Problem Set # 7

M392C: K-theory

1. Suppose $p\colon E \to B$ is a fibration. Assume E, B have basepoints e, b. For $b' \in B$ let $P_e\bigl((E; p^{-1}(b'))\bigr)$ denote the space of paths in E which begin at e and terminate on the fiber $p^{-1}(b')$. Prove that p induces a fibration
$$P_e\bigl((E; p^{-1}(b'))\bigr) \longrightarrow P_b(B; b')$$
with contractible fibers. What assumptions do you need to make on the topological spaces E, B? Conclude that p is a weak homotopy equivalence. When can you conclude that p is a homotopy equivalence?

2. Fix a positive integer n. Let E denote the space of skew-Hermitian $n \times n$ matrices with operator norm ≤ 1. (The eigenvalues $i\lambda_1, \dots, i\lambda_n$ satisfy $|\lambda_j| \le 1$.) Consider the exponential map
$$\begin{aligned} p\colon E &\longrightarrow U(n) \\ A &\longmapsto \exp(\pi A) \end{aligned}$$
   (a) For each k between 0 and n prove that the restriction of p over the subspace of $U(n)$ consisting of unitary matrices with (-1)-eigenspace of dimension k is a fiber bundle. What is the fiber?

   (b) Show that p is a quasifibration.

3. Use the contractibility of the unit sphere in Hilbert space, proved in a previous problem set, to prove that the infinite dimensional Stiefel manifold is contractible.

4. Go through the proof of Kuiper's theorem and prove that all homotopies are continuous.