbb{Z}/2\mathbb{Z})$, the group of locally constant maps $M \rightarrow \mathbb{Z}/2\mathbb{Z}$. Of course, this can be identified with the set of maps $\pi_0 M \rightarrow \mathbb{Z}/2\mathbb{Z}$.

Remark 9.3. The isomorphism class of $\mathfrak{o}(V) \rightarrow M$ is the *first Stiefel-Whitney class* $w_1(V) \in H^1(M; \mathbb{Z}/2\mathbb{Z})$. The Stiefel-Whitney classes are characteristic classes of real vector bundles. They live in the cohomology algebra $H^*(BO; \mathbb{Z}/2\mathbb{Z})$.

(9.4) Recollection of frame bundles. Recall from (6.38) that to the vector bundle $V \rightarrow M$ is associated a principal $GL_n(\mathbb{R})$ -bundle $\mathcal{B}(V) \rightarrow M$ of bases, often called the *frame bundle* of $V \rightarrow M$. If we endow $V \rightarrow M$ with a metric, then we can take orthonormal frames and so construct a principal $O(n)$ -bundle of frames $\mathcal{B}_O(V) \rightarrow M$.

Exercise 9.5. Recall the determinant homomorphism

$$(9.6) \quad GL_n(\mathbb{R}) \xrightarrow{\det} \mathbb{R}^{\neq 0}.$$

Let $GL_n^+(\mathbb{R}) \subset GL_n(\mathbb{R})$ denote the subgroup $\det^{-1}(\mathbb{R}^{>0})$. Then $GL_n^+(\mathbb{R})$ acts freely on $\mathcal{B}(V)$. Identify the quotient with $\mathfrak{o}(V)$. What is the analogous statement for orthonormal frames?

(9.7) Reduction of structure group. Let H, G be Lie groups and $\rho: H \rightarrow G$ a homomorphism. (For the discussion of orientations this is the inclusion $GL_n^+(\mathbb{R}) \hookrightarrow GL_n(\mathbb{R})$.)

Definition 9.8.

- (i) Let $Q \rightarrow M$ be a principal H -bundle. The *associated* principal G -bundle $Q_\rho \rightarrow M$ is the quotient

$$(9.9) \quad Q_\rho = (Q \times G)/H,$$

where H acts freely on the right of $Q \times G$ by

$$(9.10) \quad (q, g) \cdot h = (q \cdot h, \rho(h)^{-1}g), \quad q \in Q, \quad g \in G, \quad h \in H.$$

- (ii) Let $P \rightarrow M$ be a principal G -bundle. Then a *reduction to H* is a pair (Q, θ) consisting of a principal H -bundle $Q \rightarrow M$ and an isomorphism

$$(9.11) \quad \begin{array}{ccc} Q_\rho & \xrightarrow{\theta} & P \\ & \searrow & \swarrow \\ & M & \end{array}$$

of principal G -bundles.

Exercise 9.12.

- (i) What is the G -action on Q_ρ ?
 (ii) Define an isomorphism of reductions.
 (iii) Suppose $V \rightarrow M$ is a real vector bundle of rank n with metric. Let $\rho: O(n) \hookrightarrow GL_n(\mathbb{R})$ be the inclusion. What is $\mathcal{B}_O(V)_\rho$?
 (iv) Assume that ρ is an inclusion. Show that $Q \subset Q_\rho$ and, using θ , we can identify a reduction to H as a sub-fiber bundle $Q \subset P$. Assuming that H is a closed Lie subgroup, show that reductions are in 1:1 correspondence with sections of the G/H bundle $P/H \rightarrow M$.

(9.13) Orientations as reductions of structure group. The definitions conspire to show that an orientation of a real rank n vector bundle $V \rightarrow M$ is a reduction of structure group of $\mathcal{B}(V) \rightarrow M$ to the group $GL_n^+(\mathbb{R}) \hookrightarrow GL_n(\mathbb{R})$. In particular, this follows by combining Exercise 9.5 and Exercise 9.12(iv) together with the definition of an orientation.

