Problem Set # 9

M392C: K-theory

Please write up the solutions to 6 problems and turn in by Thursday, November 5.

In this problem set G is a compact Lie group and $\mathfrak{g}$ its Lie algebra.

1. The *Killing form* $\kappa\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ is defined for any Lie algebra $\mathfrak{g}$ by $\kappa(\xi_1,\xi_2) = \mathrm{Trace}\big(\mathrm{ad}(\xi_1)\circ\mathrm{ad}\,\xi_2)\big)$. In this problem assume $\mathfrak{g}$ is the Lie algebra of a compact Lie group.
   (a) Prove that the Killing form is negative semi-definite.
   (b) Prove that the Ad-action of G on $\mathfrak{g}$ is orthogonal for the Killing form.
   (c) Assume the Killing form induces a bi-invariant metric on G. Prove that, in fact, for any bi-invariant metric the Riemannian exponential map at the identity agrees with the exponential map defined from the Lie group structure.

2. Suppose G is a connected compact Lie group.
   (a) Let $\Omega^\bullet_{\mathrm{left}}(G)\subset\Omega^\bullet(G)$ denote the vector subspace of left-invariant differential forms. Show that $\Omega^\bullet_{\mathrm{left}}(G)$ is in fact a sub-differential graded algebra, i.e., it is closed under multiplication and the differential d.
   (b) Construct an isomorphism

$$(*)\qquad \textstyle\bigwedge^\bullet\mathfrak{g}^*\to\Omega^\bullet_{\mathrm{left}}(G).$$

   Transfer the differential on $\Omega^\bullet_{\mathrm{left}}(G)$ to $\bigwedge^\bullet\mathfrak{g}^*$ and write a formula for it. In this way you obtain a differential graded complex defined directly from the Lie algebra $\mathfrak{g}$. Observe that your definition works for *any* Lie algebra (it needn't be the Lie algebra of a compact Lie group).
   (c) Prove that the inclusion in part (a) induces an isomorphism on cohomology. A map of cochain complexes with this property is called a *quasi-isomorphism*. So you can compute the de Rham cohomology of G from this Lie algebra complex. (Hint: Average over G to construct a left-invariant form from an arbitrary form.)
   (d) Use the inverse map $g\mapsto g^{-1}$ to show that the differential of a *bi-invariant* differential form vanishes. Show that the de Rham cohomology of G is isomorphic to the algebra of bi-invariant forms.
   (e) Use these ideas to compute $H^\bullet_{\mathrm{dR}}(SU_2)$.
   (f) Endow G with a bi-invariant metric. Is there a relationship between harmonic forms and bi-invariant forms?

3. (a) Consider the adjoint action of U_n. Let $T \subset U_n$ be the maximal torus of diagonal matrices. What are the root spaces and the roots?

   (b) Repeat for SU_n.

   (c) Repeat for SO_n and Sp_n.

4. Consider the group SU_3 of 3×3 unitary matrices of determinant one.

   (a) Compute the Lie algebra $\mathfrak{su}_3$ of SU_3. What is $\dim SU_3$?

   (b) Construct an Ad-invariant bilinear form on $\mathfrak{su}_3$.

   (c) Choose a maximal torus $T \subset SU_3$ to be the diagonal matrices. What is the rank of SU_3? Identify the lattices Π and Λ as subsets of $\mathfrak{t}$ and $\mathfrak{t}^*$ respectively, where $\mathfrak{t} = \mathrm{Lie}(T)$.

   (d) Find the normalizer $N(T)$ to the torus. Identify the Weyl group $W = N(T)/T$.

   (e) Restrict the adjoint representation of SU_3 to T. Diagonalize this action by complexifying the Lie algebra and compute the function $\Lambda \to \mathbb{Z}$ which specifies the multiplicities of the weights. These are the *roots* of SU_3.

   (f) Compute the weights of the standard representation of SU_3 on $\mathbb{C}^3$.

   (g) Compute the weights of the symmetric square of the standard representation.

5. Let G be a compact Lie group. It is true that there is a countable set of isomorphism classes of irreducible complex representations. Let $\{V_i\}$ be a choice of a set of representative irreducible representations. For any finite dimensional representation V construct a canonical isomorphism

$$\bigoplus_i \mathrm{Hom}_G(V_i, V) \otimes V_i \longrightarrow V.$$

   You might even consider the meaning of 'canonical' and prove that your isomorphism is just that.

6. (a) Let V be a complex vector space. Define the complex conjugate space $\overline{V}$ to be equal to V as an abelian group and with scalar multiplication complex conjugate to that in V. In other words, if $v \in V$ equals $\bar{v} \in \overline{V}$ (recall that $V = \overline{V}$ as a set, even as an abelian group), then for any complex number c, we have $\overline{c \cdot v} = \bar{c} \cdot \bar{v}$. Here the first '$\cdot$' is scalar multiplication in V, the second in $\overline{V}$.

   (b) A *real structure* on V is a linear map $J \colon V \to \overline{V}$ which satisfies $\overline{J} \circ J = \mathrm{id}_V$. Show that the fixed points of J form a real vector space W. Produce a canonical isomorphism $W \otimes \mathbb{C} \to V$.

   (c) A *quaternionic structure* on V is a linear map $J \colon V \to \overline{V}$ which satisfies $\overline{J} \circ J = -\mathrm{id}_V$. Show that in this case V is naturally a module over the quaternions $\mathbb{H}$. It is often convenient to treat quaternionic vector spaces as complex vector spaces with this extra structure.

(d) Suppose G is a compact Lie group and V a complex representation, i.e., a complex vector space with a linear G-action. Then V is *self-conjugate* if there is a real or quaternionic structure which is preserved by the group G. Give an example of a self-conjugate representation. Give an example of a representation which is not self-conjugate. Show that the tensor product of self-conjugate representations is self-conjugate. Discuss the various cases: real $\otimes$ real, real $\otimes$ quaternionic, etc.

(e) Explore how to construct a real representation from a complex representation, a complex representation from a real one, a quaternionic representation from a complex representation, etc. You will be defining certain functors between categories of representations. Spell it out in that language. Investigate various compositions of your functors.

7. Let G be a compact Lie group and V a complex representation. We proved in lecture that V carries an invariant *hermitian* form, even one which is positive definite. Now investigate the existence of an invariant *bilinear* form. Give examples to demonstrate existence or non-existence. If V is irreducible show that the space of invariant forms is zero or one-dimensional, and in the latter case all nonzero forms are either symmetric or skew-symmetric. How does the existence of invariant forms relate to the self-conjugacy of the representation?

8. True or false. Proof or example.

   (a) There is a nontrivial homomorphism $SO_3 \to Sp_1$.

   (b) There is a nontrivial homomorphism $Sp_1 \to SO_3$.

   (c) There is a nontrivial homomorphism $SO_5 \to Sp_2$.

   (d) There is a nontrivial homomorphism $Sp_2 \to SO_5$.

   (e) There is a nontrivial homomorphism $Sp_2 \to \mathrm{Spin}_5$.

   (f) There is a nontrivial homomorphism $SU_3 \to SO_7$.

   (g) There is a nontrivial homomorphism $SO_7 \to SU_3$.

   (h) There is a nontrivial homomorphism $\mathrm{Spin}_7 \to \mathrm{Spin}_8$.

9. Apply the Weyl character formula to deduce the characters of the representations discussed in Problem 4. Explore the rank 2 groups SO_4, SO_5, Sp_2 and U_2. Learn the graphic algorithm at the end of the article of Bott (see web page) for computing the character. There is graph paper available on the web page for SU_3. I welcome graph paper for other groups!