Problem Set # 6

M392C: K-theory

Here are several problems about Clifford algebras and Clifford modules.

1. (a) Prove that there are no nontrivial graded ideals in $C\ell_n$.

   (b) Let G be the *extra-special 2-group* in $C\ell_n$ generated by $\{\pm e_1, \dots, \pm e_n\}$. (It is a finite group.) Describe its representation theory. How are its representations related to Clifford modules?

   (c) If n is even we proved that $C\ell_n^{\mathbb{C}}$ is a super matrix algebra. What can you say if n is odd?

2. (a) Prove that the opposite of $C\ell(V, Q)$ is isomorphic to $C\ell(V, -Q)$.

   (b) Construct an isomorphism of the spin groups which sit inside $C\ell_n$ and $C\ell_{-n}$. Is there an isomorphism of pin groups?

   (c) Prove that the category of modules over $C\ell_{-4}$ is isomorphic to the category of quaternionic vector spaces.

3. (a) Construct isomorphisms of *ungraded* algebras

$$C\ell_{n+2} \longrightarrow C\ell_{-n} \otimes C\ell_2$$
$$C\ell_{-(n+2)} \longrightarrow C\ell_n \otimes C\ell_{-2}$$

   The tensor product is the tensor product of ungraded algebras.

   (b) Write the underlying ungraded algebras of $C\ell_5, C\ell_{-5}$ as (sums of) matrix algebras.

4. In this problem we use only the "negative" Clifford algebras. For $n \geq 0$ define A_{-n} as the Grothendieck group, or abelian group completion, of the group of isomorphism classes of $C\ell_{-n}$-modules modulo restrictions of $C\ell_{-(n+1)}$-modules.

   (a) Compute A_{-n} for all n. What are generators?

   (b) Define a ring structure on $A = \bigoplus_{n=0}^{\infty} A_{-n}$. Find generators and relations.

   (c) Treat a complex version.

   (d) How is this related to the Atiyah-Bott-Shapiro construction? First, if a $C\ell_{-n}$-module is the restriction of a $C\ell_{-(n+1)}$-module, what can you say about the vector bundle over the sphere constructed in (6.64)?

5. In this problem you construct the canonical Dirac operator on an oriented spin Riemannian manifold M of dimension n.

   (a) Forget 'spin' for now and construct $SO(M) \to M$ the bundle of oriented orthonormal frames as a principal SO_n-bundle. Define a *spin structure* to be a lift to a principal Spin_n-bundle. Explain the meaning of 'lift'. Assume now that M is spin.

   (b) Construct the associated bundle whose fiber is the Clifford algebra $C\ell_n$ by letting Spin_n act by left multiplication. It is a $\mathbb{Z}/2\mathbb{Z}$-graded bundle $S \to M$ with a *right* $C\ell_n$-action. Show that it is equivalent to a left $C\ell_{-n}$-action.

   (c) Use the Levi-Civita connection to induce a covariant derivative on sections of $S \to M$. Compose with Clifford multiplication to construct the Dirac operator. Show it is odd skew-adjoint and commutes with the $C\ell_{-n}$-action.

6. (a) Let H be a Lie group and $P \to X$ a principal H-bundle. Suppose $\rho\colon H \to G$ is a homomorphism of Lie groups. Construct an *associated* principal G-bundle $P \times_H G \to X$.

   (b) Now suppose $Q \to X$ is a principal G-bundle. A *reduction to* H is a pair (P, θ) of a principal H-bundle $P \to X$ and an isomorphism $\theta\colon Q \to P \times_H G$ of principal G-bundles. Show that if ρ is an inclusion, then isomorphism classes of reductions (define) are in 1:1 correspondence with sections of $P/H \to X$.

   (c) Explain part (b) for X a smooth n-manifold, $Q \to X$ its bundle of frames, and $\rho\colon H \to G$ the inclusion $O_n \hookrightarrow GL_n\mathbb{R}$.

   (d) Investigate the existence and uniqueness question for reductions in two cases: (i) ρ is the inclusion of a subgroup of order 2; (ii) ρ is a 2:1 cover of Lie groups.