Spin structures

(9.14) The spin group. Let $SO(n) \subset O(n)$ be the subgroup of orthogonal matrices of determinant one. In low dimensions these are familiar groups. The group $SO(1)$ is trivial: it just has the identity

element. The group $SO(2)$ is the group of rotations in the oriented plane \mathbb{R}^2 , or more concretely the group of matrices

$$(9.15) \quad \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad \theta \in \mathbb{R},$$

which has the topology of the circle. Its fundamental group is infinite cyclic. The manifold underlying the group $SO(3)$ is diffeomorphic to $\mathbb{R}P^3$, so has fundamental group isomorphic to $\mathbb{Z}/2\mathbb{Z}$. (I gave an argument for this in a previous lecture.) In fact, $\pi_1 SO(n) \cong \mathbb{Z}/2\mathbb{Z}$ for all $n \geq 3$.

Exercise 9.16. Prove this as follows. The group $SO(n)$ acts transitively on the sphere S^{n-1} , and the stabilizer of a point is isomorphic to $SO(n-1)$. So there is a fiber bundle $SO(n) \rightarrow S^{n-1}$ with typical fiber $SO(n-1)$. (It is a *principal* bundle.) Use the long exact sequence of homotopy groups and induction to deduce the assertion.

Definition 9.17. The spin group $\text{Spin}(n)$ is the double cover group of $SO(n)$.

Thus $\text{Spin}(1) \cong \mathbb{Z}/2\mathbb{Z}$ is cyclic of order 2. The spin group $\text{Spin}(2)$ is abstractly isomorphic to the circle group: $\text{Spin}(2) \rightarrow SO(2)$ is the nontrivial double cover. The manifold underlying the group $\text{Spin}(3)$ is diffeomorphic to S^3 , and $\text{Spin}(n)$ is connected and simply connected (also called *1-connected*) for $n \geq 3$. There is an explicit realization of the spin group inside the *Clifford algebra*.

Remark 9.18. Definition 9.17 relies on a general construction in Lie groups. Namely, if G is a *connected* Lie group, $\pi: \tilde{G} \rightarrow G$ a covering space, and $\tilde{e} \in \pi^{-1}(e)$ a basepoint, then there is a unique Lie group structure on \tilde{G} such that \tilde{e} is the identity element and π is a group homomorphism. If you identify the covering space \tilde{G} with a space of homotopy classes of paths in G , then you might figure out how to define the multiplication. See [War] for details.

Definition 9.19. Let $V \rightarrow M$ be a real vector bundle of rank n with a metric. A *spin structure* on V is a reduction of structure group of the orthonormal frame bundle $\mathcal{B}_O(V) \rightarrow M$ along $\rho: \text{Spin}(n) \rightarrow O(n)$.

Here ρ is the projection $\text{Spin}(n) \rightarrow SO(n)$ followed by the inclusion $SO(n) \rightarrow O(n)$. So the reduction can be thought of in two steps: an orientation followed by a lift to the double cover.

Exercise 9.20. Make this explicit: construct an orientation of $V \rightarrow M$ from a spin structure on $V \rightarrow M$.

(9.21) *Existence for complex bundles.* We will not discuss the general existence problem here, but will instead restrict to important special case and give an example of non-existence.

(9.22) *The double cover of the unitary group.* The complex vector space \mathbb{C}^n has as its underlying real vector space \mathbb{R}^{2n} . Explicitly, to an n -tuple (z^1, \dots, z^n) of complex numbers we associate the $2n$ -tuple $(x^1, y^1, \dots, x^n, y^n)$ of real numbers, where $z^n = x^n + \sqrt{-1}y^n$. The real part of the standard hermitian metric on \mathbb{C}^n is the standard real inner product on \mathbb{R}^{2n} . So there is a homomorphism, which is an inclusion,

$$(9.23) \quad U(n) \longrightarrow O(2n)$$

of unitary transformations of \mathbb{C}^n into orthogonal transformations of \mathbb{R}^{2n} . In fact, the image lies in $SO(2n)$, which follows since complex linear transformations preserve the natural orientation of $\mathbb{C}^n = \mathbb{R}^{2n}$. (Alternatively, $U(n)$ is connected, so the image of (9.23) is a connected subgroup of $O(2n)$.) Define $\tilde{U}(n)$ to be the pullback Lie group

$$(9.24) \quad \begin{array}{ccc} \tilde{U}(n) & \dashrightarrow & \text{Spin}(2n) \\ \downarrow & & \downarrow \\ U(n) & \longrightarrow & SO(2n) \end{array}$$

