Problem Set # 2

M392C: K-theory

1. (a) A 3×3 rotation matrix always has a fixed line—that is, an eigenspace—which is actually pointwise fixed—the eigenvalue is 1. Show that this is so. (A 3×3 rotation matrix is an orthogonal matrix with determinant 1. The group of all such matrices is denoted SO_3.) Except for the identity matrix I, this line is unique. Show that the map $f\colon SO_3 \setminus \{I\} \to \mathbb{RP}^2$ so defined is a submersion. What is the inverse image of a point?

   (b) Show that $\mathbb{RP}^3$ may be constructed from the unit ball $B^3 \subset \mathbb{A}^3$ by identifying antipodal points of the boundary S^2.

   (c) Construct a diffeomorphism $f\colon \mathbb{RP}^3 \to SO_3$. Hint: Take the ball in part (b) to have radius π.

   (d) Prove that the inclusion $SO_2 \hookrightarrow SO_3$ induces a surjection on π_1.

2. Prove the following: Let n be a nonnegative integer and N a sufficiently large positive integer. Then there is an isomorphism
$$\pi_{n-1} O_N \longrightarrow \widetilde{KO}(S^n).$$
What is the minimal value of N in terms of n? (The set of N larger than or equal to this minimal value is called the *stable range*.)

3. Prove directly (not using vector bundles on the sphere, as in lecture) that for $G = U, O, Sp$ the homotopy groups $\pi_q G_N$ stabilize in the sense that there exists an affine function f such that for $N \geq f(q)$ the inclusion $G_N \hookrightarrow G_{N+1}$ induces an *isomorphism* $\pi_q G_N \to \pi_q G_{N+1}$. Here '$Sp$' denotes the symplectic group. The function f depends which series of classical groups we are using. (Hint: Generalize the lemma about fiber bundles we proved in class.)