Problem Set # 5

M392C: K-theory

1. Let H be an infinite dimensional real or complex separable Hilbert space. Prove that the sphere $S(H) \subset H$ of unit norm vectors is contractible. One approach is as follows. Choose a countable basis $\dots, e_{-1}, e_0, e_1, \dots$. Let $D(H) \subset H$ denote the closed unit ball and embed $\mathbb{R} \hookrightarrow D(H)$ so that $n \mapsto e_n$ for $n \in \mathbb{Z}$. Use the Tietze extension theorem to construct a function $D(H) \to \mathbb{R}$ with no fixed points, and then a function $D(H) \to D(H)$ with no fixed points. Imitate Hirsch's proof of the Brouwer fixed point theorem to construct a deformation retraction of $D(H)$ onto $S(H)$.

2. Let H be a separable Hilbert space.

   (a) Prove that the closure of the finite rank operators on H in the norm topology is the space of compact operators.

   (b) Let $\mathfrak{gl}$ denote the space of bounded operators, cpt the space of compact operators, GL the group of invertible operators, and GL^{cpt} the group of operators which differ from the identity by a compact operator. Prove that the group $GL/GL^{\text{cpt}} \subset \mathfrak{gl}/\text{cpt}$ is the identity component of the group of invertible operators in $\mathfrak{gl}/\text{cpt}$.

3. Let $\pi\colon E \to B$ be a *fibration* of pointed spaces with basepoints $e, \pi(e) = b$. As discussed in lecture, a fibration is a continuous map which satisfies the homotopy lifting property. Let $F = \pi^{-1}(b)$ with basepoint e. Then there is a long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_q(F,e) \longrightarrow \pi_q(E,e) \longrightarrow \pi_q(B,b) \longrightarrow \pi_{q-1}(F,e) \longrightarrow \cdots$$

   (a) Interpret and prove exactness at the very end of this sequence ($q = 0$).

   (b) What does this long exact sequence say for covering spaces?

   (c) Apply this repeatedly to the fiber bundle $U(n-1) \to U(n) \to S^{2n-1}$ to prove that the homotopy groups of the unitary group stabilize and to compute the stable range (compare to the Proposition 3.26 in the class notes).

   (d) Repeat for the orthogonal groups $O(n)$. What about the symplectic groups $Sp(n)$?