It is the unique connected double cover of $U(n)$. (The fundamental group of $U(n)$ is infinite cyclic for all n .)

(9.25) Spin structures on a complex vector bundle. Let $E \rightarrow M$ be a rank n complex vector bundle. There is an underlying rank $2n$ real vector bundle $E_{\mathbb{R}} \rightarrow M$. As manifolds $E_{\mathbb{R}} = E$ and the projection map is the same. What is different is that we forget some of the structure of $E \rightarrow M$, namely we forget scalar multiplication by $\sqrt{-1}$ and only remember the real scalar multiplication. Choose a hermitian metric on $E \rightarrow M$, a contractible choice which carries no topological information. Then there is an associated principal $U(n)$ bundle $\mathcal{B}_U(E) \rightarrow M$, the *unitary* bundle of frames.

Definition 9.26. A *spin structure* on $E \rightarrow M$ is a reduction of $\mathcal{B}_U(E) \rightarrow M$ along $\tilde{U}(n) \rightarrow U(n)$.

This is not really a definition, but rather a consequence of (9.24).

Exercise 9.27. Recast Definition 9.26 as a theorem and prove that theorem. (Hint: ... is a spin structure on $E_{\mathbb{R}} \rightarrow M$...).

Proposition 9.28. Let $E \rightarrow M$ be a complex vector bundle which admits a spin structure. Then there exists $\tilde{c} \in H^2(M; \mathbb{Z})$ such that $2c = c_1(E)$.

Corollary 9.29. The manifold $\mathbb{C}\mathbb{P}^n$ does not admit a spin structure if n is even.

For according to Proposition 7.51 we have $c_1(\mathbb{C}\mathbb{P}^n) = (n+1)y$, where $y \in H^2(\mathbb{C}\mathbb{P}^n; \mathbb{Z})$ is the generator.

I outline the proof of Proposition 9.28 in the following exercise, the first part of which should have been part of the lecture on Chern classes.

Exercise 9.30.

- (i) Let $E \rightarrow M$ be a complex vector bundle of rank n . Define the associated *determinant line bundle* $\text{Det } E \rightarrow M$. One method is to use complex exterior algebra, analogous to Exercise 2.6(i) in the real case. Another is to use principal bundles and the determinant homomorphism $\det: GL_n(\mathbb{C}) \rightarrow \mathbb{C}^\times$.
- (ii) Use the splitting principle (7.42) to prove that $c_1(\text{Det } E) = c_1(E)$.

(iii) Construct the top homomorphism in the commutative diagram of Lie group homomorphisms

$$(9.31) \quad \begin{array}{ccc} \tilde{U}(n) & \dashrightarrow & \mathbb{T} \\ \downarrow & & \downarrow \\ U(n) & \xrightarrow{\det} & \mathbb{T} \end{array}$$

in which the right vertical arrow is the squaring map.

(iv) Recall Proposition 7.19 and complete the proof of Proposition 9.28.

(9.32) Double covers and uniqueness of spin structures. We work in the context of (9.7). Let $\rho: H \rightarrow G$ be a double cover of the Lie group G . We have in mind $G = SO(n)$ and $H = Spin(n)$. Let $P \rightarrow M$ be a principal G -bundle and (Q, θ) a reduction along ρ to a principal H -bundle. Suppose $R \rightarrow M$ is a double cover, which may be viewed as a principal $\mathbb{Z}/2\mathbb{Z}$ -bundle. Then we can construct a new reduction (Q', θ') by “acting on” the reduction (Q, θ) with the double cover $R \rightarrow M$. For this, consider the fiber product $Q \times_M R \rightarrow M$, which is a principal $(H \times \mathbb{Z}/2\mathbb{Z})$ -bundle. The bundle $Q' \rightarrow M$ is obtained by dividing out by the diagonal $\mathbb{Z}/2\mathbb{Z} \subset H \times \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z} \subset H$ is the kernel of the covering map ρ . I leave you to construct θ' .

