Problem Set # 4

M392C: Bordism Old and New

Due: December 6, 2012

1. The embedding $U(m) \hookrightarrow O(2m)$ of the unitary group into the orthogonal group determines a $2m$-dimensional tangential structure $BU(m) \to BO(2m)$. Compute the integral homology $H_\bullet(MTU(m))$ of the associated Madsen-Tillmann spectrum.

2. For each of the following maps $\mathcal{F}\colon \mathrm{Man}^{\mathrm{op}} \to \mathrm{Set}$, answer : Is $\mathcal{F}$ a presheaf? Is $\mathcal{F}$ a sheaf?

   (a) $\mathcal{F}(M) =$ the set of smooth vector fields on M

   (b) $\mathcal{F}(M) =$ the set of orientations of M

   (c) $\mathcal{F}(M) =$ the set of sections of $\mathrm{Sym}^2 T^*M$

   (d) $\mathcal{F}(M) =$ the set of Riemannian metrics on M

   (e) $\mathcal{F}(M) =$ the set of isomorphism classes of double covers of M

   (f) $\mathcal{F}(M) = H^q(M; A)$ for some $q \geq 0$ and abelian group A

3. Define a sheaf $\mathcal{F}$ of categories on Man which assigns to each test manifold M a groupoid of double covers of M. Be sure to check that you obtain a presheaf—compositions map to compositions—which satisfies the sheaf condition. Describe $|\mathcal{F}|$ and $B|\mathcal{F}|$. Compute the set $\mathcal{F}[M]$ of concordance classes of double covers on M.

4. (a) Fix $q \geq 0$. Define a sheaf $\mathcal{F}$ of sets on Man which assigns to each test manifold M the set of closed differential q-forms. Compute $\mathcal{F}[M]$. Identify $|\mathcal{F}|$.

   (b) Fix $k > 0$. Fix a complex Hilbert space $\mathcal{H}$. Define a sheaf $\mathcal{F}$ of sets on Man which assigns to each test manifold M the set of rank k vector bundles $\pi\colon E \to M$ together with an embedding $E \hookrightarrow M \times \mathcal{H}$ into the vector bundle with constant fiber $\mathcal{H}$ and a flat covariant derivative operator. (The flat structure and embedding are uncorrelated.) Discuss briefly why $\mathcal{F}$ is a sheaf. Compute $\mathcal{F}[M]$. Identify $|\mathcal{F}|$.