Problem Set # 11

M392C: K-theory

1. Let T be a torus and E an irreducible 2-dimensional orthogonal (real) representation whose complexification has weights $\lambda, -\lambda$. Let T act on the Clifford algebra $C\ell(E)$ via the spin group of E, which may require lifting to a double cover torus. Prove that the character of $C\ell(E)$ is then $2(e^{i\lambda/2} + e^{-i\lambda/2})$. Prove that the $\lambda/2$-weight space is an irreducible module M for $C\ell_{\mathbb{C}}(E)$ in the sense that $C\ell_{\mathbb{C}}(E)$ is isomorphic to $\mathrm{End}(M)$ as a superalgebra.

2. A *super division algebra* is a superalgebra—that is, a $\mathbb{Z}/2\mathbb{Z}$-graded algebra—in which every nonzero homogeneous element is invertible. Which real and complex Clifford algebras are super division algebras? No Clifford algebra is isomorphic to the quaternions $\mathbb{H}$, but there are Clifford algebras Morita equivalent to $\mathbb{H}$: which ones?

3. In this problem you will demonstrate that $C\ell_8$ is a $\mathbb{Z}/2\mathbb{Z}$-graded real matrix algebra. We gave a streamlined proof in Lecture 6. This exercise is designed to give you hands-on practice with Clifford algebras. We prove that there is a $\mathbb{Z}/2\mathbb{Z}$-graded real vector space $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ with $\dim \mathbb{S}^\pm = 8$ and an isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded algebras $C\ell_8 \longrightarrow \mathrm{End}(\mathbb{S})$.

   (a) Show that it is enough to prove that $C\ell_8 \cong \mathrm{End}(\mathbb{S})$ as *ungraded* algebras. For this use the volume element $e_1 e_2 e_3 \cdots e_8$ to grade $\mathbb{S}$.

   (b) Show that $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{R}(4)$, where $\mathbb{R}(n)$ denotes the algebra of $n \times n$ matrices. (Hint: Use $\mathbb{R}(4) \cong \mathrm{Hom}_{\mathbb{R}}(\mathbb{H}, \mathbb{H})$.)

   (c) Let $C\ell_{p,q}$ denote the real Clifford algebra generated by $e_1, \dots, e_{p+q}$ where these elements anticommute and
   $$e_i^2 = \begin{cases} = 1, & i \le p; \\ +1, & i = p+1, \dots, p+q. \end{cases}$$
   So $\mathrm{Cliff}_n = C\ell_{n,0}$. Construct isomorphisms
   $$C\ell_{2,0} \cong \mathbb{H}$$
   $$C\ell_{0,2} \cong \mathbb{R}(2)$$

   (d) *In this problem we use* ungraded *tensor products*. Construct isomorphisms
   $$C\ell_{k,0} \otimes C\ell_{0,2} \cong C\ell_{0,k+2}$$
   $$C\ell_{0,k} \otimes C\ell_{2,0} \cong C\ell_{k+2,0}$$

   (e) Prove that $C\ell_4 \cong \mathbb{H}(2)$. Prove the desired statement that $C\ell_8 \cong \mathbb{R}(8)$.

(f) How does this prove a mod 8 periodicity of the real Clifford algebras?

(g) While you're at it, prove that $C\ell_{p,p}$ is a real matrix algebra. You can do this by mimicking the proof given in lecture for the complex even dimensional Clifford algebra. The idea is that the bilinear form is split: the vector space is the sum of two maximal isotropic subspaces.