Remark 9.33. There is a category of double covers of M , and it has a “product” operation which makes it a categorical analog of a group. That *Picard category* acts on the category of reductions to H .

Exercise 9.34. As in Exercise 9.2 the set of isomorphism classes of double covers of M is $H^1(M; \mathbb{Z}/2\mathbb{Z})$. How is its abelian group structure related to Remark 9.33? Show that any two reductions to H are related by a double cover in the manner described. Conclude that $H^1(M; \mathbb{Z}/2\mathbb{Z})$ acts simply transitively on the set of isomorphism classes (Exercise 9.12(ii)) of reductions.

Exercise 9.35. Just as an orientation on a manifold M with boundary induces an orientation of the boundary ∂M , show that the same is true of a spin structure. (As a spin structure includes an orientation, the statement about orientations is included.)

Exercise 9.36. Show that there are two isomorphism classes of spin structure on S^1 . Describe the principal $Spin(2)$ -bundles and the isomorphisms θ explicitly. Which occurs as the boundary of a spin structure on the disk D^2 ?

Reductions of structure group and classifying spaces

We continue in the context of (9.7), working with an arbitrary homomorphism $\rho: H \rightarrow G$. As we have only constructed classifying spaces for compact Lie groups (in (6.59)), we assume H and G are compact. Let $EH \rightarrow BH$ be the universal H -bundle. The associated G -bundle has a classifying map

$$(9.37) \quad \begin{array}{ccc} EH \times_{\rho} G & \longrightarrow & EG \\ \downarrow & & \downarrow \\ BH & \xrightarrow{B\rho} & BG \end{array}$$

which we denote¹ $B\rho$. The top horizontal arrow in (9.37) induces an isomorphism $\theta^{\text{univ}}: EH \times_\rho G \xrightarrow{\cong} (B\rho)^*(EG)$. The pair $(EG \times_\rho G, \theta^{\text{univ}})$ is the universal reduction of a G -bundle to an H -bundle.

Proposition 9.38. *Let $P \rightarrow M$ be a principal G -bundle and $f: M \rightarrow BG$ a classifying map. Then a lift \tilde{f} in the diagram*

$$(9.39) \quad \begin{array}{ccc} & & BH \\ & \nearrow \tilde{f} & \downarrow \\ M & & BG \\ & \searrow f & \end{array}$$

induces a reduction to H , and conversely a reduction to H induces a lift \tilde{f} . Isomorphism classes of reductions are in 1:1 correspondence with homotopy classes of lifts.

Here a homotopy of lifts is a map $F: \Delta^1 \times M \rightarrow BH$ such that $F(t, -): M \rightarrow BH$ is a lift of f for all $t \in \Delta^1$.

Proof. Given a lift, pull back the universal reduction $(EG \times_\rho G, \theta^{\text{univ}})$ to M . Conversely, any reduced bundle $(Q \rightarrow M, \theta)$ has a classifying map of principal H -bundles

$$(9.40) \quad \begin{array}{ccc} Q & \longrightarrow & EG \\ \downarrow & & \downarrow \\ M & \xrightarrow{g} & BH \end{array}$$

and so a map of principal G -bundles

$$(9.41) \quad \begin{array}{ccc} Q \times_\rho G & \longrightarrow & EG \times_\rho G \\ \downarrow & & \downarrow \\ M & \xrightarrow{g} & BH \end{array}$$

The isomorphism θ then induces a diagram

$$(9.42) \quad \begin{array}{ccccc} P & \longrightarrow & (B\rho)^*(EG) & \longrightarrow & EG \\ \downarrow & & \downarrow & & \downarrow \\ M & \xrightarrow{g} & BH & \xrightarrow{B\rho} & BG \end{array}$$

in which the composition $B\rho \circ g$ is a classifying map, so is necessarily homotopic to f . Construct \tilde{f} as the endpoint of a homotopy of maps $M \rightarrow BH$ which lifts the homotopy $B\rho \circ g \rightarrow f$ using the homotopy lifting property of $B\rho$. \square

Exercise 9.43. Work out the details of the last argument as well as the proof of the last assertion of Proposition 9.38.

