Problem Set # 4

M392C: K-theory

1. Let X be a topological space, $A \subset X$ a subspace, and $a \in A$ a basepoint for A and X. Suppose there exists a deformation retraction of a neighborhood U of A in X back to A; fix one.

   (a) Construct a homotopy equivalence between X/A and $CA \cup_A X$. (Recall that the cone CA is the smash product $[0,1] \wedge A$ where $[0,1]$ has basepoint 0, and we attach CA to X along $\{1\} \times A$.)

   (b) Let $i\colon A \hookrightarrow X$ be the inclusion and $q\colon X \twoheadrightarrow X/A$ the quotient map. Construct the Puppe sequence

   $$A \xrightarrow{i} X \xrightarrow{q} X/A \longrightarrow \Sigma A \xrightarrow{\Sigma i} X$$

   and prove that each of the three stretches of two consecutive maps has the form of the first stretch for some triple (X', A', a') with a deformation retraction of a neighborhood of A' to A'.

2. (a) Following the scheme of the 9/17 lecture, from the algebra $\mathbb{R}$ construct a map $g\colon S^0 \times S^0 \to S^0$ which is an H-space structure, and from that construct a map $f\colon S^1 \to S^1$. What is the degree of f? Can you recognize the mapping cone C_f?

   (b) Repeat starting with the division algebras $\mathbb{C}, \mathbb{H}, \mathbb{O}$ in turn. What spaces do you get as mapping cones?

3. (a) What is the group of automorphisms of the algebra $\mathbb{R}$? Of the algebra $\mathbb{C}$?

   (b) Determine the automorphism group of $\mathbb{H}$. You might note that the subspace $\operatorname{Im}\mathbb{H}$ of imaginary quaternions is preserved by an automorphism, and then try to see what other geometric structures are preserved by virtue of the algebra structure being preserved. Is every automorphism inner?

   (c) Let G be the automorphism group of the octonions $\mathbb{O}$. What can you deduce about G? Is it a Lie group? Is it compact? Again, observe that an automorphism preserves the 7-dimensional subspace $\operatorname{Im}\mathbb{O}$ of imaginary octonions. Does an automorphism act orthogonally on this subspace? Can you sit G as the total space of a fiber bundle, for example by examining its action on a vector in $\operatorname{Im}\mathbb{O}$? Or, perhaps, its action on two orthogonal vectors in $\operatorname{Im}\mathbb{O}$? What is the dimension of G? Is G connected? Is every automorphism inner?