Problem Set # 3

M392C: K-theory

1. Let $H \to S^2$ be the hyperplane bundle. Construct an explicit isomorphism $H \oplus H \to H^{\oplus 2} \oplus \underline{\mathbb{C}}$.

2. Let S be a set with composition laws $\circ_1, \circ_2 \colon S \times S \to S$ and distinguished element $1 \in S$. Assume (i) 1 is an identity for both $\circ_1$ and $\circ_2$; and (ii) for all $s_1, s_2, s_3, s_4 \in S$ we have

$$(s_1 \circ_1 s_2) \circ_2 (s_3 \circ_1 s_4) = (s_1 \circ_2 s_3) \circ_1 (s_2 \circ_2 s_4).$$

   Prove that $\circ_1 = \circ_2$ and that this common operation is commutative and associative.

3. Show that the projective plane $\mathbb{FP}^2$ is the mapping cone of the appropriate Hopf map. In other words, construct a CW structure on $\mathbb{FP}^2$ with exactly 3 cells (of dimensions $0, d, 2d$, where $d = \dim_{\mathbb{R}} \mathbb{F}$) and show the attaching map is the Hopf map. Your solution should be very geometric and use the geometry of the projective plane. Work the problem for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and for the octonions as well.

4. Recall that in the Atiyah-Bott *Acta* proof of the periodicity theorem if X is compact Hausdorff, $E \to X$ a complex vector bundle, and

$$p(x, \lambda) = \sum_{k=0}^{n} p_k(x)\lambda^k, \qquad p_k(x) \in \operatorname{End} E_x,$$

   is invertible for $x \in X$, $\lambda \in \mathbb{T}$, then we define

$$\mathcal{L}^n p = \begin{pmatrix} 1 & -\lambda & & & & \\ & 1 & -\lambda & & & \\ & & & \ddots & & \\ & & & & 1 & -\lambda \\ p_n & & \cdots & & p_1 & p_0 \end{pmatrix},$$

   which is an automorphism of $E^{\oplus(n+1)} \times \mathbb{T} \to X \times \mathbb{T}$. Prove this last assertion: $\mathcal{L}^n p$ is invertible. Now $\mathcal{L}^n$ maps polynomials of degree $\leq n$ to clutching functions for a bundle on $X \times \mathbb{T}$; let $(E^{\oplus(n+1)}, \mathcal{L}^n p)$ denote its isomorphism class. Noting that p above is also a polynomial of degree $\leq n+1$, prove that

$$(E^{\oplus(n+2)}, \mathcal{L}^{n+1} p) = (E^{\oplus(n+1)}, \mathcal{L}^n p) + (E, 1)$$
$$(E^{\oplus(n+2)}, \mathcal{L}^{n+1}(\lambda p)) = (E^{\oplus(n+1)}, \mathcal{L}^n p) + (E, \lambda)$$

5. (a) Let $\mathbb{E}$ be a finite dimensional complex vector space and $T \in \operatorname{End} \mathbb{E}$ a linear transformation with no eigenvalues on $\mathbb{T} \subset \mathbb{C}$. Define

$$Q = \frac{1}{2\pi i} \int_{|w|=1} (w - T)^{-1} \, dw.$$

Prove that $Q^2 = Q$ and $QT = TQ$. Prove that the image $Q\mathbb{E} \subset \mathbb{E}$ of the projection Q is the sum of the generalized eigenspaces of T for eigenvalues in the unit disk. What is the image of the complementary projection $1 - Q = \mathrm{id}_{\mathbb{E}} - Q$?

(b) What can you say if $\mathbb{E}$ is an infinite dimensional Hilbert space?

6. (a) For any space X we let $X_+ = X \sqcup \mathrm{pt}$ be the pointed space which is the union of X and a disjoint basepoint. Let X be a pointed CW complex. Construct a pointed homotopy equivalence

$$\Sigma(X_+) \simeq \Sigma(S^0 \vee X).$$

Here Σ denotes the suspension operation $S^1 \wedge -$ on pointed spaces.

(b) For pointed CW complexes X, Y construct a pointed homotopy equivalence

$$\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y).$$

What do you conclude if $X = Y = S^1$? Use this to compute the homology groups of a torus. The "stable splitting" technique is a useful one for homology computations.