¹There is a functorial construction B of classifying spaces which inspires this notation.

General tangential structures

We now generalize reductions of $O(n)$ -bundles along homomorphisms $\rho: H \rightarrow G$ to a more general and flexible notion of a tangential structure.

Recall the construction (6.24) of the classifying space $BO(n)$ as a colimit of finite dimensional Grassmannians. There are closed inclusions $Gr_n(\mathbb{R}^q) \rightarrow Gr_{n+1}(\mathbb{R}^{q+1})$ obtained by sending $W \mapsto \mathbb{R} \oplus W$ where we write $\mathbb{R}^{q+1} = \mathbb{R} \oplus \mathbb{R}^q$. These induce maps $BO(n) \rightarrow BO(n+1)$, and we define

$$(9.44) \quad BO = \operatorname{colim}_{n \rightarrow \infty} BO(n).$$

It is a classifying space for the infinite orthogonal group O defined in (5.38).

Definition 9.45. An n -dimensional tangential structure is a topological space $\mathcal{X}(n)$ and a fibration $\pi(n): \mathcal{X}(n) \rightarrow BO(n)$. A stable tangential structure is a topological space \mathcal{X} and a fibration $\pi: \mathcal{X} \rightarrow BO$. It gives rise to an n -dimensional tangential structure for each $n \in \mathbb{Z}^{\geq 0}$ by letting $\pi(n): \mathcal{X}(n) \rightarrow BO(n)$ be the fiber product

$$(9.46) \quad \begin{array}{ccc} \mathcal{X}(n) & \dashrightarrow & \mathcal{X} \\ \pi(n) \downarrow & & \downarrow \pi \\ BO(n) & \longrightarrow & BO \end{array}$$

If M is a k -dimensional manifold, then an $\mathcal{X}(n)$ -structure on M is a lift $M \rightarrow \mathcal{X}(n)$ of a classifying map $M \rightarrow BO(n)$ of \widetilde{TM} , where we have stabilized the tangent bundle TM of the m -dimensional manifold M to the rank n bundle

$$(9.47) \quad \widetilde{TM} := \underline{\mathbb{R}^{n-m}} \oplus TM.$$

An \mathcal{X} -structure on M is a family of coherent $\mathcal{X}(n)$ -structures for n sufficiently large.

Notice that an n -dimensional tangential structure induces an m -dimensional tangential structure for all $m < n$ by taking the fiber product

$$(9.48) \quad \begin{array}{ccc} \mathcal{X}(m) & \dashrightarrow & \mathcal{X}(n) \\ \pi(m) \downarrow & & \downarrow \pi(n) \\ BO(m) & \longrightarrow & BO(n) \end{array}$$

Example 9.49. The trivial tangential structure has $\mathcal{X} = BO$, so $\mathcal{X}(n) = BO(n)$, and the structure maps $\pi(n)$ are identity maps.

Example 9.50. A stable framing is the tangential structure $\mathcal{X} = EO$, a contractible space with a free O -action. What is $\mathcal{X}(n) \rightarrow BO(n)$ in this example? (Hint: It is a principal O -bundle.)

Example 9.51. An n -framing is the n -dimensional unstable tangential structure $\mathcal{X}(n) = EO(n)$. An n -framing of an m -manifold M is a trivialization of \widetilde{TM} . What is $\mathcal{X}(m)$?

