Problem Set # 3

M392C: Bordism Old and New

Due: November 20, 2012

1. Here are some problems concerning invertibility in symmetric monoidal categories, as in Lecture 17.

   (a) Construct a category of invertibility data (Definition 17.18), and prove that this category is a contractible groupoid.

   (b) Prove Lemma 17.21(i).

   (c) Let $\alpha\colon \mathrm{Bord}^{SO}_{\langle 0,1\rangle} \to C$ be a TQFT. Prove that if $\alpha(\mathrm{pt}_+)$ is invertible, then α is invertible.

2. Compute the invariants of the Picard groupoid of superlines. (See (17.27) and (17.35) in the notes.)

3. Show that a special Γ-set determines a commutative monoid. More strongly, construct a category of special Γ-sets, a category of commutative monoids, and an equivalence of these categories.

4. Let $\mathbb{S}$ denote the Γ-set $\mathbb{S}(S) = \Gamma^{\mathrm{op}}(S^0, S)$, for $S \in \Gamma^{\mathrm{op}}$ a finite pointed set. Compute $\pi_1|\mathbb{S}|$.

5. Let C be a category. An object $* \in C$ is *initial* if for every $y \in C$ there exists a unique morphism $* \to y$, and it is *terminal* if for every $y \in C$ there exists a unique morphism $y \to *$.

   (a) Prove that an initial object is unique up to unique isomorphism, and similarly for a terminal object.

   (b) Examine the existence of initial and terminal objects for the following categories: Vect, Set, Space, Set_*, Space_*, the category of commutative monoids, a bordism category, a category of topological quantum field theories.

   (c) Prove that if C has either an initial or final object, then its classifying space is contractible.

6. Let K denote the classifying spectrum of the category whose objects are finite dimensional complex vector spaces and whose morphisms are isomorphisms of vector spaces. Compute $\pi_0 K$. Compute $\pi_1 K$.

7. Let M be a *commutative* monoid. We described a general construction of the group completion of any monoid. Give a much simpler construction of the group completion $|M|$ by imposing an equivalence relation on $M \times M$. You may wish to think about the examples $M = (\mathbb{Z}^{\geq 0}, +)$ and $M = (\mathbb{Z}^{>0}, \times)$.

8. Let G be a topological group, viewed as a category C with a single object. (Normally we use 'G' in place of 'C', but for clarity here we distinguish.)

   (a) Describe the nerve NC of G explicitly.

   (b) Define a groupoid $\mathcal{G}$ whose set of objects is G and with a unique morphism between any two objects. Construct a free right action of G on $\mathcal{G}$ with quotient C. First, define carefully what that means.

   (c) Prove that the classifying space $B\mathcal{G}$ is contractible.

   (d) Show that G acts freely on $B\mathcal{G}$ with quotient BC.

   So we would like to assert that $B\mathcal{G} \to BC$ is a principal G-bundle, and by Theorem 6.45 in the notes a universal bundle, which then makes BC a classifying space in the sense of Lecture 6. The only issue is local triviality; see Segal's paper.