Example 9.52. An orientation is the stable tangential structure $\mathcal{X} = BSO$, and an orientation of \widetilde{TM} amounts to an orientation of TM since \mathbb{R}^{n-m} has a canonical orientation. In this case the space $\mathcal{X}(n)$ in (9.46) is the classifying space $BSO(n)$.

Example 9.53. A spin structure is also a stable tangential framing; the space $\mathcal{X}(n)$ is the classifying space $BSpin(n)$.

Example 9.54. There are examples which are not reductions of structure group. For example, if $\mathcal{X}(n) = BO(n) \times B\Gamma$ for some finite group Γ , then an $\mathcal{X}(n)$ -structure on M is a principal Γ -bundle over M . We can replace $B\Gamma$ by any space Y . Isomorphism classes of $\mathcal{X}(n)$ -structures then track homotopy classes of maps $M \rightarrow Y$.

Exercise 9.55. Following Proposition 9.38, define an isomorphism of $\mathcal{X}(n)$ -structures. Formulate the classification of isomorphism classes of $\mathcal{X}(n)$ -structures as a problem in homotopy theory.

(9.56) *The universal $\mathcal{X}(n)$ -bundle.* Let $\pi: \mathcal{X} \rightarrow BO$ be a stable tangential structure with induced tangential structures $\pi(n): \mathcal{X}(n) \rightarrow BO(n)$ for each $n \in \mathbb{Z}^{\geq 0}$. Let

$$(9.57) \quad S(n) \longrightarrow BO(n)$$

be the universal real vector bundle of rank n , as in (6.27). Its pullback to $\mathcal{X}(n)$ has a tautological $\mathcal{X}(n)$ -structure, the identity map $\text{id}_{\mathcal{X}(n)}$ lifting $\pi(n)$ in

$$(9.58) \quad \begin{array}{ccc} \pi(n)^*(S(n)) & \longrightarrow & S(n) \\ \downarrow & & \downarrow \\ \mathcal{X}(n) & \xrightarrow{\pi(n)} & BO(n) \end{array}$$

so is the universal real rank n bundle with $\mathcal{X}(n)$ -structure. By abuse of notation we denote this pullback $\pi(n)^*(S(n))$ as simply

$$(9.59) \quad S(n) \longrightarrow \mathcal{X}(n)$$

Suppose M is an m -manifold, $m \leq n$. Then an $\mathcal{X}(n)$ -structure on M is an $\mathcal{X}(n)$ -structure on its stabilized tangent bundle $\widetilde{TM} \rightarrow M$, as stated in Definition 9.45, which is more simply a classifying map

$$(9.60) \quad \begin{array}{ccc} \widetilde{TM} & \longrightarrow & S(n) \\ \downarrow & & \downarrow \\ M & \longrightarrow & \mathcal{X}(n) \end{array}$$

(9.61) *Manifolds with boundary.* If M is a manifold with boundary and it is equipped with an $\mathcal{X}(n)$ -structure (9.60), then there is an induced $\mathcal{X}(n)$ -structure on the boundary. Namely, we just restrict (9.60) to ∂M ; recall the exact sequence (1.12) at the boundary, which is split by the discussion in (5.3); and use Definition 9.45 which involves the stabilized tangent bundle (9.47) of the boundary: $T(\widetilde{\partial M}) := \underline{\mathbb{R}} \oplus T(\partial M)$.

\mathcal{X} -bordism

(9.62) *Involutions.* The classifying space $BO(n)$ is a colimit (6.24) of Grassmannians $Gr_n(\mathbb{R}^q)$. Endow \mathbb{R}^q with the standard inner product. Then the map $W \mapsto W^\perp$ to the orthogonal subspace induces inverse diffeomorphisms

$$(9.63) \quad Gr_n(\mathbb{R}^m) \longleftrightarrow Gr_{m-n}(\mathbb{R}^m)$$

which exchange the tautological subbundles S with the tautological quotient bundles Q . The double colimit of (9.63) as $n, m \rightarrow \infty$ yields an involution

$$(9.64) \quad \iota: BO \longrightarrow BO$$

If $\mathcal{X} \rightarrow BO$ is a stable tangential structure, we define its pullback by ι to be a new stable tangential structure

$$(9.65) \quad \begin{array}{ccc} \mathcal{X}^\perp & \dashrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ BO & \xrightarrow{\iota} & BO \end{array}$$

If $f: M \rightarrow BO$ is the stable classifying map of a vector bundle $V \rightarrow M$, and there is a complementary bundle $V^\perp \rightarrow M$ such that $V \oplus V^\perp \cong \underline{\mathbb{R}}^m$, then $\iota \circ f: M \rightarrow BO$ is a stable classifying map for $V^\perp \rightarrow M$.

(9.66) *Stable normal structures from stable tangential structures.* We reconsider the discussion in (5.15), only instead of embedding in the sphere we embed in affine space (which is what we were doing anyhow). Fix a stable tangential structure $\pi: \mathcal{X} \rightarrow BO$. Let M be a smooth n -manifold. A stable \mathcal{X} -structure on M is an $\mathcal{X}(n+q)$ -structure on $TM \rightarrow M$ for sufficiently large q , i.e., compatible classifying maps

$$(9.67) \quad \begin{array}{ccccccc} & & S(n) & \longrightarrow & S(n) & \longrightarrow & \dots \\ & \nearrow & \downarrow & & \downarrow & & \\ TM & & \mathcal{X}(n, n+q) & \longrightarrow & \mathcal{X}(n, n+q+1) & \longrightarrow & \dots \\ \downarrow & \nearrow & & & & & \\ M & & & & & & \end{array}$$

In the diagram we $\mathcal{X}(n, n+q)$ is the pullback of $\mathcal{X}(n) \rightarrow BO(n)$ to the Grassmannian $Gr_n(\mathbb{R}^{n+q}) \hookrightarrow BO(n)$. Now suppose $M \hookrightarrow \mathbb{A}^{n+q}$ is an embedding with normal bundle $\nu \rightarrow M$ of rank q . We use the Euclidean metric to identify $\nu \cong TM^\perp$ so $TM \oplus \nu \cong \underline{\mathbb{R}^{n+q}}$. Then using the perp map (9.63) we obtain a classifying map

$$(9.68) \quad \begin{array}{ccc} \nu & \longrightarrow & Q(q) \\ \downarrow & & \downarrow \\ M & \longrightarrow & \mathcal{X}^\perp(q, n+q) \end{array}$$

Here $\mathcal{X}^\perp(q, n+q)$ is the pullback of $\mathcal{X}^\perp \rightarrow BO$ to the Grassmannian $Gr_q(\mathbb{R}^{n+q})$. Stabilizing we obtain a classifying map of the stable normal bundle. It is simply $\iota \circ f$, where f is the stable classifying map (9.67) of the tangent bundle. Note that $\iota \circ f$ is defined without choosing an embedding.

In this way we pass back and forth between stable tangential \mathcal{X} -structures and stable normal \mathcal{X}^\perp -structures.

(9.69) \mathcal{X} -bordism groups. We now imitate Definition 1.19 to define a bordism of closed manifolds equipped with an \mathcal{X} -structure on the stable tangent bundle, or equivalently an \mathcal{X}^\perp -structure on the stable normal bundle. Bordism is an equivalence relation, and we denote the bordism group of closed n -dimensional \mathcal{X} -manifolds as $\Omega_n^\mathcal{X}$.

Exercise 9.70. Prove that bordism is an equivalence relation. Pay attention to the symmetry argument: see (2.20).

Exercise 9.71. Show that for $\mathcal{X} = BSO$, as in Example 9.52, this reproduces the oriented bordism group defined in Lecture 2. Quite generally, if $\mathcal{X} = BG$, then we use the notation Ω_\bullet^G in place of $\Omega_\bullet^\mathcal{X}$.

References

- [War] Frank W. Warner, *Foundations of differentiable manifolds and Lie groups*, Graduate Texts in Mathematics, vol. 94, Springer-Verlag, New York, 1983. Corrected reprint of the 1971 edition. 3